Problem 1. 15 points. Oscillating external force.

For a linear one-dimensional motion of particle in an storage ring with circumference C consider an oscillating force applied to the particle:

\[
\frac{d}{ds} \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1(s) & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ f(s) \cos \omega t \end{bmatrix},
\]

\[K_1(s + C) = K_1(s); f(s + C) = f(s); t = \frac{s}{v_o}; \mu_e = C \frac{O}{v_o} = 2\pi Q_e.\]

Betatron tune and the eigen vectors are known function

\[\mu_x = 2\pi Q_x; \ Y(s) = Y(s + C) = \begin{bmatrix} w(s) \\ w'(s) + \frac{i}{w(s)} \end{bmatrix}\]

are considered to be known.

(a) Find solution in a form

\[x = x_e(s) + x_o(s)\]

where is \(x_o(s)\) well know free oscillations and forced oscillations

\[x_e(s) = b(s) \cos(\omega t + \phi); b(s + C) = b(s)\]

Find expression for \(b(s)\) in a form of integral over the ring circumference.

Hint: use class notes for a general case and apply it to 1D

(b) Find and write down resonant conditions, when amplitude of oscillation is unlimited.

Solution: Let’s write this as:

\[X = \begin{bmatrix} x \\ p \end{bmatrix} \rightarrow X' = DX + \Re F; F = \begin{bmatrix} 0 \\ f(s) \end{bmatrix} e^{iks}; k = \frac{\omega}{v_o};\]

\[X = \Re Z; \ Z' = DZ + F\]

\[Z = \tilde{U}A'; \tilde{U}' = D\tilde{U} \rightarrow \tilde{U}A' = F\]

\[A = A_o + \int_{\tilde{\xi}}^{\xi} \tilde{U}^{-1}(\xi) F(\xi) d\tilde{\xi}\]

\[Z(s) = c \left( A_o + \int_{\tilde{\xi}}^{\xi} \tilde{U}^{-1}(\xi) F(\xi) d\tilde{\xi} \right)\]

and
\[ Z(s + C) = \tilde{U}(s + C) \left[ A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right] = \]
\[ \tilde{U}(s) \Lambda \left[ A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi + \int_{s}^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right] \]

Since the external force is oscillating with a given frequency, let’s try to find a particular solution which has both: a complex amplitudes periodic with \( s \) and oscillating with the external frequency. Since the general solution of non-homogeneous linear OD equation is a combination of a particular solution of the inhomogeneous equation plus a general solution of homogenous equation. Thus, let’s set

is comprise

\[ \tilde{U}(s + C) \left[ A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right] = \tilde{U}(s) \left[ A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right] e^{\mu C}; \]

\[ \tilde{U}(s + C) = \tilde{U}(s) \Lambda; \Lambda = \begin{bmatrix} \epsilon^{i\mu} & 0 \\ 0 & \epsilon^{-i\mu} \end{bmatrix} \rightarrow \]

\[ \Lambda \int_{s}^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi = (\mathbf{I} e^{i\mu} - \Lambda) \left[ A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right] = \]

with a simple solution of

\[ \left( A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right) = (\mathbf{I} e^{i\mu} - \Lambda)^{-1} \Lambda \int_{s}^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi; \]

\[ \tilde{U}(s) = \tilde{U}(s) \Phi(s); \Phi(s) = \begin{bmatrix} \epsilon^{i\nu(s)} & 0 \\ 0 & \epsilon^{-i\nu(s)} \end{bmatrix}; \]

\[ Z = \tilde{U}(s) (\mathbf{I} e^{i\mu} - \Lambda)^{-1} \Lambda \int_{s}^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi = e^{i\nu} g(s) \]

\[ g(s) = U(s) \left( I - \Lambda e^{-i\mu} \right)^{-1} \int_{s-C}^{s} \Phi(s - \xi) U^{-1}(\xi) \left[ \begin{array}{c} 0 \\ f(\xi) \end{array} \right] e^{-i(\nu-x) d\xi}. \]

It is easy to see that is periodic function of \( s \): \( g(s + C) = g(s) \). Thus we fond the solution
Finally, for such a system located at 

\[ \psi(s) \]

between 

\[ \Lambda^{-1} \]

and corresponding derivative. 

Second part: let’s look onto the denominators 

\[ 1 - e^{i(\psi(s) - \psi(\xi))} \]

\[ 1 - e^{-i(\psi(s) - \psi(\xi))} \]

- they turned into zeros (e.g. amplitude of excited oscillation grow to infinity – in linear approximation) when

\[ \mu \pm kC = 2N\pi \rightarrow Q_{\text{exc}} = N \pm Q; \quad Q_{\text{exc}} = \frac{kC}{2\pi}; Q = \frac{k\mu}{2\pi}. \]

where N is an arbitrary integer. It is called a resonant excitation of oscillations – it allows to measure the non-integer part of the tune. But note, it can not distinguish between \( \pm Q \). Finally, for such a system located at \( s_o \):

\[ f(s) = f_o \delta(s - s_o) \]

\[ x(s) = \text{Re} \left[ \frac{e^{i\xi s}w(s)w(s_o)f_o}{2i} \left\{ \frac{e^{i(\psi(s) - \psi(\xi))}}{1 - e^{i(\psi(s) - \psi(\xi))}} \frac{e^{-i(\psi(s) - \psi(\xi))}}{1 - e^{-i(\psi(s) - \psi(\xi))}} \right\} \right] \]