PHY 554

Fundamentals of Accelerator Physics

Lecture 9: Introduction to RF Accelerators

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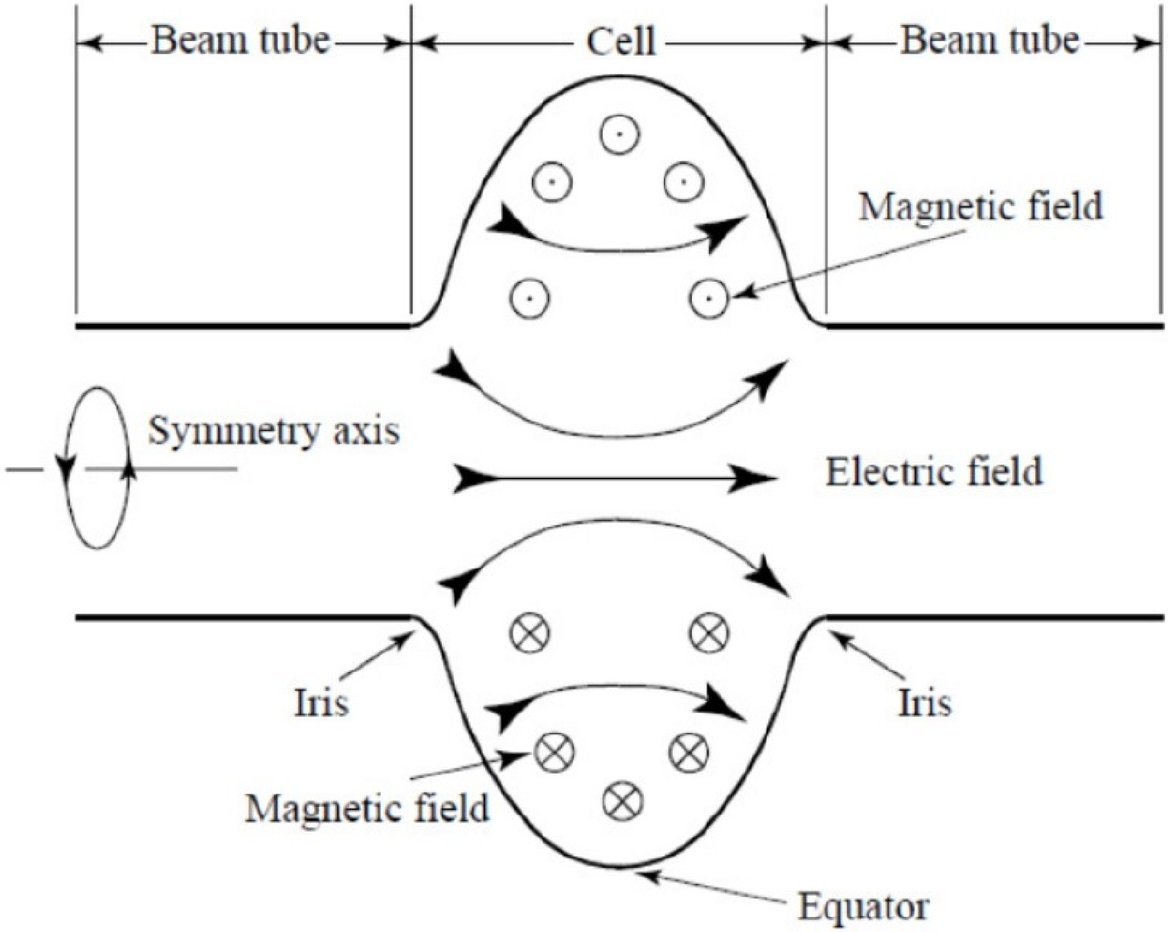


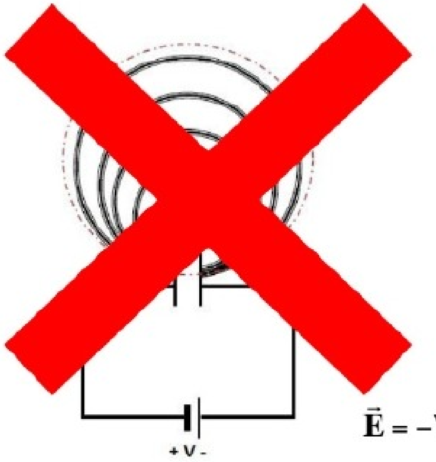
Electrostatic acceleration: what is the limit?

- In a static electric field, the energy gain is determined by the potential difference:

$$\Delta\mathcal{E} = q\int_A^B \mathbf{E}\cdot \mathrm{d}\mathbf{l} = q(V_A - V_B).$$

- Returning through a DC accelerating structure cannot provide unlimited repeated energy gain.
- Maxwell's equations and energy conservation cannot be bypassed.

$$\Delta E = e\oint \vec{\mathrm{E}}\cdot d\vec{l} = -e\frac{\partial}{\partial t}\left(\int \cancel{\vec{\mathrm{H}}\cdot d\vec{s}}\right)$$

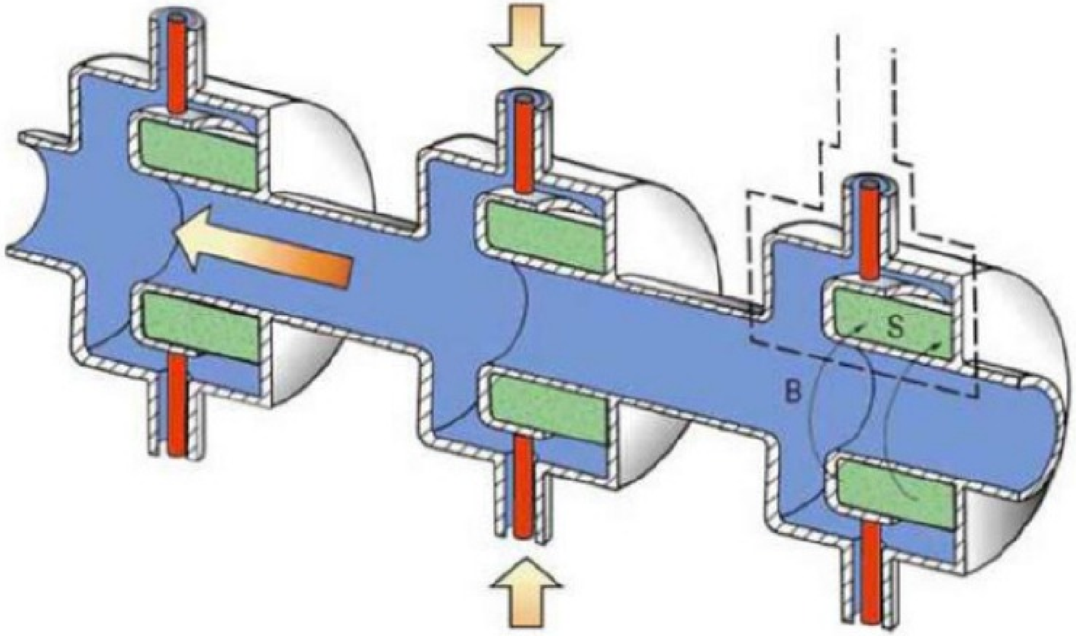


$$\Delta E = e\oint \vec{\mathrm{E}}\cdot d\vec{l} = 0$$

$$\vec{\mathrm{E}} = -\vec{\nabla}\varphi \quad \rightarrow \quad E(\vec{r}) = E(0) - e\varphi(\vec{r})$$

Can one gain energy again and again by passing through a DC accelerating gap? No.

Induction linacs: linear betatrons



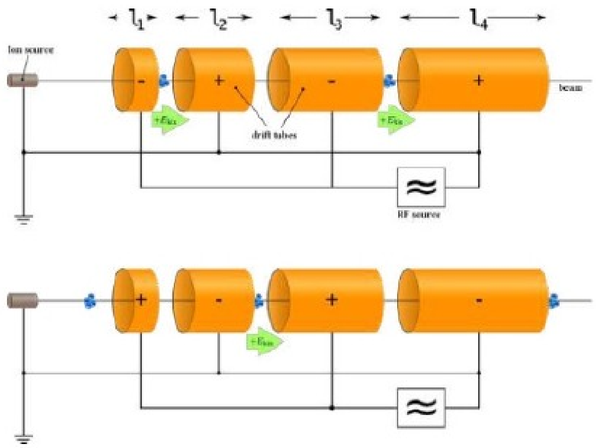
$$\oint \vec{\mathbf{E}} \cdot d\vec{\ } = -\frac{\partial}{\partial t}\left(\int \vec{\mathbf{H}} \cdot d\vec{s}\right)$$

- Acceleration is produced by a time-varying magnetic flux.
- Useful for high-power and high-current beams.
- Accelerating field is limited.
- Pulsed by nature, with relatively low repetition rate (typically kHz).

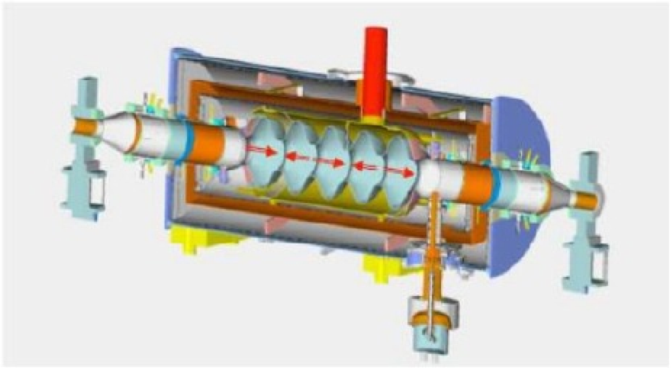
How an RF accelerator works

- The accelerating field oscillates in time, typically sinusoidally, and has a longitudinal component along the particle trajectory.
- The structure is divided into cells. Cell geometry and RF phase synchronize the field with the particle arrival.
- Energy gain from successive cells therefore adds coherently.

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = q\,\mathbf{E}\cdot\mathbf{v}.$$



Wideröe linac: $\beta = v/c$ changes along the linac.



Electron linac: $\beta \simeq 1$.

Simple things to remember

- DC electrostatic acceleration is limited by the available terminal potential difference.
- RF linear accelerators are not fundamentally limited in beam energy by a single terminal voltage.
- Coherent energy gain from cell to cell is achieved by matching the RF oscillation period to the particle travel time between cells.
- Accurate synchronization of RF phase and beam arrival time is essential for any RF linac.

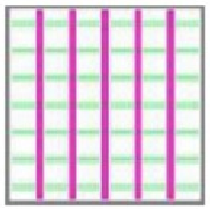
Maxwell's equations in CGS (Gaussian) units

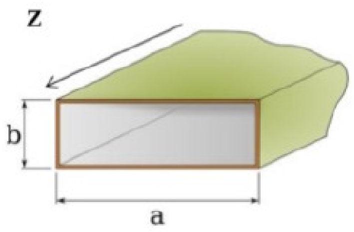
$$\nabla\cdot\mathbf{E} = 4\pi\rho, \qquad \nabla\cdot\mathbf{B} = 0,$$

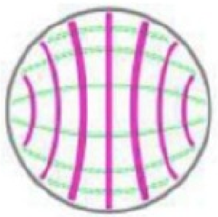
$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}.$$

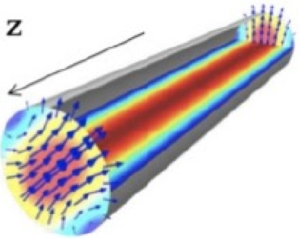
Waveguides

Rectangular | **Circular**









$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = 0, \qquad \Delta \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

$$\vec{A} = \mathrm{Re}\left\{\vec{A}(\vec{r}_\perp)\exp[i(\omega t - k_z z)]\right\},$$

$$\nabla_\perp^2 \vec{A} + \left(k_0^2 - k_z^2\right)\vec{A} = 0, \qquad k_0 = \frac{\omega}{c},$$

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

At the surfaces

$$\vec{n} \times \vec{E}\Big|_s = 0, \qquad \vec{n}\cdot\vec{B} = 0 \quad \Longrightarrow \quad E_z\Big|_s = 0, \qquad \frac{\partial B_z}{\partial n}\Big|_s = 0.$$

TE and TM waves: transverse fields from E_z and B_z

$$k_0 = \frac{\omega}{c}, \qquad k_c^2 = k_0^2 - k_z^2, \qquad \nabla_\perp = \hat{\boldsymbol{x}}\,\partial_x + \hat{\boldsymbol{y}}\,\partial_y.$$

In source-free CGS units, Maxwell's curl equations become

$$\nabla \times \boldsymbol{E} = \mathrm{i}k_0\boldsymbol{B}, \qquad \nabla \times \boldsymbol{B} = -\mathrm{i}k_0\boldsymbol{E}.$$

Writing $\boldsymbol{E} = \boldsymbol{E}_\perp + \hat{\boldsymbol{z}}E_z$ and $\boldsymbol{B} = \boldsymbol{B}_\perp + \hat{\boldsymbol{z}}B_z$, their transverse parts give

$$\begin{aligned} \mathrm{i}k_z\,\hat{\boldsymbol{z}}\times\boldsymbol{E}_\perp - \hat{\boldsymbol{z}}\times\nabla_\perp E_z &= \mathrm{i}k_0\boldsymbol{B}_\perp, \\ \mathrm{i}k_z\,\hat{\boldsymbol{z}}\times\boldsymbol{B}_\perp - \hat{\boldsymbol{z}}\times\nabla_\perp B_z &= -\mathrm{i}k_0\boldsymbol{E}_\perp. \end{aligned}$$

Solving these two coupled equations,

$$\boxed{\boldsymbol{E}_\perp = \frac{\mathrm{i}}{k_c^2}\left(k_z\nabla_\perp E_z - k_0\,\hat{\boldsymbol{z}}\times\nabla_\perp B_z\right)},$$

$$\boxed{\boldsymbol{B}_\perp = \frac{\mathrm{i}}{k_c^2}\left(k_z\nabla_\perp B_z + k_0\,\hat{\boldsymbol{z}}\times\nabla_\perp E_z\right)}.$$

TE: $E_z = 0,\ B_z \neq 0$ **TM:** $B_z = 0,\ E_z \neq 0$

E_z and B_z therefore completely determine the transverse EM fields.

Cut-off frequency

The longitudinal field satisfies a transverse eigenvalue equation,

$$\nabla_\perp^2 \Psi + k_c^2 \Psi = 0,$$

while the longitudinal propagation constant obeys

$$k_z^2 = \frac{\omega^2}{c^2} - k_c^2.$$

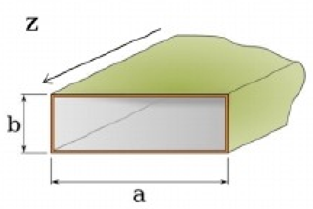
Hence the cut-off frequency is

$$\boxed{\omega_c = c\, k_c}.$$

- $\omega > \omega_c$: k_z is real and the mode propagates.
- $\omega < \omega_c$: $k_z = \mathrm{i}\alpha$ and the field is evanescent, $\sim e^{-\alpha z}$.
- TE and TM modes obey different boundary conditions and therefore have different eigenvalues k_c.

Cut-off frequency: common waveguides

Rectangular



$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi+\gamma_{mn}^2\psi=0$$

$$\mathrm{TE}:\ \psi_{mn}^{\mathrm{TE}}=\psi_0\cos k_m x\cos k_n y,\quad m+n\geq 1,$$

$$\mathrm{TM}:\ \psi_{mn}^{\mathrm{TM}}=\psi_0\sin k_m x\sin k_n y,\quad m,n\geq 1,$$

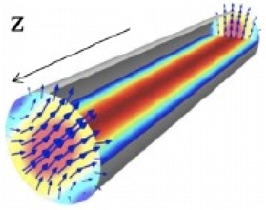
$$k_m=\frac{m\pi}{a},\quad k_n=\frac{n\pi}{b},\quad \gamma_{mn}^2=k_m^2+k_n^2.$$

Lowest cut-off angular frequency

$$\mathrm{TE}_{10}:\ a>b,\qquad \boxed{\omega_c=\frac{\pi c}{a}},$$

$$\mathrm{TM}_{11}:\qquad \boxed{\omega_c=\pi c\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}.$$

Circular



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\gamma_{mn}^2\psi=0,$$

$$\psi_{mn}=\varphi_{mn}(r)e^{in\theta},\qquad \varphi_{mn}=J_n(\gamma_{mn}r).$$

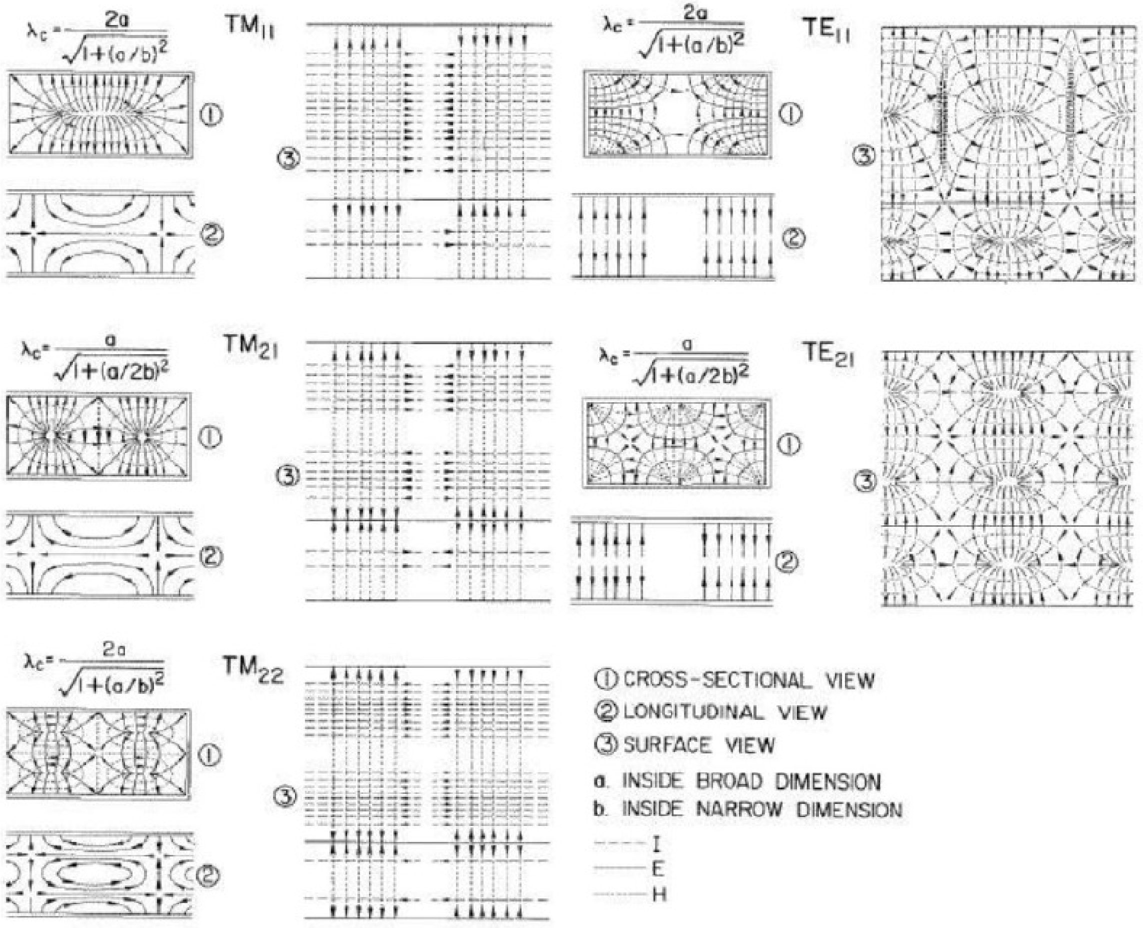
Boundary conditions: $\mathrm{TM}:\ \psi|_s=0,\qquad \mathrm{TE}:\ \left.\frac{\partial\psi}{\partial n}\right|_s=0.$

Lowest cut-off angular frequency

$$\mathrm{TM}_{01}:\ \gamma_{01}\simeq\frac{2.40483}{R},\qquad \boxed{\omega_c\simeq\frac{2.40c}{R}},$$

$$\mathrm{TE}_{11}:\ \gamma_{11}\simeq\frac{1.84118}{R},\qquad \boxed{\omega_c\simeq\frac{1.84c}{R}}.$$

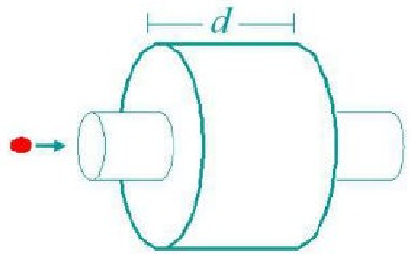
Modes in a rectangular waveguide

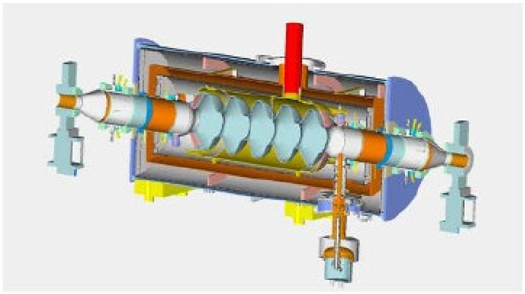


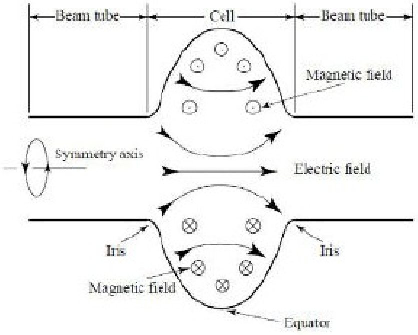
Each TE_{mn} or TM_{mn} mode has its own transverse field pattern and cut-off frequency.

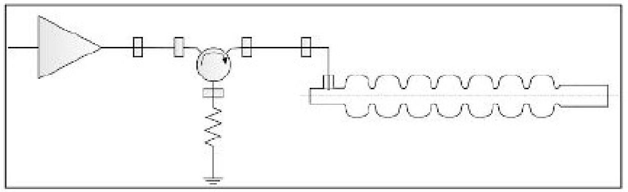
RF cavities

- An RF cavity is a resonator designed to confine electromagnetic energy.
- Beam pipes attached to the cavity are chosen so that the operating mode is below their cut-off frequency.
- The RF field then decays evanescently along the beam pipes instead of propagating away.
- RF power is delivered through waveguides/coaxial lines and an input coupler; circulators and loads protect the RF source.









RF cavity modes: the lowest accelerating mode is TM_{010}

Inside a perfectly conducting cylindrical cavity, the fields satisfy Maxwell's equations and conducting-wall boundary conditions. There are infinitely many TE and TM eigenmodes.

- Acceleration requires $E_z \neq 0$, so a TM mode is required.
- The lowest-frequency accelerating mode of a pillbox cavity is TM_{010}.

For TM_{010}, using the field expressions from the source PPTX,

$$\boxed{E_z(r,t) = E_0 J_0\left(\frac{2.405\, r}{R}\right) e^{i\omega t}}$$

$$\boxed{H_\phi(r,t) = -iE_0 J_1\left(\frac{2.405\, r}{R}\right) e^{i\omega t}}$$

$$\boxed{\omega_{010} = \frac{2.405\, c}{R}}, \qquad \lambda_{010} = 2.61R.$$

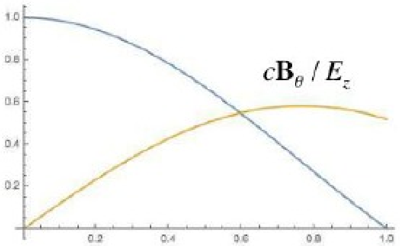
A notable result: for an ideal pillbox without beam tubes, ω_{010} depends on the radius R, not the cavity length d.

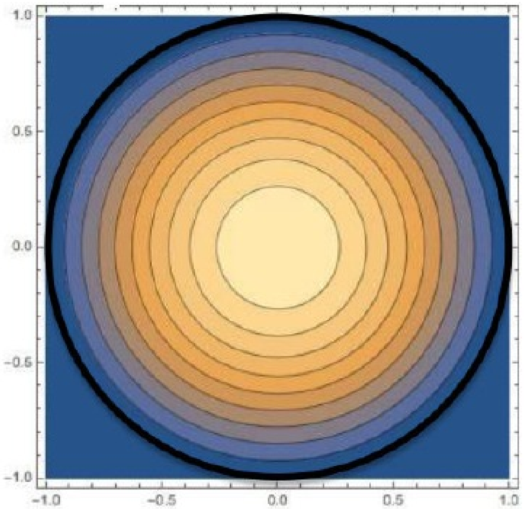
Pillbox cavity

- Similarly to the waveguide case, choose a TM mode to have non-zero E_z.
- Select the TM_{010} mode with $k_z = 0$.

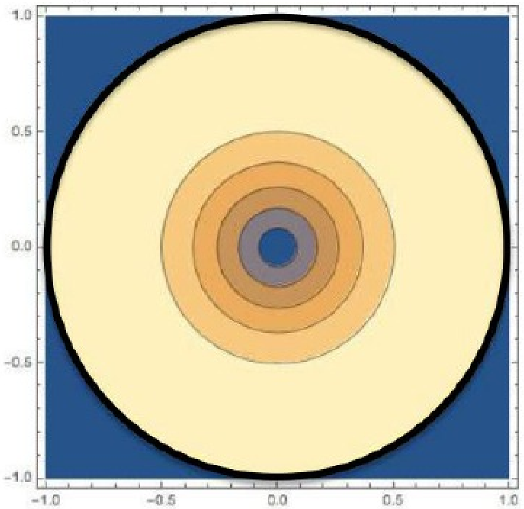
$$\mathbf{E}_z = E_z \cdot J_o\left(2.405\frac{x}{-}\right)\sin(\omega t);$$

$$\mathbf{B}_\theta = \frac{1}{c}E_z \cdot J_1\left(2.405\frac{x}{-}\right)\cos(\omega t)$$





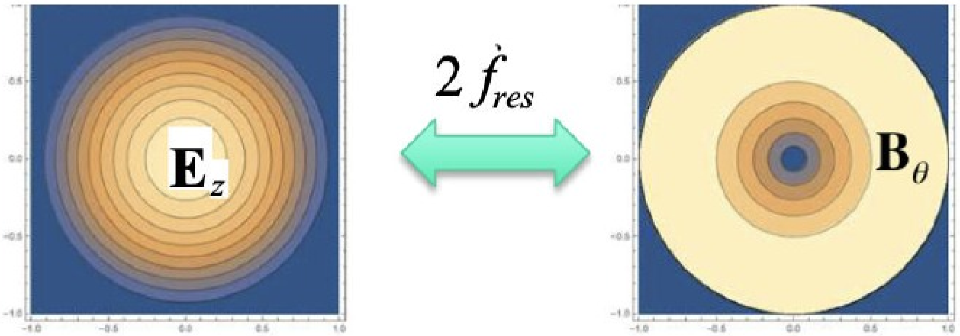
$\mathbf{B}_\theta$



Electromagnetic energy oscillates between electric and magnetic fields, peaking at the same stored energy in an ideal lossless cavity.

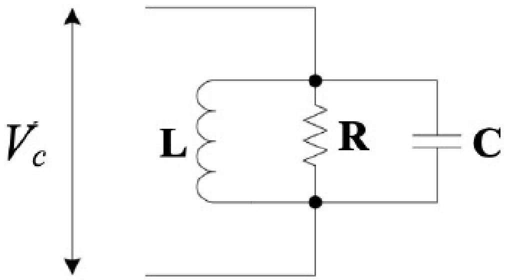
EM cavity

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_o(\vec{r})\cos(\omega t + \varphi(\vec{r}))$$
$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_o(\vec{r})\sin(\omega t + \psi(\vec{r}))$$
$$\int \vec{\mathbf{E}}_o^{\,2}\, dV = \int \vec{\mathbf{B}}_o^{\,2}\, dV \quad (\cdot c^2 \; for\; SI)$$

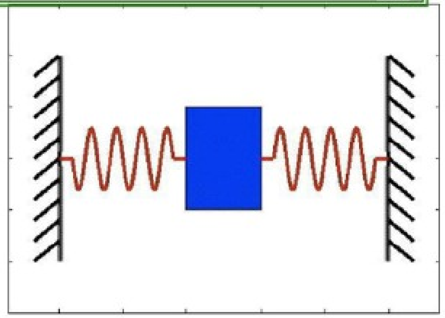


- Each cavity mode has a full analogy with a resonant LC circuit or a mechanical oscillator.
- Energy stored in the electric field corresponds to potential energy; energy stored in the magnetic field corresponds to kinetic energy.

$$E_M = \frac{LI^2}{2}; E_E = \frac{Q^2}{2C}; \omega_o = \frac{1}{\sqrt{LC}}$$



$$E_K = \frac{M\mathrm{v}^2}{2}; E_P = \frac{Kx^2}{2}; \omega_o = \sqrt{\frac{K}{M}}$$



Quality factor - definition

Consider a stand-alone cavity without external couplers. Let W be the stored electromagnetic energy and P_{loss} the dissipated power. The quality factor is

$$\boxed{Q = \omega_0 \frac{W}{P_{\mathrm{loss}}}}.$$

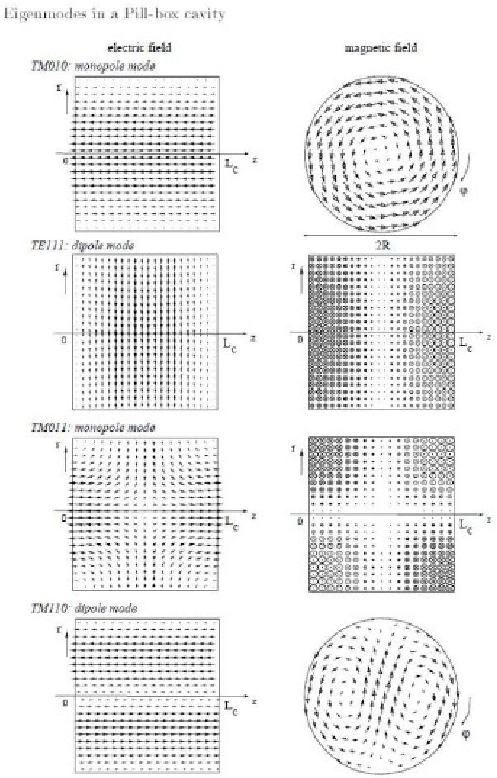
With

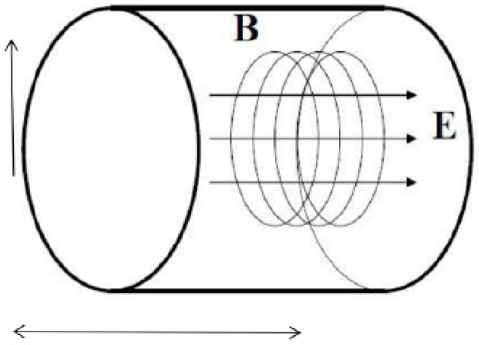
$$W = \int_V \left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) dV,$$

Q measures how slowly the stored energy decays. Equivalently, it is 2π times the number of RF oscillations associated with an e-fold reduction of energy.

Fundamental and higher-order modes (HOMs)

- A cavity supports an infinite spectrum of eigenmodes.
- The desired accelerating mode is the **fundamental accelerating mode**.
- Other resonances are higher-order modes (HOMs).
- Beam excitation of HOMs can produce unwanted longitudinal or transverse fields and may degrade beam quality or stability.





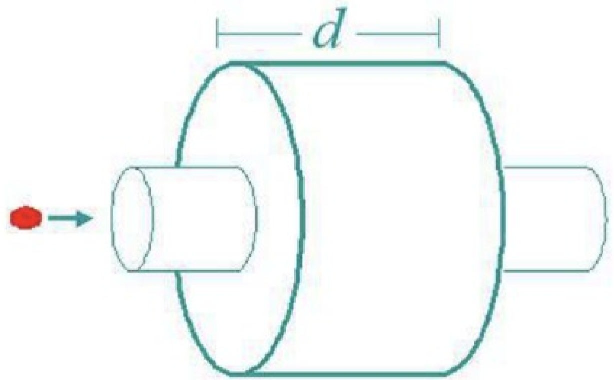
$$TM: \varphi_{mnl} = J_n(\gamma_{mn} r)\cos k_{z,l} z;\ k_{z,l} = l\frac{\pi}{d};\ J_n(\gamma_{mn} R) = 0;$$

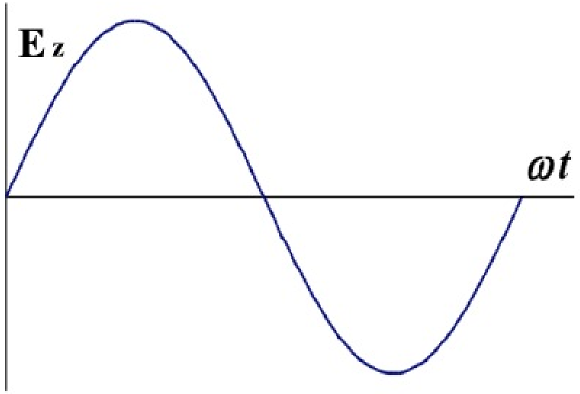
$$\omega_{res} = c\sqrt{\gamma_{mn}^2 + l^2\frac{\pi^2}{d^2}};\ l = 0,1,2....$$

$$TE: \varphi_{mnl} = J_n(\kappa_{mn} r)\sin k_{z,l} z;\ k_{z,l} = l\frac{\pi}{}\ \ J_n'(\kappa_{mn} R) = 0;$$

$$\omega_{res} = c\sqrt{\kappa_{mn}^2 + l^2\frac{\pi^2}{d^2}};\ l = 1,2....$$

Accelerating voltage and transit-time factor





For a particle moving on axis with speed $v = \beta c$,

$$\boxed{V_c = \left| \int E_z(0,z) e^{\mathrm{i}\omega z/(\beta c)}\, \mathrm{d}z \right|}.$$

For a uniform pillbox field of length d,

$$V_c = E_0 d\, T, \qquad \boxed{T = \frac{\sin(\omega d/2\beta c)}{\omega d/2\beta c}}.$$

Choosing $d = \beta\lambda/2$ gives

$$T = \frac{2}{\pi}.$$

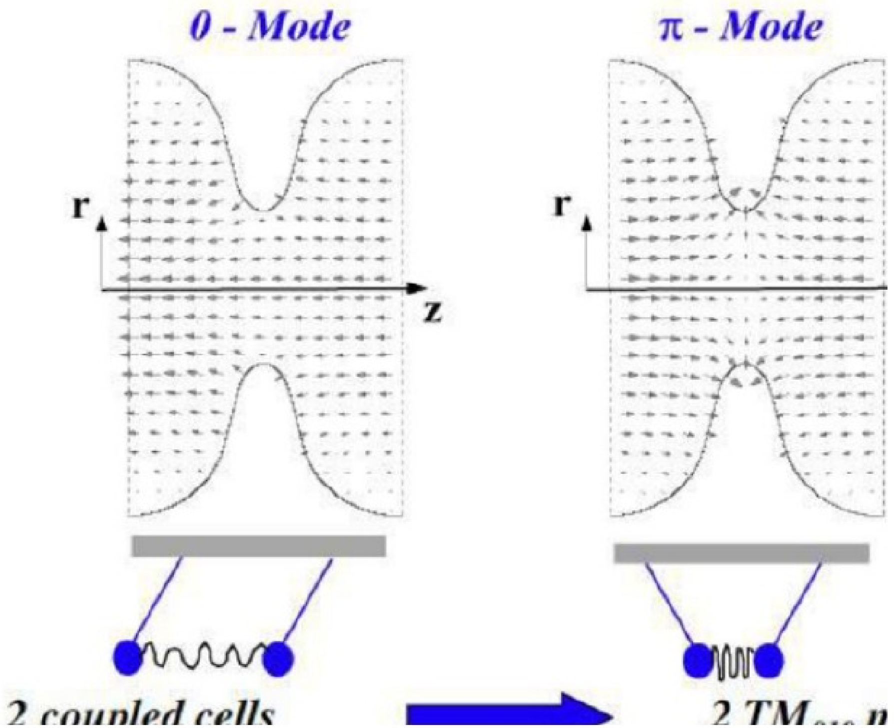
The accelerating gradient is conventionally

$$E_{\mathrm{acc}} = \frac{V_c}{d}.$$

For practical multicell cavities one often uses $d = \beta\lambda/2$ per cell.

Multicell cavities: coupled oscillators

- Several cells can be connected to form one multicell cavity.
- The TM_{010}-like mode of each cell couples to neighboring cells through the irises.
- Coupling splits a single-cell resonance into a set of closely spaced normal modes - a **passband**.
- An N-cell cavity has N modes in each passband.



Multi-cell cavities

A single RF cell has limited accelerating voltage.

More energy can be obtained by using many independent cavities, but that requires many RF transmitters, waveguides and controls and gives a lower real-estate gradient.

Therefore, where high accelerating gradient is important, accelerator systems commonly use multicell cavities.

The cell length is chosen to keep the beam synchronized with the accelerating RF phase.

$$\boxed{Max(V_{RF}) = \frac{\mathbf{E_o}\lambda_{RF}}{\pi}}$$

9-cell Tesla design

7-cell

5-cell

What we learned

- Resonant cavity modes belong to TE and TM families; an infinite number of eigenmodes exist.
- Acceleration requires longitudinal electric field, so the lowest-frequency TM mode is commonly used as the fundamental accelerating mode.
- Other cavity modes are HOMs and can be parasitic for the beam.
- Important cavity figures of merit include accelerating voltage, transit-time factor and quality factor Q.
- In an N-cell cavity, each single-cell mode splits into a passband containing N modes; the passband width is set by cell-to-cell coupling.
- Accelerating cavities operate below the cut-off frequency of their beam pipes, so the RF field decays evanescently along the pipes.
- Coaxial lines and rectangular waveguides are widely used to deliver RF power to cavities.