The focusing function is piecewise constant!

$$K(s) = \begin{cases} +K \\ -K \\ 0 \end{cases} \qquad y = \begin{cases} A\sin\sqrt{K}s + B\cos\sqrt{K}s \\ A\sinh\sqrt{K}s + B\cosh\sqrt{K}s \\ A + Bs \end{cases}$$

$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}(s-s_0) & \frac{1}{\sqrt{K}}\sin\sqrt{K}(s-s_0) \\ -\sqrt{K}\sin\sqrt{K}(s-s_0) & \cos\sqrt{K}(s-s_0) \end{pmatrix}, \quad \ldots$$

$$\ell = s - s_0.$$

$$M(s|s_0) = \begin{cases} \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell \end{pmatrix} & K > 0\text{: focusing quad.} \\ \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} & K = 0\text{: drift space} \\ \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell \end{pmatrix} & K < 0\text{: defocussing quad.} \end{cases}$$

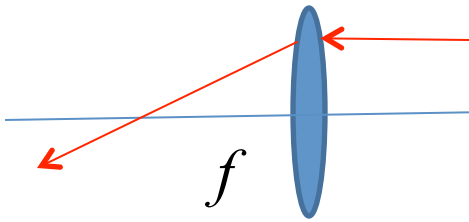
In thin-lens approximation with ℓ → 0, the transfer matrix for a quadrupole reduces to

$$M_{\text{focusing}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}, \quad M_{\text{defocussing}} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \qquad f = \lim_{\ell\to 0}\frac{1}{|K|\ell}$$

$$X'' + K(s)X = 0, \qquad K_x(s) = \frac{1}{\rho^2} \mp \frac{1}{B\rho}\frac{\partial B_z}{\partial x}, \; K_z(s) = \pm \frac{1}{B\rho}\frac{\partial B_z}{\partial x},$$

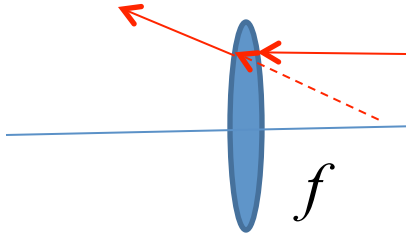
Thin lens approximation: Let $|K|\ell \to 1/f$ as $\ell \to 0$.

1. focusing quadrupole:



$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

2. de-focusing quadrupole:



$$M(s,s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

3. Dipole: $K_x(s)=1/\rho^2$.

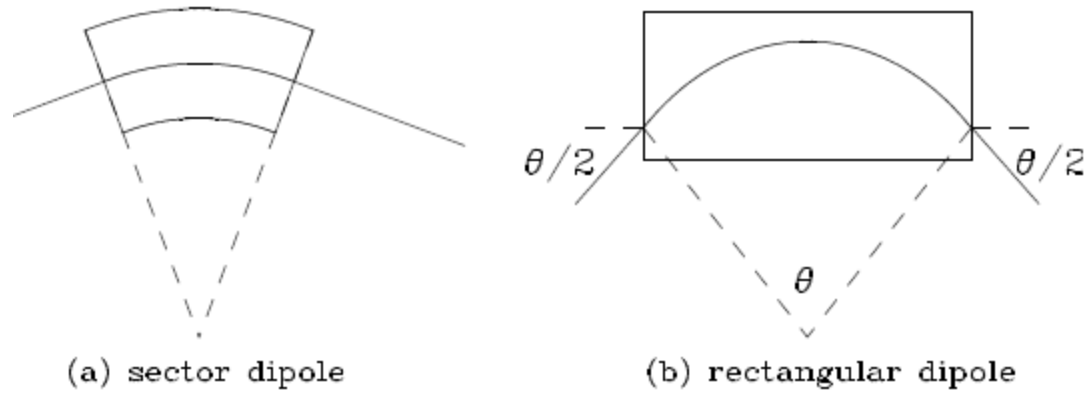
$$M(s,s_0) = \begin{pmatrix} \cos\frac{\ell}{\rho} & \rho\sin\frac{\ell}{\rho} \\ -\frac{1}{\rho}\sin\frac{\ell}{\rho} & \cos\frac{\ell}{\rho} \end{pmatrix} \to \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

4. Drift space: K=0

$$M(s,s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

Sector dipole

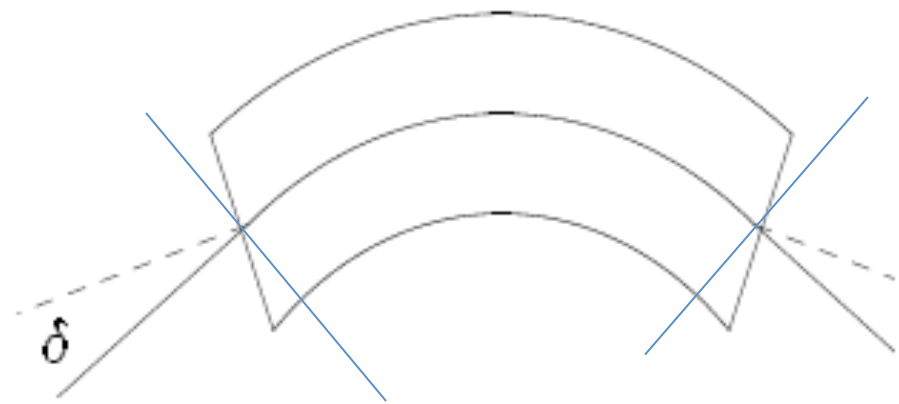
$$M(s,s_0) = \begin{pmatrix} \cos\frac{\ell}{\rho} & \rho\sin\frac{\ell}{\rho} \\ -\frac{1}{\rho}\sin\frac{\ell}{\rho} & \cos\frac{\ell}{\rho} \end{pmatrix}$$



(a) sector dipole (b) rectangular dipole

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{\tan\delta}{\rho} & 1 \end{pmatrix} \quad M_z = \begin{pmatrix} 1 & 0 \\ -\frac{\tan\delta}{\rho} & 1 \end{pmatrix}$$

where δ is the entrance or the exit angle of the particle with respect to the normal direction of the dipole edge. Thus the edge effect with $\delta > 0$ gives rise to horizontal defocussing and vertical focusing.



Using edge focusing, the zero-gradient synchrotron (ZGS) was designed and constructed in the 1960's at Argonne National Laboratory. The ZGS was made of 8 dipoles with a circumference of 172 m attaining the energy of 12.5 GeV. Its first proton beam was commissioned on Sept. 18, 1963. See L. Greenbaum, A Special Interest (Univ. of Michigan Press, Ann Arbor, 1971).

The most general representation of the matrix **M**(s) with **unit modulus** is given by the **Courant-Snyder** parameterization.

$$\mathrm{M}(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} = I\cos\Phi + J\sin\Phi$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad J^2 = -I, \quad \text{or} \quad \beta\gamma = 1 + \alpha^2$$

The ambiguity in the sign of sin can be resolved by requiring β to be a positive definite number if |Trace(M)| ≤ 2, and by requiring Im(sinΦ)>0 if |Trace(M)| > 2. The definition of the phase factor is still ambiguous up to an integral multiple of 2π. This ambiguity will be resolved when the matrix is tracked along the accelerator elements. Using the property of matrix J, we obtain the De Moivere's theorem:

$$\mathbf{M}^k = (\mathbf{I}\cos\Phi + \mathbf{J}\sin\Phi)^k = \mathbf{I}\cos k\Phi + \mathbf{J}\sin k\Phi,$$
$$\mathbf{M}^{-1} = \mathbf{I}\cos\Phi - \mathbf{J}\sin\Phi.$$

The necessary and sufficient condition for **stable orbital motion** is that all matrix elements of the matrix $[\mathbf{M}(s)]^{mP}$ remain bounded as m increases. Let λ_1, λ_2 be the eigenvalues and v_1, v_2 be the corresponding eigenvectors of the matrix **M**. Since M has a unit determinant, the eigenvalues are the reciprocals of each other, i.e. $\lambda_1=1/\lambda_2$, and $\lambda_1+\lambda_2=\mathrm{Trace}(\mathbf{M})$. The eigenvalue satisfies the equation

$$\lambda^2 - Trace(M)\lambda + 1 = 0$$

Let Trace(**M**) = 2 cos(Φ), where Φ is real if Trace(**M**) ≤ 2, and complex if Trace(**M**) > 2. The eigenvalues are $\lambda_1=e^{j\Phi}$ and $\lambda_2=e^{-j\Phi}$, where Φ is the betatron phase advance of a periodic cell. Expressing the initial condition of beam coordinates (X_0, X'_0) as a linear superposition of the eigenvectors:

$$\begin{pmatrix} X_0 \\ X'_0 \end{pmatrix} = a\upsilon_1 + b\upsilon_2$$

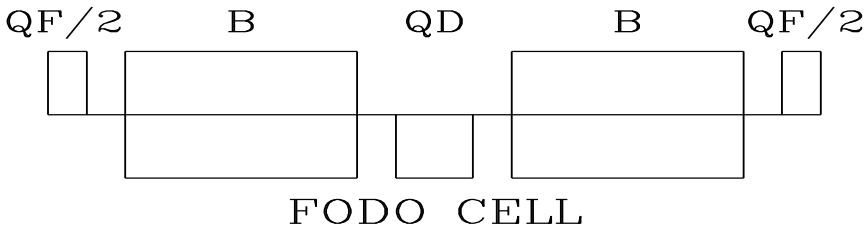
we find that the particle coordinate after the mth revolution becomes

$$\begin{pmatrix} X_m \\ X'_m \end{pmatrix} = M^m \begin{pmatrix} X_0 \\ X'_0 \end{pmatrix} = a\lambda_1^m \upsilon_1 + b\lambda_2^m \upsilon_2$$

The stability of particle motion requires that λ_1^m and λ_2^m not grow with m. Thus a necessary condition for orbit stability is to have a real betatron phase advance Φ, or

$$Trace(M) \le 2$$

Example: FODO cell



A FODO cell is a basic block in beam transport, where the transfer matrices for dipoles (B) can be approximated by drift spaces, and QF and QD are the focusing and defocusing quadrupoles.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix}$$

$$\mathrm{M}(s) = \begin{pmatrix} \cos\Phi + \alpha \sin\Phi & \beta \sin\Phi \\ -\gamma \sin\Phi & \cos\Phi - \alpha \sin\Phi \end{pmatrix}$$

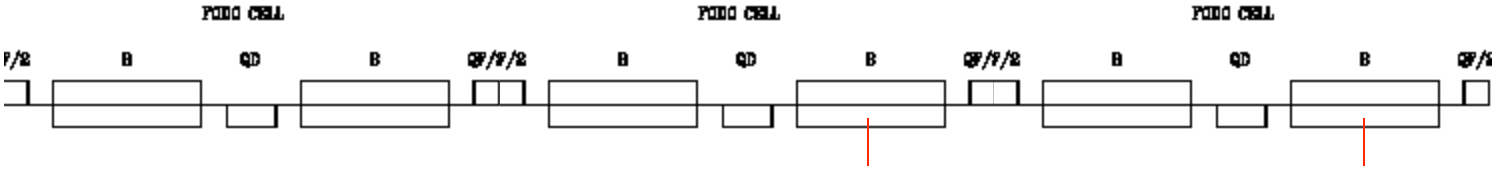
$$\cos\Phi = \tfrac{1}{2}\mathrm{Tr}(\mathrm{M})$$

$$\cos\Phi = 1 - \frac{L_1^2}{2f^2}, \quad \sin\frac{\Phi}{2} = \frac{L_1}{2f}$$

$$\beta = \frac{2L_1(1 + \frac{L_1}{2f})}{\sin\Phi} = \frac{2L_1(1 + \sin\frac{\Phi}{2})}{\sin\Phi}$$

$$\alpha = 0$$

Example: FODO cell



$$\begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{L}{4} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{3L}{4} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

Questions:

1) Will Φ of these above matrix be identical?

2) Will α and β of these matrices be identical?

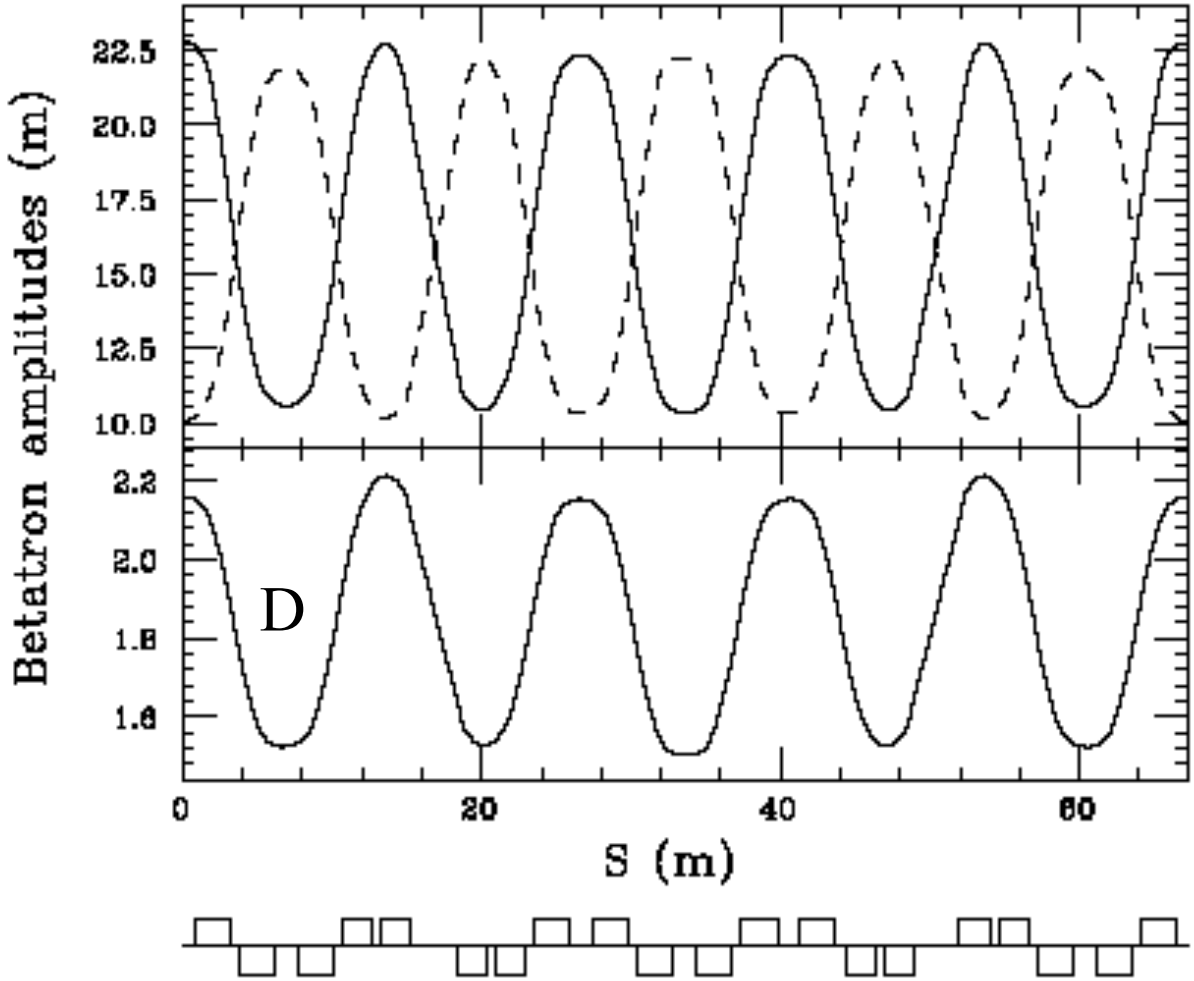
3) What are the meanings of these parameters?

AGS

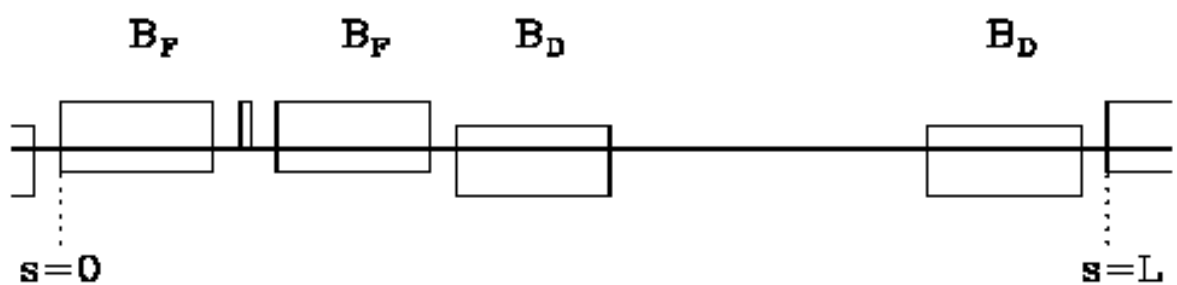
$$\beta = \frac{2L_1(1+\sin\frac{\Phi}{2})}{\sin\Phi}$$

$$\alpha = 0$$

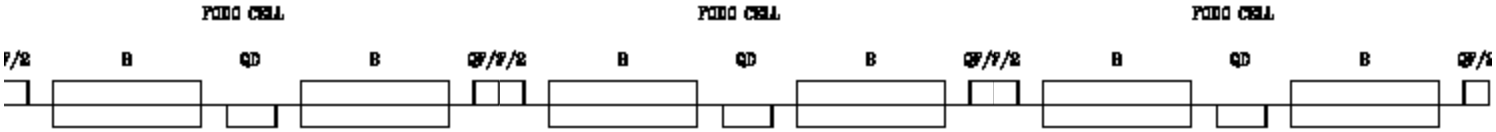
$$\sin\frac{\Phi}{2} = \frac{L_1}{2f}$$



Fermilab Booster



Stability of accelerator cells: FODO cell example



$$\begin{pmatrix} 1 & 0 \\ \frac{-1}{f_1} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi_x + \alpha_x \sin\Phi_x & \beta_x \sin\Phi_x \\ -\gamma_x \sin\Phi_x & \cos\Phi_x - \alpha_x \sin\Phi_x \end{pmatrix}$$

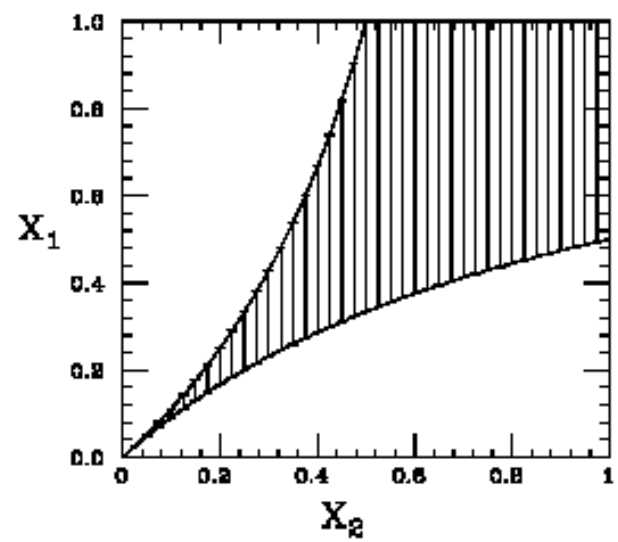
$$\begin{pmatrix} 1 & 0 \\ \frac{1}{f_1} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi_z + \alpha_z \sin\Phi_z & \beta_z \sin\Phi_z \\ -\gamma_z \sin\Phi_z & \cos\Phi_z - \alpha_z \sin\Phi_z \end{pmatrix}$$

$$\cos\Phi_x = 1 + \frac{L}{f_2} - \frac{L}{f_1} - \frac{L^2}{2f_1 f_2} = 1 + 2X_2 - 2X_1 - 2X_1 X_2$$

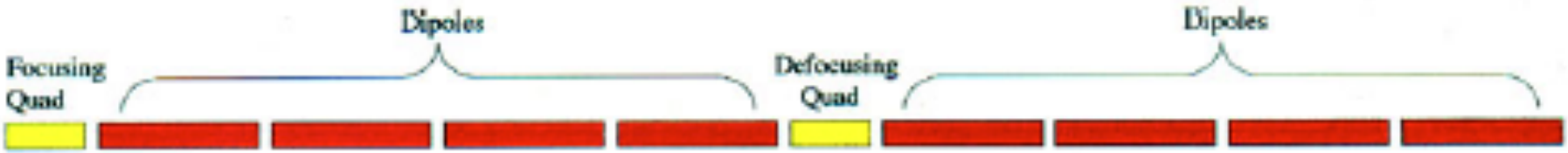
$$\cos\Phi_z = 1 - \frac{L}{f_2} + \frac{L}{f_1} - \frac{L^2}{2f_1 f_2} = 1 - 2X_2 + 2X_1 - 2X_1 X_2$$

Stability condition: (necktie diagram)

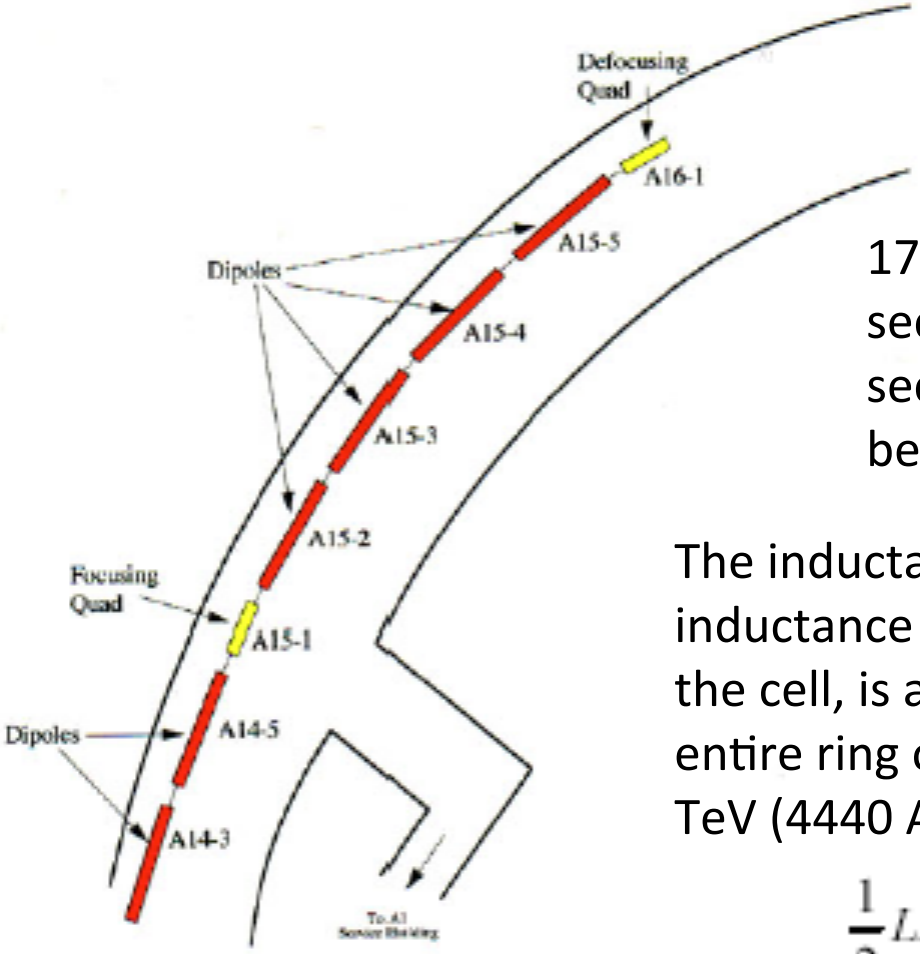
$$|\cos\Phi_x| \le 1, \quad |\cos\Phi_z| \le 1.$$

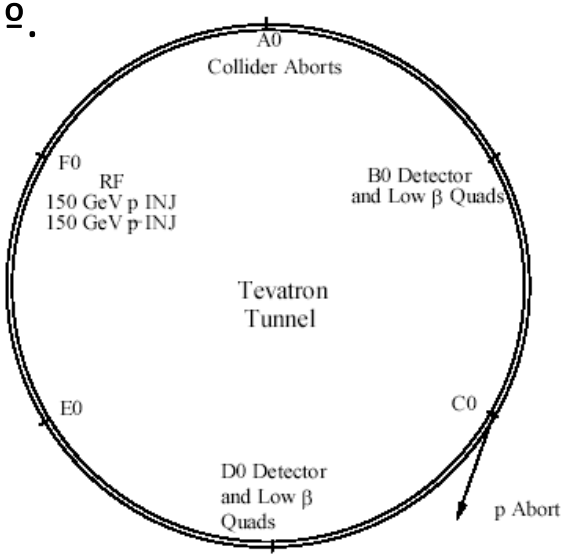



TEVATRON



A half-cell is composed of a quadrupole, a mini-straight section for correction coil spool piece, and 4 dipoles. A cell deflects the beam by 3.5º.





17 cells for each sector. Since each sector bends the beam 60 degrees.

The inductance of a typical "half cell", that is the inductance of either the upper **or** lower bus through the cell, is about 0.18 H. The inductance for the entire ring of 36 H. The inductive stored energy at 1 TeV (4440 A) is 350MJ.

$$\frac{1}{2}LI^2 = \frac{1}{2}(36H)(4440A)^2 = 3.50x10^8 Joules.$$