Homework 4

Problem 1. 4x5 points. Matrix of an ideal solenoid.
Consider particles with momentum p_o propagating along the axis of idealized solenoid with

$$B_s = \left\{ \begin{array}{c} 0, s < 0 \\ B_o, 0 \le s \le l \\ 0, s > 1 \end{array} \right\}$$

All other components of the field are zero, e.g. s=z, not curvature.

(a) Use Sylvester formula and calculate 4x4 transport matrix of the solenoid;
(b) Show that resulting matrix can be presented is form of focusing matrix in each direction and a rotation

$$M_s = \begin{bmatrix} I\cos\varphi & I\sin\varphi \\ -I\sin\varphi & I\cos\varphi \end{bmatrix} \cdot \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; F = \begin{bmatrix} a & b \\ c & a \end{bmatrix}; ab - cd = 1$$

are 2x2 matrices and F is focusing one. Write expressions for φ, F through $p_o, B_o, l...,$

(c) Finally use one tricks available for you since we can use torsion and decouple x and y motion:

$$\tilde{h}_n = \frac{\pi_1^2 + \pi_3^2}{2} + f\frac{x^2}{2} + g\frac{y^2}{2} + L(x\pi_3 - y\pi_1)$$

$$f = \left(\frac{eB_s}{2p_o c}\right)^2; g = \left(\frac{eB_s}{2p_o c}\right)^2; L = \kappa + \frac{e}{2p_o c}B_s;$$

by choosing $\kappa = -\dfrac{e}{2p_o c}B_s$. Show that matrix in this coordinates system is block diagonal (e.g. de-coupled)

$$M_s = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$$

with F identical to that in the problem (b) above. Show also that rotation is angle aroin z-axis is $\kappa l = -\varphi$.

(d) Finally, explain why a simple trajectory x=const and y=const (which intuitively is trajectory parallel to the magnetic lines) is not a solution?

$$\mathrm{v}_{x,y} = 0; \rightarrow \vec{\mathrm{v}} = \hat{z}\mathrm{v}_o; \vec{f} = \frac{e}{c}\left[\hat{z}\mathrm{v}_o \times \hat{z}B_0\right] = 0$$

Hint: consider what is happening at the entrance and exit to the solenoid.

Let's start for introducing convenient variables:

$$\tilde{h}_n = \frac{\pi_x^2 + \pi_y^2}{2} + \Omega^2 \frac{x^2}{2} + \Omega^2 \frac{y^2}{2} + (\kappa + \Omega)\left(x\pi_x - y\pi_y\right);\;\; \Omega = \frac{e}{2p_o c} B_s;$$

$$\mathbf{H} = \begin{bmatrix} \Omega^2 & 0 & 0 & \kappa+\Omega \\ 0 & 1 & -\kappa-\Omega & 0 \\ 0 & -\kappa-\Omega & \Omega^2 & 0 \\ \kappa+\Omega & & 0 & 1 \end{bmatrix}; \mathbf{D} = \begin{bmatrix} 0 & 1 & -\kappa-\Omega & 0 \\ -\Omega^2 & 0 & 0 & -\kappa-\Omega \\ \kappa+\Omega & 0 & 0 & 1 \\ 0 & \kappa+\Omega & -\Omega^2 & 0 \end{bmatrix}$$

(a) In regular Cartesian coordinate system κ =0.
Equation for eigen values (as we remember form class) is by-quadratic:

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & -\Omega & 0 \\ -\Omega^2 & 0 & 0 & -\Omega \\ \Omega & 0 & 0 & 1 \\ 0 & \Omega & -\Omega^2 & 0 \end{bmatrix}; \mathbf{D}^2 = \begin{bmatrix} -2\Omega^2 & 0 & 0 & -2\Omega \\ 0 & -2\Omega^2 & 2\Omega^3 & 0 \\ 0 & 2\Omega & -2\Omega^2 & 0 \\ -2\Omega^3 & 0 & 0 & -2\Omega^2 \end{bmatrix}$$

$$\det\begin{bmatrix} -\lambda & 1 & -\Omega & 0 \\ -\Omega^2 & -\lambda & 0 & -\Omega \\ \Omega & 0 & -\lambda & 1 \\ 0 & \Omega & -\Omega^2 & -\lambda \end{bmatrix} = \lambda^4 + 4\Omega^2\lambda^2 = 0$$

which can be simplified to

$$\lambda_{1,2} = 0; \lambda_{3,4} = \pm i2\Omega;$$

The only unpleasantly is that we have degenerated case with vector high (degeneracy) up to two. We have to use generalized Sylvester formula: we did in lecture 13.

$$\exp[\mathbf{D}s] = \sum_{k=1}^{m}\left[e^{\lambda_k s} \prod_{i\neq k} \left\{ \frac{\mathbf{D}-\lambda_i\mathbf{I}}{\lambda_k - \lambda_i} \sum_{j=0}^{n_k-1}\left(\frac{\mathbf{D}-\lambda_k\mathbf{I}}{\lambda_i-\lambda_k}\right)^j \right\}^{n_i} \sum_{p=0}^{n_k-1}\frac{s^p}{p!}(\mathbf{D}-\lambda_k\mathbf{I})^p \right]$$

$$\exp[\mathbf{D}s] = e^{\lambda_1 s}\prod_{i=3,4}\left\{\frac{\mathbf{D}-\lambda_i\mathbf{I}}{\lambda_1-\lambda_i}\sum_{j=0}^{1}\left(\frac{\mathbf{D}-\lambda_1\mathbf{I}}{\lambda_i-\lambda_1}\right)^j\right\}\sum_{p=0}^{1}\frac{s^p}{p!}(\mathbf{D}-\lambda_1\mathbf{I})^p +$$

$$e^{\lambda_3 s}\prod_{i=1,4}\left\{\frac{\mathbf{D}-\lambda_i\mathbf{I}}{\lambda_3-\lambda_i}\right\}^{n_i} + e^{\lambda_4 s}\prod_{i=1,4}\left\{\frac{\mathbf{D}-\lambda_i\mathbf{I}}{\lambda_4-\lambda_i}\right\}^{n_i};\;\; n_1 = 2; n_{3,4} = 1$$

$$\sum_{p=0}^{1}\frac{s^p}{p!}(\mathbf{D}-\lambda_1\mathbf{I})^p = \mathbf{I} + \mathbf{D}s$$

$$\frac{\mathbf{D}-\lambda_i\mathbf{I}}{-\lambda_i}\sum_{j=0}^{1}\left(\frac{\mathbf{D}}{\lambda_i}\right)^j = -\frac{\mathbf{D}-\lambda_i\mathbf{I}}{-\lambda_i}\left(\mathbf{I}+\frac{\mathbf{D}}{\lambda_i}\right) = -\frac{\mathbf{D}^2-\lambda_i^2\mathbf{I}}{\lambda_i^2}, i = 3,4$$

$$\exp[\mathbf{D}s] = \left(\frac{\mathbf{D}^2 + 4\Omega^2\mathbf{I}}{4\Omega^2}\right)^2(\mathbf{I}+\mathbf{D}s) + \left(e^{2i\Omega s}\frac{\mathbf{D}+2i\Omega\mathbf{I}}{2i\Omega} - e^{-2i\Omega s}\frac{\mathbf{D}-2i\Omega\mathbf{I}}{2i\Omega}\right)\left(\frac{\mathbf{D}}{2i\Omega}\right)^2$$

Let's note that $\left(\mathbf{I}+\mathbf{A}^2\right)^2 = \left(\mathbf{I}^2+\mathbf{A}^2\right)^2 = \mathbf{I}+2\mathbf{A}^2+\mathbf{A}^4 = \mathbf{I}+\mathbf{A}^2+\mathbf{A}^2(\mathbf{I}+\mathbf{A}^2)$, e.g.

$$\left(\frac{\mathbf{D}^2+4\Omega^2\mathbf{I}}{4\Omega^2}\right)^2 = \left(\mathbf{I}+\frac{\mathbf{D}^2}{4\Omega^2}\right)^2 = \mathbf{I}+\frac{\mathbf{D}^2}{4\Omega^2}+\frac{\mathbf{D}^2}{4\Omega^2}\left(\mathbf{I}+\frac{\mathbf{D}^2}{4\Omega^2}\right)$$

Because of the Hamilton-Kelly theorem, $\mathbf{D}^2\left(\mathbf{D}^2+4\Omega^2\mathbf{I}\right)=0$ the second term disappers. Combining remaining terms

$$\exp[\mathbf{D}s] = \left(\mathbf{I}+\frac{\mathbf{D}^2}{4\Omega^2}\right)(\mathbf{I}+\mathbf{D}s) - \frac{\mathbf{D}^2}{4\Omega^2}\left(\mathbf{I}\cos(2\Omega s)+\frac{\mathbf{D}}{2\Omega}\sin(2\Omega s)\right)$$

Further simplification comes from the simplification is this fact (which is indication of eigen vector multiplicity, but the fact that matrix D actually can be diagonolised). It is easy prove by multiplication that

$$\left(\mathbf{I}+\frac{\mathbf{D}^2}{4\Omega^2}\right)\mathbf{D}=0$$

$$\exp[\mathbf{D}s] = \mathbf{I}+\frac{\mathbf{D}^2}{4\Omega^2} - \frac{\mathbf{D}^2}{4\Omega^2}\left(\mathbf{I}\cos 2\varphi+\frac{\mathbf{D}}{2\Omega}\sin 2\varphi\right); \varphi=\Omega s$$

$$\mathbf{M}_s = \frac{1}{2}\begin{bmatrix} 1+\cos 2\varphi & \dfrac{\sin 2\varphi}{\Omega} & -\sin\varphi & \dfrac{\cos 2\varphi - 1}{\Omega} \\ -\Omega\sin 2\varphi & 1+\cos 2\varphi & \Omega(1-\cos 2\varphi) & -\sin 2\varphi \\ \sin 2\varphi & \Omega(1-\cos 2\varphi) & 1+\cos 2\varphi & \dfrac{\sin 2\varphi}{\Omega} \\ \dfrac{\cos 2\varphi - 1}{\Omega} & \sin 2\varphi & -\Omega\sin 2\varphi & 1+\cos 2\varphi \end{bmatrix} \qquad (1)$$

While this matrix has a clear structure, it is not obvious why?

(b) Let's use our choice of coordinate system and use rotation (torsion) around z-axis that

$$\kappa+\Omega=0; \kappa=-\Omega$$

$$\tilde{h}_n = \frac{\pi_x^2+\pi_y^2}{2}+\Omega^2\frac{x^2}{2}+\Omega^2\frac{y^2}{2}; \;\; \Omega=\frac{e}{2p_o c}B_s;$$

$$\mathbf{H} = \begin{bmatrix} \Omega^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \Omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\Omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\Omega^2 & 0 \end{bmatrix}$$

i.e. a simple focusing element we had seen many times:

$$\tilde{\mathbf{M}} = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}; F = \begin{bmatrix} \cos\varphi & \dfrac{\sin\varphi}{\Omega} \\ -\Omega\sin\varphi & \cos\varphi \end{bmatrix}; \varphi=\Omega s$$

Let go to the end of the solenoid at s=L. Since we were rotating with angular rate $\kappa = -\Omega$, we had accumulated rotation angle

$$\theta = \kappa L = -\Omega L$$

while our matrix (still in rotate coordinates) is

$$\tilde{\mathbf{M}} = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}; F = \begin{bmatrix} \cos\varphi & \dfrac{\sin\varphi}{\Omega} \\ -\Omega\sin\varphi & \cos\varphi \end{bmatrix}; \varphi = \Omega L$$

What we need, is just rotate the coordinate system back by opposite angle: $-\theta = \varphi = \Omega L$:

$$\mathbf{R} = \begin{bmatrix} \mathbf{I}\cos\varphi & -\mathbf{I}\sin\varphi \\ \mathbf{I}\sin\varphi & \mathbf{I}\cos\varphi \end{bmatrix}; \mathbf{M} = \mathbf{R}\tilde{\mathbf{M}}; \begin{bmatrix} F\cos\varphi & -F\sin\varphi \\ F\sin\varphi & F\cos\varphi \end{bmatrix}$$

$$F\cos\varphi = \begin{bmatrix} \cos^2\varphi & \dfrac{\sin 2\varphi}{\Omega} \\ -\Omega\sin 2\varphi & \cos^2\varphi \end{bmatrix}$$

This form of the matrix makes a lot of sense: it is product of a uniform (symmetric) focusing and rotation. Naturally, it coincides with (1). Thus, we proved (b) and (c).

The fact that the rotation angle and the oscillation angles are equal, creates unusual (constant) terms in matrix. In fact this constant term is relevant to question (d).

We understand that inside the solenoid (where filed is constant) the force acting on particle propagating parallel to z-axis is zero:

$$\mathrm{v}_{x,y} = 0; \rightarrow \vec{\mathrm{v}} = \hat{z}\mathrm{v}_o; \vec{f} = \frac{e}{c}\left[\hat{z}\mathrm{v}_o \times \hat{z}B_0\right] = 0$$

since the velocity and magnetic field are parallel. Hence a x=const and y=const (parallel) to the axis is a solution.

But looking at transport matrix of the solenoid, we see that particle coming with transverse displacement x or y, parallel to the z-axis ($p_{x,y}$=0) will focused and rotated…. The paradox is in the fact that particle comes from area with zero longitudinal magnetic field into the field of solenoid, it experience change in transverse momentum proportions to transverse vector potential: the canonical momentum is preserve, but mechanical does jump! This can be seen that solenoidal (z-component) of magnetic field corresponds to vector potential of:

$$\hat{z}B_0 = \vec{\nabla} \times \vec{A} \rightarrow \vec{A} = B_0\left(\hat{x}y - \hat{y}x\right)$$

It means that a particle passing through an “edge” of the solenoid, experiences jump of the mechanical momentum of:

$$\vec{P} = \vec{p} + \frac{e}{c}\vec{A} = const \Rightarrow \Delta\vec{p} = -\frac{e}{c}\Delta\vec{A} = -\frac{eB_0}{c}\left(\hat{x}y - \hat{y}x\right),$$

e.g. particle coming with a radial displacement get a rotating kick. It will result on

Larmore rotation in the field. At the exit of the solenoid it get angular kick in opposite direction. Of cause, if our particle comes to the solenoid edge with angular momentum

$$\vec{p}_o = \frac{eB_0}{c}\left(\hat{x}y - \hat{y}x\right) \rightarrow \vec{p}_{sol} = \vec{p}_o + \Delta\vec{p} = 0$$

then it will propagate inside solenoid parallel to the z-axis, and will receive exactly the same angular momentum at he exit of the solenoid – this is eigen colution with zero eigen values: x=const, y=const in solenoid, but its transverse Cannonical momentum is not zero:

$$\vec{P} = \frac{eB_0}{c}\left(\hat{x}y - \hat{y}x\right).$$

To make the picture even more clear, lest consider a realistic edge of the solenoid with

$$\hat{z} \cdot \vec{B} = B_0\left(s\right)$$

We know that divergence of the magnetic field must be zero:

$$div\vec{B} = \frac{\partial B_s}{\partial s} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

Since solenoid has axial symmetry, transverse gradients are equal

$$\frac{\partial B_x}{\partial x} = \frac{\partial B_y}{\partial y} = -\frac{1}{2}\frac{\partial B_s}{\partial s}$$

and the field near the axis of the solenoid can approximated by

$$\vec{B} \cong \hat{z} \cdot B_0\left(s\right)$$