Homework 14. Due October 28

Problem 1. 15 points. Oscillating external force.

For a linear one-dimensional motion of particle in an storage ring with circumference C consider an oscillating force applied to the particle:

$$\frac{d}{ds}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1(s) & 0 \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ f(s)\cos\omega t \end{bmatrix};$$

$$K_1(s+C) = K_1(s); f(s+C) = f(s); t = \frac{s}{\mathrm{v}_o}; \mu_e = C\frac{\omega}{\mathrm{v}_o} = 2\pi Q_e.$$

Betatron tune and the eigen vectors are known function

$$\mu_x = 2\pi Q_x;\;\; Y(s) = Y(s+C) = \begin{bmatrix} w(s) \\ w'(s) + \dfrac{i}{w(s)} \end{bmatrix}$$

are considered to be known.

(a) Find solution in a form

$$x = x_e(s) + x_o(s)$$

where is $x_o(s)$ well know free oscillations and forced oscillations

$$x_e(s) = b(s)\cos(\omega t + \varphi); b(s+C) = b(s)$$

Find expression for $b(s)$ in a form of integral over the ring circumference.

Hint: use class notes for a general case and apply it to 1D

(b) Find and write down resonant conditions, when amplitude of oscillation is unlimited.

Solution: Let's write this as:

$$X = \begin{bmatrix} x \\ p \end{bmatrix} \Longrightarrow X' = DX + \operatorname{Re} F; F = \begin{bmatrix} 0 \\ f(s) \end{bmatrix} e^{iks}; k = \frac{\omega}{\mathrm{v}_o};$$

$$X = \operatorname{Re} Z;\;\; Z' = DZ + F$$

$$Z = \tilde{U}A'; \tilde{U}' = D\tilde{U} \to \tilde{U}A' = F$$

$$A = A_o + \int_o^s \tilde{U}^{-1}(\xi)F(\xi)d\xi$$

$$Z(s) = c\left(A_o + \int_o^s \tilde{U}^{-1}(\xi)F(\xi)d\xi\right)$$

and

$$Z(s+C) = \tilde{U}(s+C)\left(A_o + \int_o^{s+C} \tilde{U}^{-1}(\xi)F(\xi)d\xi\right) =$$

$$\tilde{U}(s)\Lambda\left(A_o + \int_o^{s} \tilde{U}^{-1}(\xi)F(\xi)d\xi + \int_s^{s+C} \tilde{U}^{-1}(\xi)F(\xi)d\xi\right)$$

Since the external force is oscillating with a given frequency, let's try to find a particular solution which has both: a complex amplitudes periodic with s and oscillating with the external frequency. Since the general solution of non-homogeneous linear OD equation is a combination of a particular solution of the inhomogeneous equation plus a general solution of homogenous equation. Thus, let's set

is comprise

$$\tilde{U}(s+C)\left(A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi)F(\xi)d\xi\right) = \tilde{U}(s)\left(A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi)F(\xi)d\xi\right)e^{ikC};$$

$$\tilde{U}(s+C) = \tilde{U}(s)\Lambda; \Lambda = \begin{bmatrix} e^{i\mu} & 0 \\ 0 & e^{i\mu} \end{bmatrix} \rightarrow$$

$$\Lambda \int_s^{s+C} \tilde{U}^{-1}(\xi)F(\xi)d\xi = \left(\mathbf{I}e^{ikC} - \Lambda\right)\left(A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi)F(\xi)d\xi\right) =$$

with a simple solution of

$$\left(A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi)F(\xi)d\xi\right) = \left(\mathbf{I}e^{ikC} - \Lambda\right)^{-1} \Lambda \int_s^{s+C} \tilde{U}^{-1}(\xi)F(\xi)d\xi;$$

$$\tilde{U}(s) = \tilde{U}(s)\Phi(s); \Phi(s) = \begin{bmatrix} e^{i\psi(s)} & 0 \\ 0 & e^{-i\psi(s)} \end{bmatrix};$$

$$Z = \tilde{U}(s)\left(\mathbf{I}e^{ikC} - \Lambda\right)^{-1} \Lambda \int_s^{s+C} \tilde{U}^{-1}(\xi)F(\xi)d\xi = e^{iks}g(s)$$

$$g(s) = U(s)\left(\mathbf{I} - \Lambda e^{-ikC}\right)^{-1} \int_{s-C}^{s} \Phi(s-\xi)U^{-1}(\xi)\begin{bmatrix} 0 \\ f(\xi) \end{bmatrix} e^{-ik(s-\xi)}d\xi.$$

It is easy to see that is periodic function of s: $g(s+C) = g(s)$. Thus we fond the solution

$$\left(\mathbf{I}-\Lambda e^{-ikC}\right)^{-1}=\begin{bmatrix}\frac{1}{1-e^{i(\mu-kC)}} & 0\\ 0 & \frac{1}{1-e^{-i(\mu+kC)}}\end{bmatrix};\quad \Phi(s-\xi)=\begin{bmatrix}e^{i(\psi(s)-\psi(\xi))} & 0\\ 0 & e^{-i(\psi(s)-\psi(\xi))}\end{bmatrix};$$

$$e^{-i(\psi(s)-\psi(\xi))-1}(\xi)\begin{bmatrix}0\\ f(\xi)\end{bmatrix}=-\frac{1}{2i}\begin{bmatrix}\mathrm{w}'\text{-}\frac{i}{\mathrm{w}} & -\mathrm{w}\\ -\mathrm{w}'\text{-}\frac{i}{\mathrm{w}} & \mathrm{w}\end{bmatrix}\begin{bmatrix}0\\ f(\xi)\end{bmatrix}=\frac{f(\xi)\mathrm{w}(\xi)}{2i}\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\left(\mathbf{I}-\Lambda e^{-ikC}\right)^{-1}\Phi(s-\xi)U^{-1}(\xi)\begin{bmatrix}0\\ f(\xi)\end{bmatrix}=\frac{f(\xi)\mathrm{w}(\xi)}{2i}\begin{bmatrix}\frac{e^{i(\psi(s)-\psi(\xi))}}{1-e^{i(\mu-kC)}}\\ -\frac{e^{-i(\psi(s)-\psi(\xi))}}{1-e^{-i(\mu+kC)}}\end{bmatrix}$$

$$z(s)=\frac{e^{iks}\mathrm{w}(s)}{2i}\left\{\int_s^{s+C} f(\xi)\mathrm{w}(\xi)\left(\frac{e^{i(\psi(s)-\psi(\xi))}}{1-e^{i(\mu-kC)}}-\frac{e^{-i(\psi(s)-\psi(\xi))}}{1-e^{-i(\mu+kC)}}\right)e^{-ik(s-\xi)}\,d\xi\right\}$$

and

$$x(s)=\mathrm{Re}\left[\frac{e^{iks}\mathrm{w}(s)}{2i}\left\{\begin{array}{l}\frac{1}{1-e^{i(\mu-kC)}}\int_s^{s+C} d\xi f(\xi)\mathrm{w}(\xi)e^{i(\psi(s)-\psi(\xi))}e^{-ik(s-\xi)}-\\ -\frac{1}{1-e^{-i(\mu+kC)}}\int_s^{s+C} d\xi f(\xi)\mathrm{w}(\xi)e^{-i(\psi(s)-\psi(\xi))}e^{-ik(s-\xi)}\end{array}\right\}\right]$$

and corresponding derivative.

Second part: let's look onto the denominators $1-e^{i(\mu-kC)}$ and $1-e^{-i(\mu+kC)}$ - they turned into zeros (e.g. amplitude of excited oscillation grow to infinity – in linear approximation) when

$$\mu\pm kC=2N\pi\rightarrow Q_{exc}=N\pm Q;\quad Q_{exc}=\frac{kC}{2\pi};Q=\frac{k\mu}{2\pi}.$$

where N is an arbitrary integer. It is called a resonant excitation of oscillations – it allows to measure the non-integer part of the tune. But note, it can not distinguish between $\pm Q$.

Finally, for such a system located at s_o: $f(s)=f_o\delta(s-s_o)$

$$x(s)=\mathrm{Re}\left[\frac{e^{iks_o}\mathrm{w}(s)\mathrm{w}(s_o)f_o}{2i}\left\{\frac{e^{i(\psi(s)-\psi(s_o))}}{1-e^{i(\mu-kC)}}-\frac{e^{-i(\psi(s)-\psi(s_o))}}{1-e^{-i(\mu+kC)}}\right\}\right]$$