

## Homework 13

### Problem 1. 15 points. Turning the beam around – ultimate storage rings

Let's consider that we build a storage ring (magnets only), where ultra-relativistic charged particles traveling in circle of constant radius  $R$  while radiating synchrotron radiation. It means that the magnetic field is adjusted to the loss of its energy.

- Find the energy of the particle as function of the traveled distance  $s$  or angle  $s/R$ ;
- Find the distance when the particle's energy is reduced by a factor 2.
- Loosing half of the energy is considered to be "dead-end" for recirculating the beams – than linear accelerators have to do the job. For  $R$  being 6,371 kilometers – that of the Earth, find critical energy of electrons, muons and protons when particles are loosing  $\frac{1}{2}$  of the energy in a single turn.

### Problem 2. 10 points. Circulating particle in magnetic field

Consider ultra-relativistic charged particle with initial energy circulating in an uniform constant magnetic field  $\mathbf{B}_y$ .

- Find energy of the particle as function of time.
- What will be its trajectory?

Note: Neglect non-relativistic effects

Solution:

**Problem 1:** since we are considering ultra-relativistic particles, we can assume that  $s=ct$ , e.g. neglect  $(1-\beta) \ll 1$ . (a) Losses for radiation with fixed radius are

$$\frac{dE_{SR}}{ds} = -mc^2 \frac{d\gamma}{ds} \cong \frac{2}{3} \gamma^4 \frac{e^2}{R^2}; \quad (22-12)$$

where we used obvious:  $E = \gamma mc^2$ ;  $dE = -dE_{SR}$ . Solution is straightforward:

$$-\frac{d\gamma}{\gamma^4} = \frac{2}{3} \frac{r_c}{R^2} ds; \quad r_c = \frac{e^2}{mc^2}; \quad \frac{\gamma^{-3} - \gamma_o^{-3}}{3} = \frac{2}{3} \frac{r_c}{R^2} s = \frac{2}{3} \frac{r_c}{R} \theta; \quad \theta = \frac{s}{R};$$

$$\gamma = \frac{\gamma_o}{\sqrt[3]{1 + 2\gamma_o^3 \frac{r_c}{R^2} s}} = \frac{\gamma_o}{\sqrt[3]{1 + 2\gamma_o^3 \frac{r_c}{R} \theta}};$$

(b)  $\gamma = \gamma_o/2$  means

$$\sqrt[3]{1 + 2\gamma_o^3 \frac{r_c}{R^2} s} = 2 \rightarrow s_{1/2} = \frac{7}{2} \frac{R^2}{\gamma_o^3 r_c}.$$

(c) with  $R = 6.371 \times 10^6$  m one turn is  $s = 2\pi R$  and we have the relativistic factor of a particle loosing  $\frac{1}{2}$  of its energy in one turn around the Earth:

$$(d) \quad s_{1/2} = 2\pi R = \frac{7}{2} \frac{R^2}{\gamma_{cr}^3 r_c} \rightarrow \gamma_{cr} = \sqrt[3]{\frac{7}{4\pi} \frac{R}{r_c}}$$

Classical radius of the electron is  $2.82 \times 10^{-15}$  m we get critical  $\gamma_{cr} = 1.08 \times 10^7$ . The rest energy of electron is  $m_e c^2 = 0.511 \times 10^6$  eV (0.511 MeV), it means that the dead-end energy of electron storage ring at Earth is

$$E_{cre} = 2\gamma_{cr} m_e c^2 = 5.52 \cdot 10^{12} \text{ eV} = 5.52 \text{ TeV}$$

Rest energy of a muon is  $m_\mu c^2 = 1.057 \times 10^8$  eV (106 MeV), classical radius of  $1.36 \times 10^{-17}$  m,  $\gamma_{cr} = 6.39 \times 10^7$  and

$$E_{cr\mu} = 2\gamma_{cr} m_\mu c^2 = 6.75 \cdot 10^{15} \text{ eV} = 6,747 \text{ TeV}$$

For proton with  $m_p c^2 = 1.057 \times 10^8$  eV (106 MeV), classical radius of  $1.53 \times 10^{-18}$  m,  $\gamma_{cr} = 1.32 \times 10^8$  and

$$E_{crp} = 2\gamma_{cr} m_p c^2 = 1.24 \cdot 10^{17} \text{ eV} = 1.24 \cdot 10^5 \text{ TeV}$$

Note, that the later will require average bending magnetic field of 65 T, which is not within reach of current technology.

### Problem 2. 10 points. Circulating particle in magnetic field

The losses of ultra-relativistic charge particle circulating in constant magnetic fields is (a)

$$\frac{1}{R} = \frac{eB}{pc} \cong \frac{eB}{E} = \frac{eB}{\gamma mc^2} = \frac{1}{\gamma} \frac{1}{\rho_m}; \frac{d\gamma}{dt} \cong \frac{1}{c} \frac{d\gamma}{ds}; \rho_m = \frac{mc^2}{eB};$$

$$\frac{d\gamma}{dt} \cong -\frac{2}{3c} \gamma^2 \frac{r_c}{\rho_m^2} \rightarrow \frac{d\gamma}{\gamma^2} = \frac{2}{3c} \frac{r_c}{\rho_m^2} dt = \frac{2}{3} \frac{r_c}{\rho_m^2} ds;$$

$$\gamma^{-1} - \gamma_o^{-1} = \frac{2}{3} \frac{r_c}{\rho_m^2} s = \frac{2}{3c} \frac{r_c}{\rho_m^2} t \rightarrow \gamma(t) = \frac{\gamma_o}{1 + \frac{2}{3c} \frac{r_c}{\rho_m^2} \gamma_o t}$$

$$\gamma(s) = \frac{\gamma_o}{1 + \frac{2\gamma_o}{3} \frac{r_c}{\rho_m^2} s}$$

(b) The easiest is to describe it as radius dependence on the bending angle in parametric form:

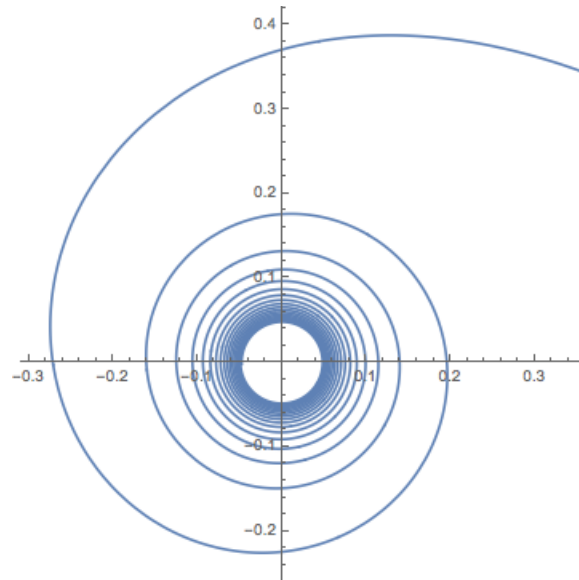
$$d\theta = \frac{ds}{R(s)} = \frac{1 + \frac{2\gamma_o}{3} \frac{r_c}{\rho_m^2} s}{\gamma_o \rho_m} ds;$$

$$R(s) = \gamma \rho_m = \frac{\gamma_o \rho_m}{1 + \frac{2\gamma_o}{3} \frac{r_c}{\rho_m^2} s}; \theta = \frac{s + \frac{\gamma_o}{3} \frac{r_c}{\rho_m^2} s^2}{\gamma_o \rho_m}.$$

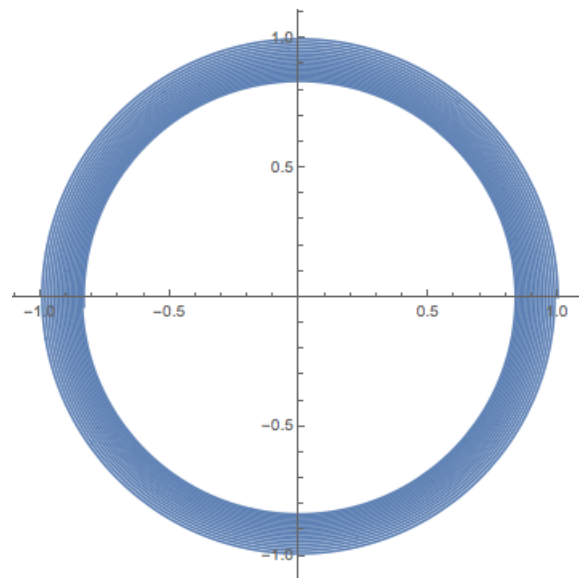
In dimensionless form it will be

$$\frac{R(s)}{\gamma_o \rho_m} = \frac{R(s)}{R_o} = \frac{1}{1+2\zeta}; \theta = \alpha \zeta (1+\zeta); \zeta = \frac{\gamma_o r_c}{3 \rho_m^2} s; \alpha = \frac{3 \rho_m}{\gamma_o^2 r_c};$$

For  $\alpha = 1$ :



For  $\alpha = 1000$ :



Since energy is lost, it is always a collapsing