USPAS “Hadron Beam Cooling in Particle Accelerators”

Problem 1: Re-distribution of decrements/increments. 25 points

Let’s consider a storage ring with plane orbit (no torsion) and absence of elements coupling horizontal and vertical degrees of freedom. In this case, transvers components of the vector potential can be set to zero with Canonical momenta coinciding with mechanical momenta:

$$\pi_x = \frac{P_x}{p_o} = x'.$$

Horizontal and longitudinal oscillations in a storage ring remain coupled in all places where dispersion is non-zero. As we discuss in class, in the absence of dissipative processes,

$$\frac{dX}{ds} = \mathbf{D}(s)X;\ X = \begin{bmatrix} x \\ x' \\ \tau \\ \delta \end{bmatrix}; \mathbf{D}(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K_1(s) & 0 & 0 & K_o(s) \\ -K_o(s) & 0 & 0 & 1/(\gamma\beta)^2 \\ 0 & 0 & -U(s) & 0 \end{bmatrix}$$

and slow synchrotron oscillations (Q_s<<*1*) it is described by two actions and two phases with associated periodic eigen vectors eq. (**M3.33**):

$$X = \mathrm{Re}\left(a_x Y_x(s) e^{i(\psi_x+\varphi_x)} + a_s Y_s(s) e^{i(\psi_s+\varphi_s)}\right);\ \psi_x{}' = \frac{1}{\mathrm{w}_x(s)^2} \equiv \frac{1}{\beta_x(s)};\ \psi_s{}' \cong 2\pi Q_s \frac{s}{C};$$

$$Y_x = \begin{bmatrix} \mathrm{w}_x \\ \mathrm{w}'_x + \dfrac{i}{\mathrm{w}_x} \\ \eta\left(\mathrm{w}'_x + \dfrac{i}{\mathrm{w}_x}\right) - \eta' \mathrm{w}_x \\ 0 \end{bmatrix};\ Y_s = \begin{bmatrix} \dfrac{i}{\mathrm{w}_s}\eta \\ \dfrac{i}{\mathrm{w}_s}\eta' \\ \mathrm{w}_s \\ \dfrac{i}{\mathrm{w}_s} \end{bmatrix};\ \mathrm{w}_s = \sqrt{|\eta_\tau / \mu_3|};$$

Let’s add a distributed weak cooling process with linear drag forces:

$$\frac{dX}{ds} = \left(\mathbf{D}(s) + \delta\mathbf{D}(s)\right)X;\ \delta\mathbf{D}(s) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\chi_x(s) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_{xs}(s) & 0 & 0 & -\chi_s(s) \end{bmatrix};$$

$$Trace(\delta\mathbf{D}) = -\left(\chi_x(s) + \chi_s(s)\right) < 0.$$

Assuming that this is a weak perturbation, calculate one turn decrements. Assuming that both friction forces are dissipative $\chi_{x,y}(s) > 0$ find if it is possible that one degree of freedom experiences growth instead of damping. In other words what an off-diagonal tear $g_{xs}(s)$ (asymmetric, i.e. non-Hamiltonian, x-δ coupling) contributes to distribution of decrements?

Solution: We derive it in equation (M3-56):

$$a_k e^{i\varphi} \cong a_{k0} \cdot e^{\frac{s}{C}(i\Delta\mu - \xi_k)};\ i\Delta\mu_k - \xi_k = \frac{1}{2i}\int_0^C \left(Y_k^{*T}\mathbf{S}\delta\mathbf{D}Y_k\right)ds$$

and opening expression under integral

$$\mathbf{S}\delta\mathbf{D}\cdot Y = -\begin{bmatrix} 0 & \chi_x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_{xs} & 0 & 0 & \chi_l \\ 0 & 0 & 0 & 0 \end{bmatrix}\cdot Y = -\begin{bmatrix} \chi_x y_2 \\ 0 \\ g_{xs}y_1 + \chi_l y_4 \\ 0 \end{bmatrix}$$

$$Y^{T^*}\cdot\mathbf{S}\cdot\delta\mathbf{D}\cdot Y = -y_1^*\chi_x y_2 - y_3^*\left(g_{xs}y_1 + \chi_l y_4\right)$$

$$\varsigma_x = -\frac{y_{x1}^* y_{x2}\cdot\chi_x + g_{xs}y_{x3}^* y_{x1}}{2i} = -\frac{\chi_x - g_{xs}\eta}{2} + i\frac{\chi_x \mathrm{w}_x \mathrm{w}_x' + g_{xs}\eta \mathrm{w}_x \mathrm{w}_x' - g_{xs}\eta' \mathrm{w}_x^2}{2}$$

$$\varsigma_s = -\frac{y_{s1}^* y_{s2}\cdot\chi_x + y_{s3}^*\left(g_{xs}y_{s1} + \chi_l y_{s4}\right)}{2i} = -\frac{\chi_l + g_{xs}\eta}{2} - i\frac{\eta\eta'\chi_x}{2\mathrm{w}_s^2}$$

we get:

$$\xi_x = \frac{1}{2}\int_o^C \left(\chi_x(s) - g_{xs}(s)\eta(s)\right)ds;\ \xi_x = \frac{1}{2}\int_o^C \left(\chi_s(s) + g_{xs}(s)\eta(s)\right)ds;$$

$$\xi_x + \xi_x = \frac{1}{2}\int_o^C \left(\chi_x(s) + \chi_s(s)\right)ds = \frac{1}{2}\int_o^C Trace\left(\delta D(s)\right)ds;$$

$$\Delta\mu_x = \frac{\chi_x \mathrm{w}_x \mathrm{w}_x' + g_{xs}\eta \mathrm{w}_x \mathrm{w}_x' - g_{xs}\eta' \mathrm{w}_x^2}{2};\ \Delta\mu_s = -\frac{\eta\eta'\chi_x}{2\mathrm{w}_s^2}.$$

It means that , as expected, sum of the decrements is equal to the half of the integrated trace of of D-matrix. What is interesting is that there is a term, which can redistribute decrements:

$$\xi_x = \xi_{xo} - \xi_{xs}; \xi_s = \xi_{so} + \xi_{xs};$$

$$\xi_{xo} = \frac{1}{2}\int_o^C \chi_x(s)ds;\ \xi_x = \frac{1}{2}\int_o^C \chi_s(s)ds;\ \xi_{xs} = \frac{1}{2}\int_o^C g_{xs}(s)\eta(s)ds,$$

with ξ_{xs} depending both on the damping coupling (g_{xs}) and horizontal dispersion η. It means that one can redistribute damping between coupled degrees of freedom or even make one of the direction unstable with exponentially growing amplitudes when $\xi_{xs} > \xi_{xo}$ or when $\xi_{xs} < -\xi_{so}$.