PHY 564
Advanced Accelerator Physics
Lectures 19

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Solutions of standard accelerator problems

Q: Why we need parameterization? -> A: To comfortably solve typical accelerator problems

Lecture 17

Applications of parameterization to standard problems

Complete parameterization developed in previous lecture can be used to solve most (if not all) of standard problems in accelerator. Incomplete list is given below:

1. Dispersion
2. Orbit distortions
3. AC dipole (periodic excitation)
4. Tune change with quadrupole (magnets) changes
5. Chromaticity
6. Beta-beat
7. Weak coupling
8. Synchro-betatron coupling
9. ……..

We do not plan to go through all these examples while focusing on general methodology and use selected examples to demonstrate power of the symplectic linear parameterization.

It is not all… But already, not too shabby for a single parameterization

$$X(s) = \frac{1}{2}\tilde{\mathbf{U}}(s)\cdot A(s) = \operatorname{Re}\sum_{k=1}^{n} Y_k(s)e^{i(\psi_k(s)+\varphi_k)}a_k(s);$$

$$\frac{d}{ds}\tilde{\mathbf{U}}(s) = \mathbf{D}(s)\cdot\tilde{\mathbf{U}}(s);\tilde{\mathbf{U}} = \left[\ldots Y_k e^{i\psi_k}, Y_k^* e^{-i\psi_k}, \ldots\right]; k = 1,..,n$$

$$\tilde{\mathbf{U}}^T\mathbf{S}\tilde{\mathbf{U}} = 2i\mathbf{S}.$$

Lecture 19.

During last lecture we discuss a number of way how both the parameterization of the motion in linear Hamiltonian system can be used to solve variety of standard problems arising in accelerator physics. Some of them were exact solutions (like orbit distortions or dispersion function), but some of them were clearly perturbative and relied on averaging over fast oscillations. The later, while intuitively understandable, requires some more discussions – and this is what we start doing today. Let's consider an additional (not necessarily a simple, constant or linea, but definitely a weak) term in our equations of motion

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X + \varepsilon F(X,s); \tag{19-1}$$

Using our already well established parameterization, we can always write:

$$X = \frac{1}{2}\tilde{\mathbf{U}}(s)A(s) = \mathrm{Re}\sum_{k=1}^{n} a_k(s)Y_k(s)e^{i(\psi_k(s)+\varphi_k(s))}; A(s) = \begin{bmatrix} \dots \\ a_k e^{i\varphi_k} \\ a_k e^{-i\varphi_k} \\ \\ \end{bmatrix};$$

$$\tilde{\mathbf{U}}(s)\frac{d}{ds}A = \varepsilon F\left(\tilde{\mathbf{U}}(s)A(s),s\right) \Leftrightarrow \frac{d}{ds}a_k e^{i\varphi_k} = \varepsilon\frac{e^{-i\psi_k(s)}}{i}Y_k^{*T}(s)\mathbf{S}F\left(\tilde{\mathbf{U}}(s)A(s),s\right) \tag{19-2}$$

If one likes real form of the equations, it can be written as

$$\frac{d}{ds}a_k e^{i\varphi_k} = \left(a_k' + i\varphi_k' a_k\right)e^{i\varphi_k} = \varepsilon\frac{e^{-i\psi_k}}{i}Y_k^{*T}\mathbf{S}F;$$
$$\frac{d}{ds}a_k e^{-i\varphi_k} = \left(a_k' - i\varphi_k' a_k\right)e^{-i\varphi_k} = -\varepsilon\frac{e^{i\psi_k}}{i}Y_k^{T}\mathbf{S}F; \qquad (19\text{-}3)$$
$$\frac{da_k}{ds} = \varepsilon\,\mathrm{Im}\left[e^{i(\psi_k+\varphi_k)}Y_k^T\mathbf{S}F\right]; a_k\cdot\frac{d\varphi_k}{ds} = \varepsilon\,\mathrm{Re}\left[e^{i(\psi_k+\varphi_k)}Y_k^T\mathbf{S}F\right];$$

In analytical mechanics, these equations for constant of motion in linear system are called "reduced" or "slow" equations when ε is so small that it significantly affect the motion only when right side of equation has constant terms, e.g. either the phase or amplitude of oscillations can grow in time, not just simply oscillate.

As an example, let's consider a 1D motion with write side having a power of x:

$$F = \begin{bmatrix} 0 \\ f(s)x^m \end{bmatrix}; x = a\mathrm{w}\cos(\psi+\varphi)$$

$$\frac{da}{ds} = \varepsilon f a^m \mathrm{w}^m \sin(\psi+\varphi)\cdot\cos^m(\psi+\varphi); \frac{d\varphi}{ds} = \varepsilon f \mathrm{w}^m a^{m-1}\cos^{m+1}(\psi+\varphi); \qquad (19\text{-}4)$$

The equations (19-4) are non-linear and do not have explicit analytical solution in general case (we know that it can be parameterized for n=1). Let's now consider a periodical system:

$$\psi(s)+\mu \rightarrow \psi(s)=\chi(s)+\frac{\mu s}{C};$$

$$\chi(s+C)=\chi(s)\mathrm{w}(s+C)=\mathrm{w}(s); f(s+C)=f(s); \psi(s+C);$$

$$\frac{da}{ds}=\varepsilon f a^m \mathrm{w}^m \sin\left(\frac{\mu s}{C}+\chi+\varphi\right)\cdot\cos^m\left(\chi(s)+\frac{\mu s}{C}+\varphi\right); \frac{d\varphi}{ds}=\varepsilon f \mathrm{w}^m a^{m-1}\cos^{m+1}\left(\frac{\mu s}{C}+\chi+\varphi\right);$$

Considering that slow variable are nearly constant, we have on the right side terms oscillating with phase advancing as $(k\mu \pm 2\pi j)\frac{s}{C}=2\pi\frac{s}{C}(kQ\pm j); -m\le k\le m; j-\mathrm{int}\,eger$. Only when $kQ\pm j=0$ (or close to zero – see next) we have a stationary growth. Otherwise, the oscillating terms will average.

One can intuitively expand the variation of constants a power of the infinitesimal ε

$$a = a_o + \sum_{k=1} a_k \varepsilon^k; \varphi = \varphi_o + \sum_{k=1} \varphi_k \varepsilon^k$$

$$\frac{da}{ds} = \varepsilon a^m \mathrm{w}^m \sin(\psi+\varphi)\cdot\cos^m(\psi+\varphi); \frac{d\varphi}{ds} = \varepsilon \mathrm{w}^m a^{m-1}\cos^{m+1}(\psi+\varphi); \tag{19-5}$$

and explore it further. But this will bring us to a method developed by Bogolyubov and Metropolsky (N. Bogolubov N. (1961). *Asymptotic Methods in the Theory of Non-Linear Oscillations*. Paris: Gordon & Breach. ISBN 978-0-677-20050-7.) in analytical mechanics. You can find a straightforward, but rather long derivation in the book – here we will only discuss the results. Let's start from an equation of motion with a small (infinitesimally) perturbation for a linear system with deduced equation of

$$\frac{dA}{ds} = \varepsilon F(X,s); \tag{19-6}$$

than the first order perturbation can be written as

$$A = \xi(s) + \varepsilon\tilde{F}(\xi,s); \frac{d}{ds}\xi(s) = \langle F(\xi,s)\rangle;$$

$$\langle F(A,s)\rangle = \frac{1}{S}\int_s^{s+S}\langle F(A=const,s)\rangle ds;\ \ \tilde{F} = \int\left(F-\langle F\rangle\right)ds; \tag{19-7}$$

What is quite remarkable, that they also derived a second order perturbation:

$$A = \xi(s) + \varepsilon\tilde{F}(\xi,s) + \varepsilon^2 \overbrace{\left\{\left(\tilde{F}\frac{\partial}{\partial\xi}\right)F\right\}}^{\sim} - \varepsilon^2 \frac{\partial\tilde{\tilde{F}}}{\partial\xi}\left\langle F(\xi,s)\right\rangle;$$

$$\frac{d}{ds}\xi(s) = \varepsilon\left\langle F(\xi,s) + \varepsilon\tilde{F}\right\rangle \approx \varepsilon\left\langle\left(1 + \varepsilon\left(\tilde{F}\frac{\partial}{\partial\xi}\right)\right)F(\xi,s)\right\rangle. \quad (19\text{-}8)$$

These equations were used and still used to derive a number of analytical expressions for nonlinear

Let's consider a case we already studies during last class: **small variation of the quadrupole gradient**. It can come from errors in quadrupoles or from a deviation of the energy from the reference value. In 1D case (reduced) it is simple addition to the Hamiltonian: (including sextupole term!)

$$\delta H = \delta K_1(s)\frac{x^2}{2} = I \cdot \delta K_1(s)\beta(s)\cos^2\left(\psi(s)+\varphi\right)$$

$$\frac{d\varphi}{ds} = \frac{\partial \delta H}{\partial I} = \delta K_1 \beta \cos^2(\psi+\varphi) = \delta K_1 \beta \frac{1+\cos 2(\psi+\varphi)}{2}; \qquad (19\text{-}9)$$

$$\frac{dI}{ds} = -\frac{\partial \delta H}{\partial \varphi} = I \cdot \delta K_1(s)\beta(s)\sin 2(\psi+\varphi);$$

Using first order approximation we get:

$$\frac{d\langle\varphi\rangle}{ds} = \frac{1}{S}\int_s^{s+S} \delta K_1(s)\beta(s)\frac{1+\cos 2(\psi+\varphi)}{2} =$$

$$\frac{\langle \delta K_1(s)\beta(s)\rangle}{2} + \frac{\langle \delta K_1(s)\beta(s)\cos 2(\psi+\varphi_o)\rangle}{2} \qquad (19\text{-}10)$$

$$\frac{d\langle I\rangle}{ds} = I_o \cdot \langle \delta K_1(s)\beta(s)\sin 2(\psi+\varphi_o)\rangle;$$

We already got the first term average term

$$\frac{\left\langle \delta K_1(s)\beta(s)\right\rangle}{2} = \frac{1}{2C}\int_0^C \delta K_1(s)\beta(s) \qquad (19\text{-}10)$$

while the amplitude does not have obvious non-oscillating term. Oscillating terms are also of some interest - let's explore them:

$$\frac{d\tilde{\varphi}}{ds} = \mathrm{Re}\frac{\delta K_1(s)\beta(s)}{2} e^{i\chi(s)} e^{i\frac{4\pi Q}{C}s} e^{2i\varphi_o} = \mathrm{Re}\sum_{k=-\infty}^{\infty} c_k e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o}$$

$$\frac{d\tilde{I}}{ds} = I_o \cdot \mathrm{Im}\frac{\delta K_1(s)\beta(s)}{2} e^{i\chi(s)} e^{i\frac{4\pi Q}{C}s} e^{2i\varphi_o} = I_o \cdot \mathrm{Im}\sum_{k=-\infty}^{\infty} c_k e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o} \qquad (19\text{-}11)$$

$$\frac{\delta K_1(s)\beta(s)}{2} e^{i\chi(s)} = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i\frac{k}{C}s}.$$

where we simply expanded periodic complex function into a Fourier series. (19-11) is easy to integrate

$$\tilde{\varphi} = -\frac{C}{2\pi}\mathrm{Im}\sum_{k=-\infty}^{\infty}\frac{c_k}{2Q+k} e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o} = \mathrm{Re}\,\phi(s)e^{2i(\psi+\varphi_o)};\;\; \phi(s+C) = \phi(s)$$

$$\tilde{I} == -I_o \cdot \frac{C}{2\pi}\mathrm{Re}\sum_{k=-\infty}^{\infty}\frac{c_k}{2Q+k} e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o} = I_o \cdot \mathrm{Re}\,\upsilon(s)e^{2i(\psi+\varphi_o)};\;\; \upsilon(s+C) = \upsilon(s) \qquad (19\text{-}12)$$

Unless the accelerator is “sitting” at a parametric resonance $2Q = \pm k$, there oscillating term simply oscillating with double betatron frequency. Otherwise, at the parametric resonance $2Q = \pm k$ both the amplitude and the phase can grow – e.g. it is an instability we have to stay away from.