Coherent electron Cooling

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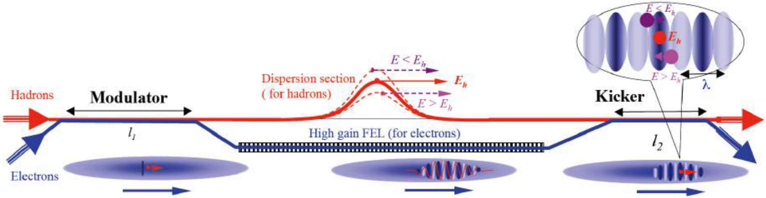
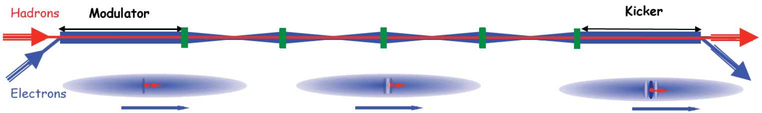
Introduction

- In the Electron-Ion Collider (EIC), Strong Hadron Cooling (SHC) is needed to reach high luminosity. Present baseline approach for SHC is based on Coherent electron Cooling (CeC).
- A general CeC scheme consists of three main sections: Modulator, Amplifier, Kicker

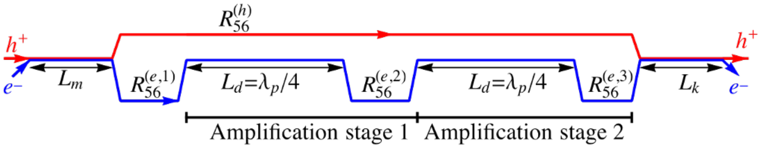


(a) CeC with free electron laser (FEL) amplifier

Other implementations of amplifier



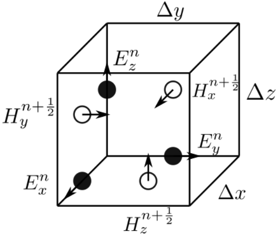
(a) Plasma cascade amplifier (PCA)



(b) Microbunched coherent electron cooling (MBEC)

Simulation tool

- The SPACE code is a parallel, relativistic, three-dimensional (3D) electromagnetic (EM) Particle-in-Cell (PIC) code.
  Finite-difference time-domain (FDTD) or Yee's method



  Uniform mesh, adaptive mesh, adaptive Particle-in-Cloud
- The GENESIS code is a three-dimensional, time-dependent code developed for high-gain FEL simulations.

Analytical tools for modulation process

Cold uniform electron beam ©V. N. Litvinenko

$$q = -Ze \cdot (1 - \cos\varphi_1) \qquad \varphi_1 = \omega_p l_1 / c\gamma_0$$

(a) Density modulation

$$\left\langle \frac{\delta E}{E} \right\rangle \cong -2Z \frac{r_e}{a^2} \cdot \frac{L_{pol}}{\gamma} \cdot \left(\frac{z}{|z|} - \frac{z}{\sqrt{a^2/\gamma^2 + z^2}} \right)$$

(b) Energy modulation

G. Wang, and M. Blaskiewicz. Physical Review E 78.2 (2008): 026413.
Linearized Vlasov Equation

$$\frac{\partial}{\partial t} f_1(\vec{x},\vec{v},t) + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} f_1(\vec{x},\vec{v},t) - \frac{e\vec{E}}{m_e} \cdot \frac{\partial}{\partial \vec{v}} f_0(\vec{v}) = 0$$

$$\vec{\nabla} \cdot \vec{E}(\vec{x},t) = \frac{\rho(\vec{x},t)}{\epsilon_0}$$

$$\rho(\vec{x},t) = Z_i e\,\delta(\vec{x}) - e\tilde{n}_1(\vec{x},t)$$

$$\tilde{n}_1(\vec{x},t) = \int f_1(\vec{x},\vec{v},t) d^3 v$$

Analytical tools for modulation process

Fourier transform

$$\frac{\partial}{\partial t} f_1(\vec{k},\vec{v},t) + i\vec{k}\cdot\vec{v} f_1(\vec{k},\vec{v},t) + i\frac{e\Phi(\vec{k},t)}{m_e}\vec{k}\cdot\frac{\partial}{\partial \vec{v}} f_0(\vec{v}) = 0$$

$$\Phi(\vec{k},t) = \frac{e}{\epsilon_0 k^2}[Z_i - \tilde{n}_1(\vec{k},t)]$$

Multiply both sides by $e^{i\vec{k}\cdot\vec{v}t}$

$$\frac{\partial}{\partial t}[e^{i\vec{k}\cdot\vec{v}t} f_1(\vec{k},\vec{v},t)] = -i\frac{e}{m_e}\Phi(\vec{k},t)e^{i\vec{k}\cdot\vec{v}t}\left(\vec{k}\cdot\frac{\partial}{\partial \vec{v}} f_0(\vec{v})\right)$$

Initial condition $f_1(\vec{k},0) = 0$

$$f_1(\vec{k},\vec{v},t) = -i\frac{e}{m_e}\int_0^t \Phi(\vec{k},t_1)e^{i\vec{k}\cdot\vec{v}(t_1-t)}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0(\vec{v})dt_1$$

Note relation

$$i\int\frac{\vec{k}}{k^2}\cdot\frac{\partial}{\partial\vec{v}}f_0(\vec{v})e^{i\vec{k}\cdot\vec{v}\tau}d^3v = \int f_0(\vec{v})e^{i\vec{k}\cdot\vec{v}\tau}\tau\,d^3v$$

Analytical tools for modulation process

We have

$$\tilde{n}_1(\vec{k},t) = \omega_p^2 \int_0^t [\tilde{n}_1(\vec{k},t_1) - Z_i](t_1 - t) g(\vec{k}(t - t_1)) dt_1$$

$$g(\vec{u}) \equiv \frac{1}{n_0} \int f_0(\vec{v}) e^{-i\vec{u}\cdot\vec{v}} d^3v$$

$$\omega_p = \sqrt{n_0 e^2 / m_e \epsilon_0}$$

Analytical tools for modulation process

For cold electrons, the velocity distribution in the rest frame of the ion reads $f_0(\vec{v}) = n_0\delta^3(\vec{v})$, which gives $g(\vec{u}) = 1$
The integral equation reduces to 2nd order ODE

$$\frac{d^2}{dt^2}\tilde{n}_1(\vec{k}, t) = -\omega_p^2\tilde{n}_1(\vec{k}, t) + Z_i\omega_p^2$$

Analytical tools for modulation process

Without ion, with initial perturbation

$$\frac{d^2}{dt^2}\tilde{n}_1\left(\vec{k},t\right)=-\omega_p^2\tilde{n}_1\left(\vec{k},t\right)$$

$$\Rightarrow \tilde{n}_1\left(\vec{k},t\right)=\tilde{n}_1\left(\vec{k},0\right)\cos\left(\omega_p t\right)+\frac{\dot{\tilde{n}}_1\left(\vec{k},0\right)}{\omega_p}\sin\left(\omega_p t\right)$$

With ion, without initial perturbation

$$\tilde{n}_1\left(\vec{k},t\right)=Z_i\left[1-\cos\left(\omega_p t\right)\right]$$

Analytical tools for modulation process

Warm uniform electron beam with $\kappa - 2$ velocity distribution:

$$f_0(\vec{v}) = \frac{1}{\pi^2 \beta_x \beta_y \beta_z} \left(1 + \frac{v_x^2}{\beta_x^2} + \frac{v_y^2}{\beta_y^2} + \frac{v_z^2}{\beta_z^2} \right)^{-2}$$

(a) $\kappa - 2$

G. Wang, and M. Blaskiewicz. Physical Review E 78.2 (2008): 026413.

$$\tilde{n}_1(\vec{x},t) = \frac{Z_i}{\pi^2 a_x a_y a_z} \int_0^{\omega_p t} \frac{\tau \sin\tau \cdot d\tau}{\left[\tau^2 + \left(\frac{x}{a_x} + \frac{v_{0,x}}{\beta_x}\tau \right)^2 + \left(\frac{y}{a_y} + \frac{v_{0,y}}{\beta_y}\tau \right)^2 + \left(\frac{z}{a_z} + \frac{v_{0,z}}{\beta_z}\tau \right)^2 \right]^2}$$

(a) Density modulation

Analytical tools for modulation process

Warm uniform electron beam with $\kappa - 2$ velocity distribution. G. Wang, V. N. Litvinenko, and M. Blaskiewicz. "Energy Modulation in Coherent Electron Cooling." Proceedings of IPAC (2013).

$$\left\langle \frac{\delta E}{E_0} \right\rangle = \frac{\langle v_z \rangle}{c} = -\frac{1}{e n_0 \pi a^2 c} I_d\left(\gamma_0 z_l, \frac{L_{\mathrm{mod}}}{\beta_0 \gamma_0 c} \right)$$

(a) Energy modulation

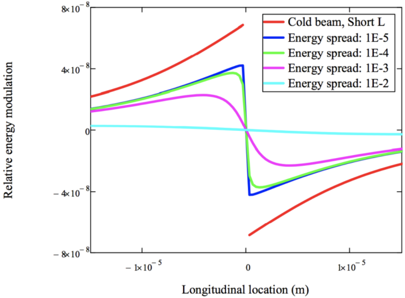
$$I_d(z,t) = -\frac{Z_i e \omega_p^2}{\pi} \int_0^t d\tau \left(z + v_{0,z}\tau\right) \left\{ \frac{a_z \sin(\omega_p \tau)}{\left[\bar{\beta}^2\tau^2 + \left(z+v_{0,z}\tau\right)^2\right]\left[1+\bar{\beta}^2\tau^2 + \left(z+v_{0,z}\tau\right)^2 / a^2\right]} - \cos(\omega_p \tau) \left[\frac{\arctan\left(\left|z+v_{0,z}\tau\right| / \left(\bar{\beta}\tau\right)\right)}{\left|z+v_{0,z}\tau\right|} - \frac{\arctan\left(\sqrt{\left(z+v_{0,z}\tau\right)^2 + a^2} / \left(\bar{\beta}\tau\right)\right)}{\sqrt{\left(z+v_{0,z}\tau\right)^2 + a^2}} \right] \right\}$$

(b) Energy modulation

The warm beam result reduces to the previously derived cold beam result at the corresponding limits

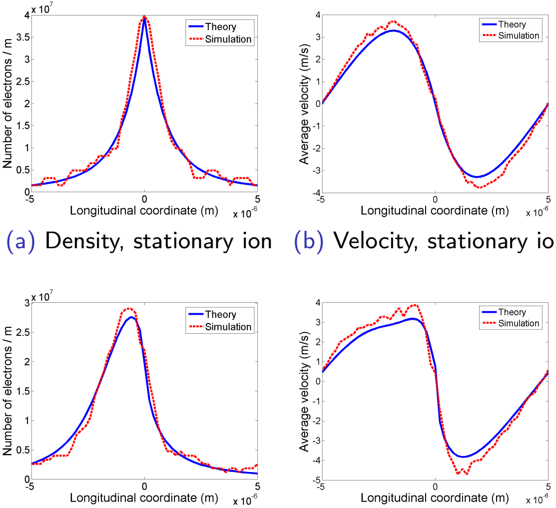
$$\bar{\beta} = 0 \quad v_{0,z} = 0 \quad L_{\text{mod}} << \beta_0 \gamma_0 c / \omega_p$$

(a)



(b) Energy modulation

Simulation using uniform beam



(a) Density, stationary ion (b) Velocity, stationary ion

(c) Density, moving ion (d) Velocity, moving ion

Continuous focusing field

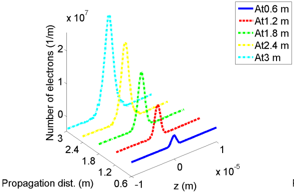
$$\vec{E}_1(\vec{r}) = \frac{m_e}{e}\frac{\sigma_v^2}{\sigma_r^2}(\vec{r} - \vec{r}_0)$$

$$\vec{E}_2(\vec{r}) = \frac{q}{2\pi\varepsilon_0|\vec{r} - \vec{r}_0|}\left(1 - e^{-|\vec{r}-\vec{r}_0|^2/2\sigma_r^2}\right)$$

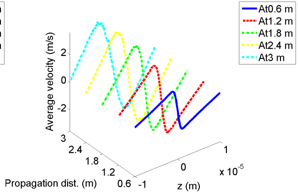
where $\vec{r} = (x, y)$ is the radial coordinate in transverse plane, $\vec{r}_0 = (x_0, y_0)$ is the center of the Gaussian distribution, σ_r is the RMS of the Gaussian distribution in both horizontal and vertical directions and σ_v is the RMS velocity of the electron distribution.

Transverse beam size is constant in the modulator.

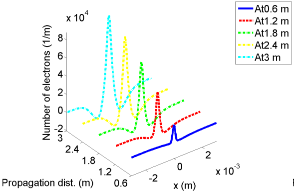
Simulation using Gaussian beam, continuous focusing



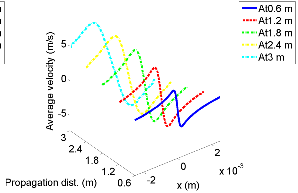
(a) Longitudinal density



(b) Longitudinal velocity

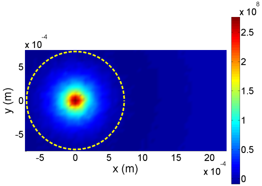


(c) Transverse density

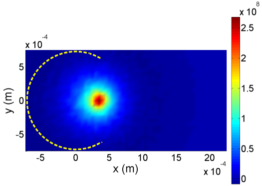


(d) Transverse velocity

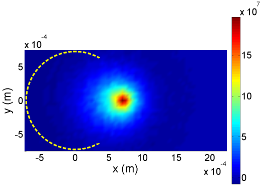
Simulation using Gaussian beam, continuous focusing



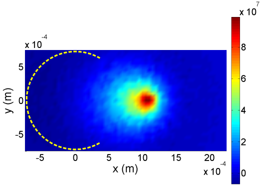
(a) Ion at center



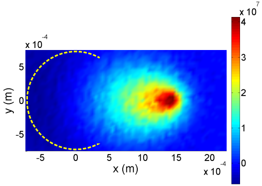
(b) Ion 0.5σ off center



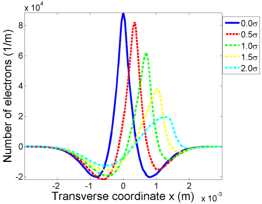
(c) Ion 1.0σ off center



(d) Ion 1.5σ off center

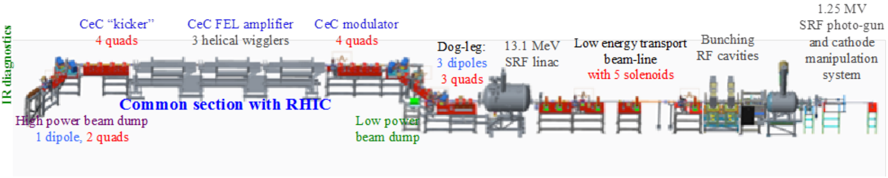


(e) Ion 2.0σ off center

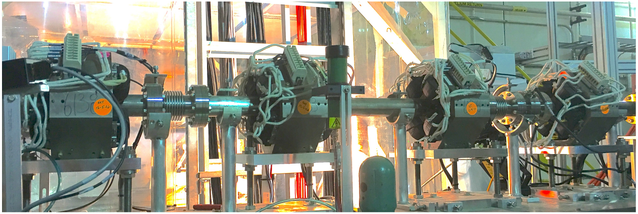


(f) Transverse density

FEL-based CeC experiment



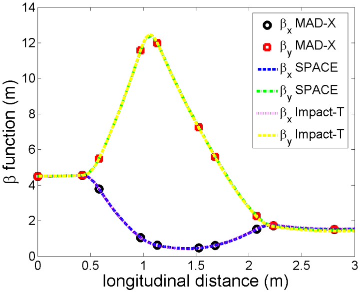
Modulator of FEL-based CeC experiment



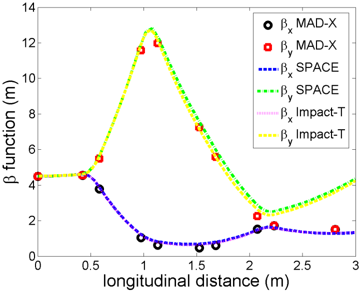
Q4 Q3 Q2 Q1

$$
\begin{aligned}
B_x &= G \cdot y \\
B_y &= G \cdot x \\
\kappa &= \frac{G}{B\rho} \\
B\rho(T \cdot m) &= 3.3356pc(GeV)
\end{aligned}
$$

Modulator, quadrupole beam line

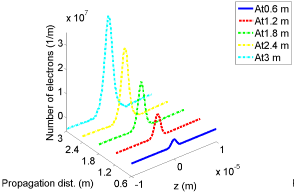


(a) No space charge

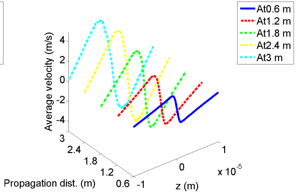


(b) With space charge

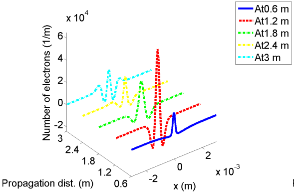
Modulation, quadrupole beam line



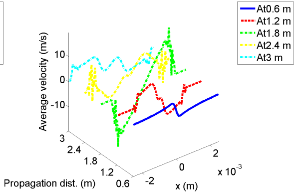
(a) Longitudinal density



(b) Longitudinal velocity



(c) Transverse density



(d) Transverse velocity

$$\langle x_o \delta x'_o \rangle = -\varepsilon, \varepsilon > 0.$$

(a) Initial correlation

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \begin{pmatrix} x_o \\ x'_o \end{pmatrix}, \quad ad - bc = 1$$

(b) Transport

$$\begin{pmatrix} \delta x(s) \\ \delta x'(s) \end{pmatrix} = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \begin{pmatrix} 0 \\ \delta x'_o \end{pmatrix}$$

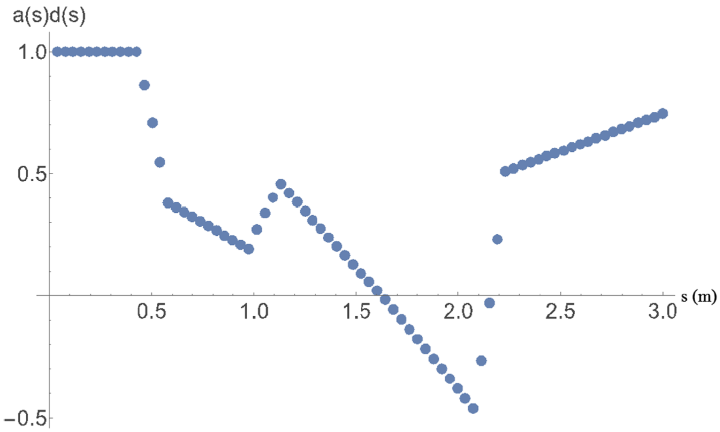
(c) Transport

$$\begin{aligned} x &= a x_o + b x'_o \\ \delta x' &= d \delta x'_o \\ \langle x \delta x' \rangle &= ad \cdot \langle x_o \delta x'_o \rangle \\ &= -ad \cdot \varepsilon \end{aligned}$$

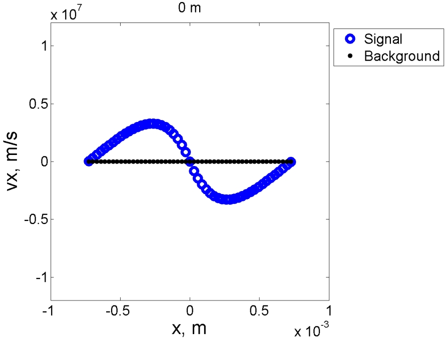
(d) Final correlation

Transport in quadrupole channel

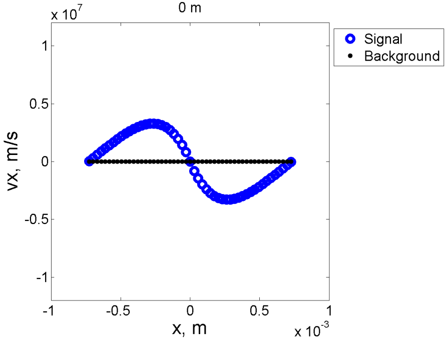
J. Ma, et al. Physical Review Accelerators and Beams 21.11 (2018): 111001.



Transverse phase advance in quadrupole beam line



(a) No space charge



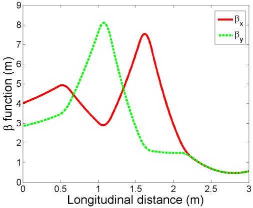
(b) With space charge

Bunching factor

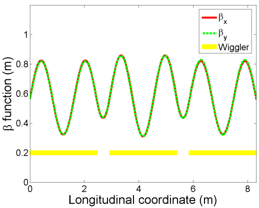
$$b \equiv \frac{1}{N_\lambda} \sum_{k=1}^{N_\lambda} e^{i \frac{2\pi}{\lambda_{opt}} z_k}, -\frac{\lambda_{opt}}{2} \leq z_k \leq \frac{\lambda_{opt}}{2},$$

where λ_{opt} is the optical wavelength, the sum is taken over a slice of λ_{opt} width, centered at the location of the ion, and N_λ is the total number of electrons within that slice.

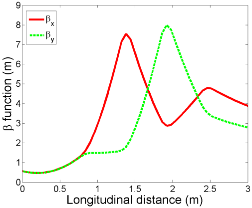
Beam envelope in FEL-based CeC



(a) Modulator

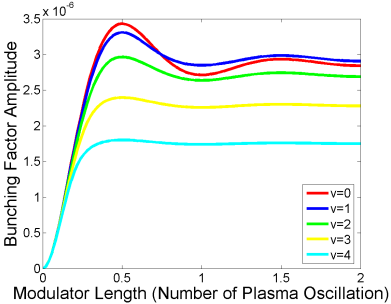


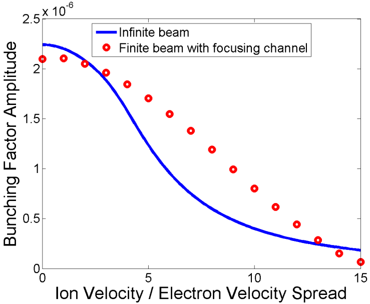
(b) FEL amplifier



(c) Kicker

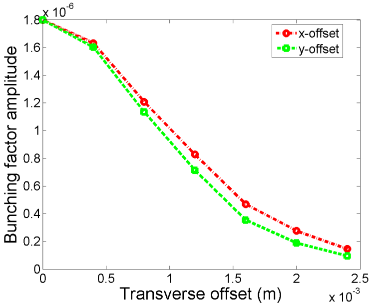
Dependence on ion velocity and modulator length

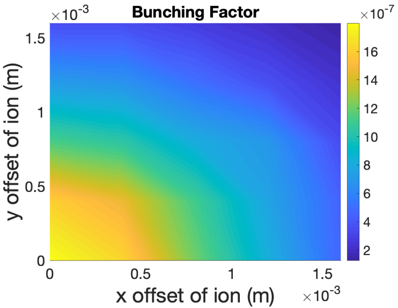




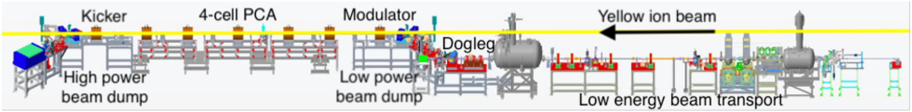
The ion velocity is in unit of electron longitudinal velocity spread.

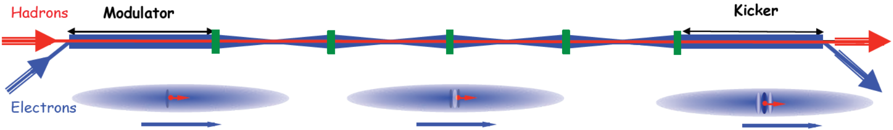
Dependence on ion transverse offset





PCA-based CeC





Solenoid field

An example of on-axis magnetic field:

$$B_{z,0} = \frac{B_0}{2}\left(\frac{L/2 - z}{\sqrt{(z - L/2)^2 + R^2}} + \frac{L/2 + z}{\sqrt{(z + L/2)^2 + R^2}}\right)$$

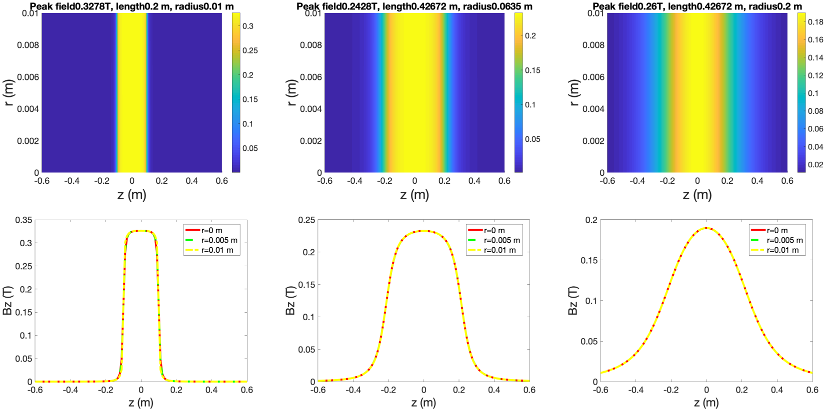
Off-axis magnetic field:

$$\begin{aligned} B_z(r) &= B_{z,0} - \frac{r^2}{4}B''_{z,0} + \frac{r^4}{64}B''''_{z,0} - \frac{r^6}{2304}B''''''_{z,0} \cdots \\ B_r(r) &= -\frac{r}{2}B'_{z,0} + \frac{r3}{16}B'''_{z,0} - \frac{r^5}{384}B'''''_{z,0} \cdots \end{aligned}$$

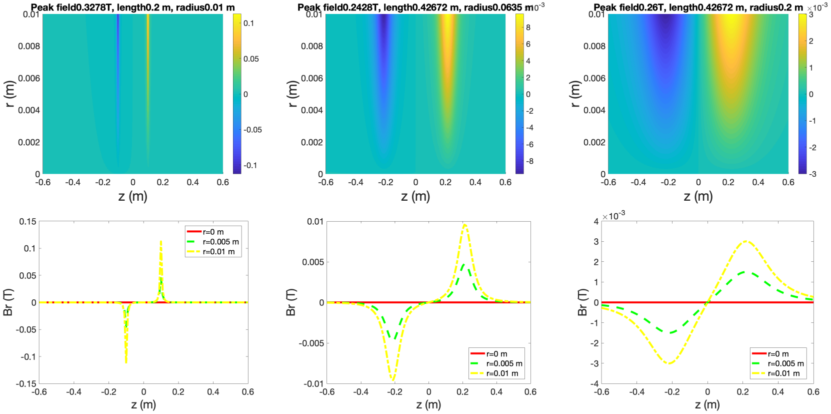
Lorentz transformation of the fields

$$
\begin{aligned}
E_x^* &= \gamma E_x - \gamma\beta c B_y \\
E_y^* &= \gamma E_y + \gamma\beta c B_x \\
E_z^* &= E_z \\
B_x^* &= \gamma B_x + \frac{\gamma\beta}{c} E_y \\
B_y^* &= \gamma B_y - \frac{\gamma\beta}{c} E_x \\
B_z^* &= B_z
\end{aligned}
$$

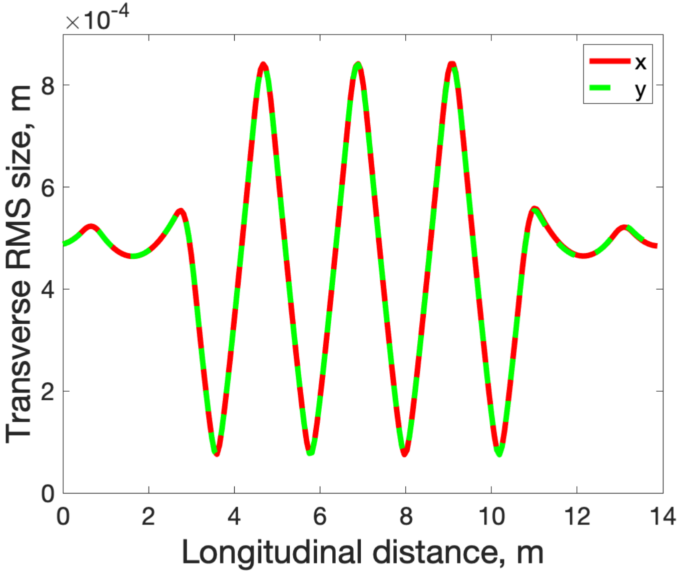
Solenoid field B_z



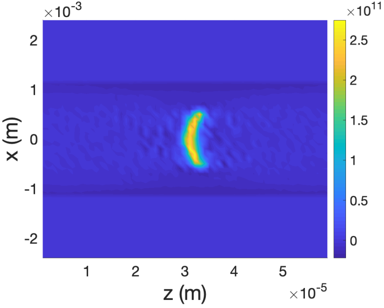
Solenoid field B_r



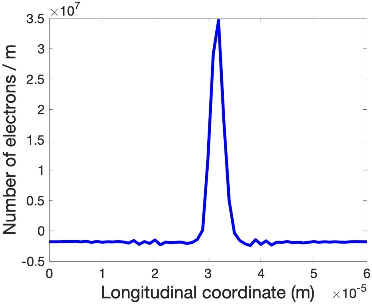
Beam envelope in PCA-based CeC



Density modulation in PCA-based CeC

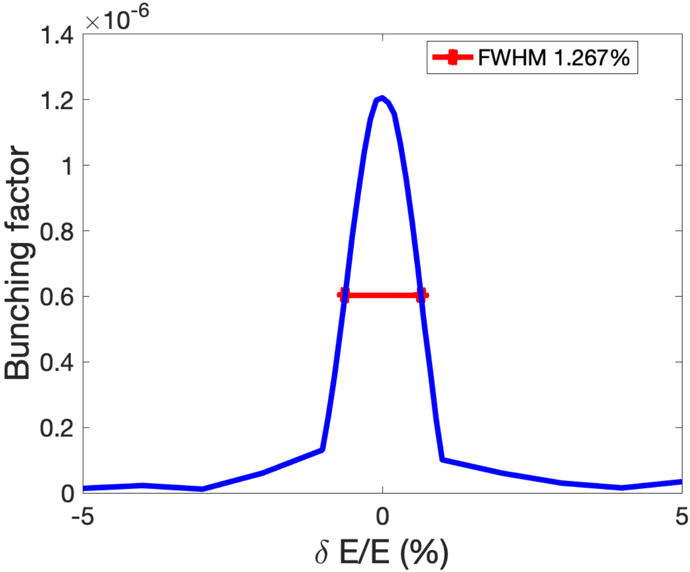


(a) 2D plot

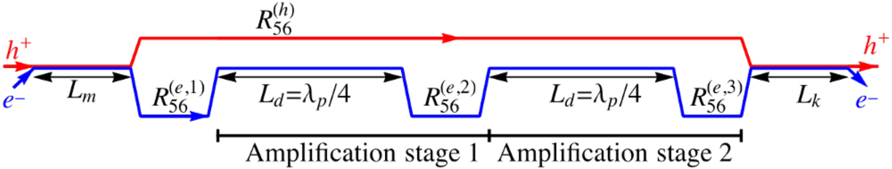


(b) 1D plot

Dependence on energy difference

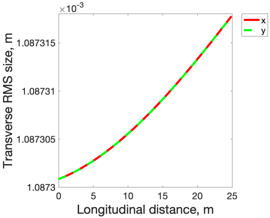


MBEC

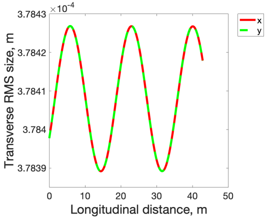


$R_{56}^{(h)}$
h^+
h^+
e^-
e^-
L_m
$R_{56}^{(e,1)}$
$L_d = \lambda_p/4$
$R_{56}^{(e,2)}$
$L_d = \lambda_p/4$
$R_{56}^{(e,3)}$
L_k
Amplification stage 1
Amplification stage 2

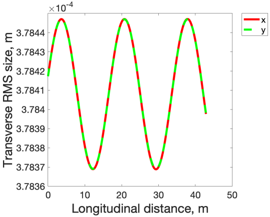
Beam envelope in MBEC



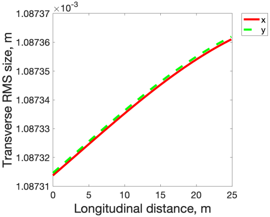
(a) Modulator



(b) First stage

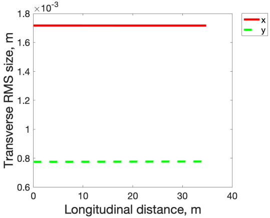


(c) Second stage

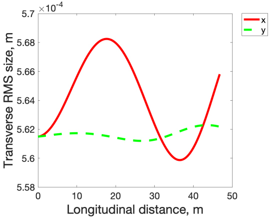


(d) Kicker

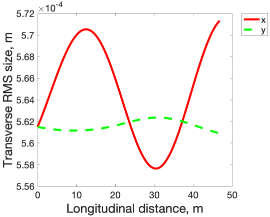
Beam envelope in MBEC



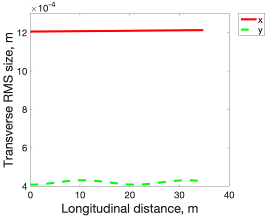
(a) Modulator



(b) First stage

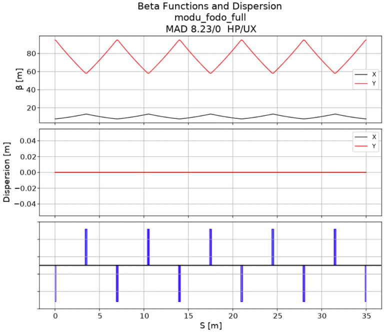


(c) Second stage

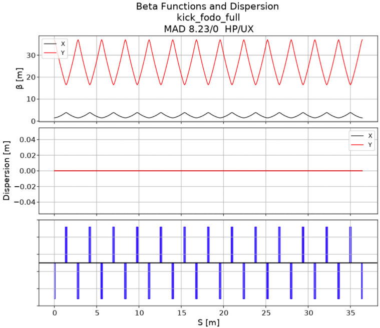


(d) Kicker

Beam envelope in MBEC

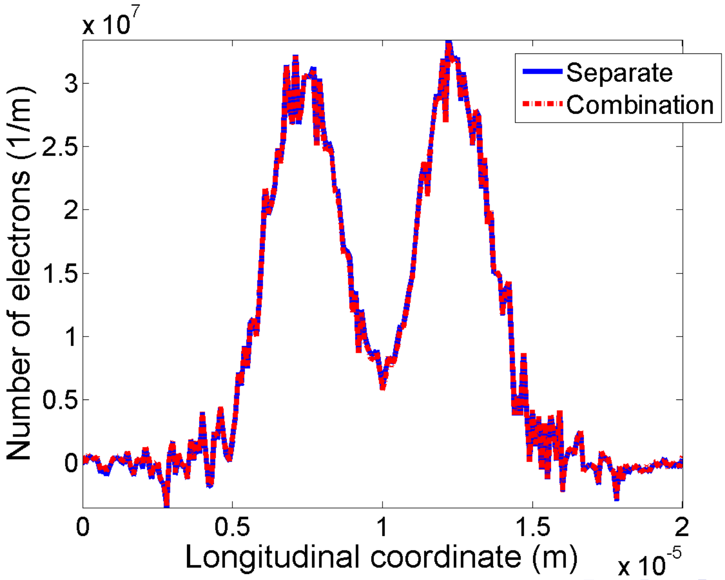


(a) Modulator

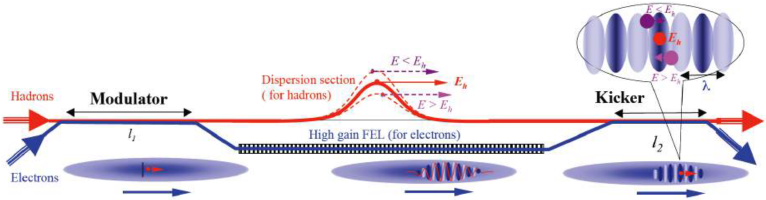


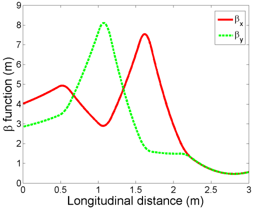
(b) Kicker

Superposition principle in density modulation

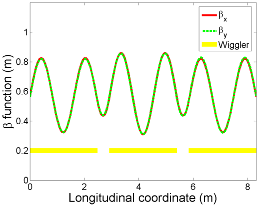


FEL-based CeC

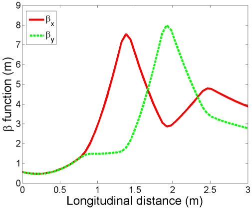




(a) Modulator

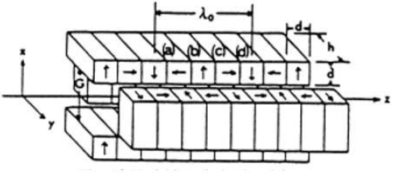


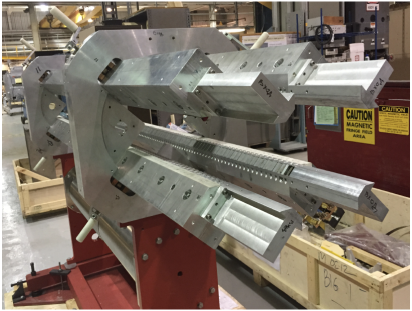
(b) FEL amplifier



(c) Kicker

Helical undulator





$$B_x(x,y,z) = B_0\cos(k_u z)$$
$$B_y(x,y,z) = B_0\sin(k_u z)$$

Electron motion in helical wiggler without radiation field

$$\vec{B}_w(x,y,z) = B_w\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right]$$

$$\vec{F}(x,y,z) = -e\vec{v}\times\vec{B} = -ev_z\hat{z}\times\vec{B} = -ev_z B_w\left[\cos(k_u z)\hat{y} + \sin(k_u z)\hat{x}\right]$$

$$\frac{d(m\gamma v_x)}{dt} = m\gamma\frac{dv_x}{dt} = -ev_z B_w \sin(k_u z) \qquad \frac{d(m\gamma v_y)}{dt} = m\gamma\frac{dv_y}{dt} = -ev_z B_w \cos(k_u z)$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \qquad v = \sqrt{v_x^2 + v_y^2 + v_z^2} \qquad \tilde{v} \equiv v_x + iv_y$$

$$m\gamma\frac{d\tilde{v}}{dt} = -iev_z B_w\left(\cos(k_u z) - i\sin(k_u z)\right) = -iev_z B_w e^{-ik_u z}$$

$$m\gamma\frac{d\tilde{v}}{dt} = m\gamma\frac{dz}{dt}\frac{d\tilde{v}}{dz} = -iev_z B_w e^{-ik_u z} \Rightarrow m\gamma\frac{d\tilde{v}}{dz} = -ieB_w e^{-ik_u z}$$

Electron motion in helical wiggler without radiation field

$$\frac{\tilde{v}(z)}{c} = \frac{-ieB_w}{mc\gamma}\int e^{-ik_u z_1}\,dz_1 = \frac{eB_w}{mc\gamma k_u}e^{-ik_u z} = \frac{K}{\gamma}e^{-ik_u z}$$

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right] \qquad v_z = const.$$

$$K \equiv \frac{eB_w\lambda_w}{2\pi mc} \qquad \theta_s = K/\gamma$$

Energy change of electrons due to radiation field

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right]$$

$$\vec{E}_\perp(z,t) = E\left[\cos(kz - \omega t)\hat{x} + \sin(kz - \omega t)\hat{y}\right] \qquad E_z = 0$$

$$= E\left[\cos(k(z - ct))\hat{x} + \sin(k(z - ct))\hat{y}\right] \qquad \omega = kc$$

$$\frac{d\mathcal{E}}{dt} = \vec{F}\cdot\vec{v} = -e\vec{v}_\perp \cdot \vec{E}_\perp$$

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \cos\left[\left(k_w + k - k\frac{c}{v_z}\right)z + \psi_0\right]$$

Resonant radiation wavelength

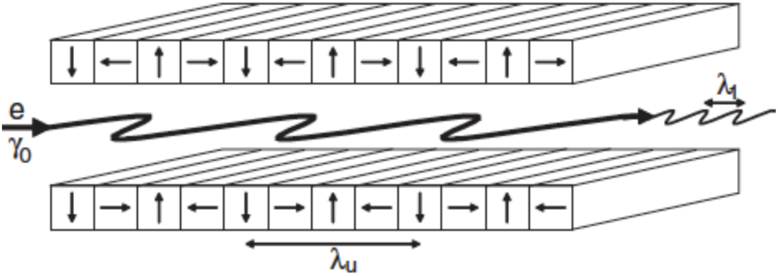
$$k_w + k_0 - k_0 \frac{c}{v_z} = 0 \Rightarrow \lambda_0 = \lambda_w \left(\frac{c}{v_z} - 1 \right) \approx \frac{\lambda_w}{2\gamma_z^2}$$

$$\gamma_z^{-2} \equiv 1 - v_z^2 / c^2 = 1 - \left(v_z^2 + v_\perp^2 \right) / c^2 + v_\perp^2 / c^2 = \gamma^{-2} + \theta_s^2 = \gamma^{-2} \left(1 + K^2 \right)$$

$$\lambda_0 \approx \frac{\lambda_w \left(1 + K^2 \right)}{2\gamma^2}$$

$$K \equiv \frac{eB_w \lambda_w}{2\pi mc}$$

Planar undulator



$$
\begin{aligned}
B_y(x,y,z) &= B_0 \sin(k_u z) \\
\lambda_0 &= \frac{\lambda_w}{2\gamma^2}\left(1+\frac{K^2}{2}\right)
\end{aligned}
$$

FEL gain

Low gain

$$g_l = \frac{(E_{ext} + \Delta E)^2 - E_{ext}^2}{E_{ext}^2} \approx \frac{2\Delta E}{E_{ext}} = \tau \cdot f(\hat{C}_l) = 2\Gamma^3 l_w^3 f_l(\hat{C}_l)$$

$$f_l(\hat{C}_l) = \frac{2}{\hat{C}_l^3}\left(1 - \cos\hat{C}_l - \frac{\hat{C}_l}{2}\sin\hat{C}_l\right)$$

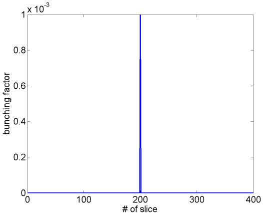
High gain

$$g_h(\hat{C}_l) = \frac{\tilde{E}^2 - E_{ext}^2}{E_{ext}^2} = \left|\frac{\lambda_2\lambda_3 e^{\lambda_1 \hat{l}_w}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2 \hat{l}_w}}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3 \hat{l}_w}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\right|^2 - 1$$

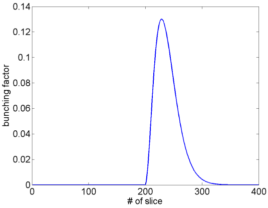
$$= 2\Gamma^3 l_w^3 f_h(\hat{C}_l) \qquad \hat{l}_w = l_w \Gamma$$

$$f_h(\hat{C}_l) = \frac{1}{2\hat{l}_w^3}\left\{\left|\frac{\lambda_2\lambda_3 e^{\lambda_1 \hat{l}_w}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2 \hat{l}_w}}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3 \hat{l}_w}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\right|^2 - 1\right\}$$

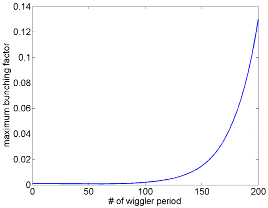
FEL gain, no saturation



(a) Initial

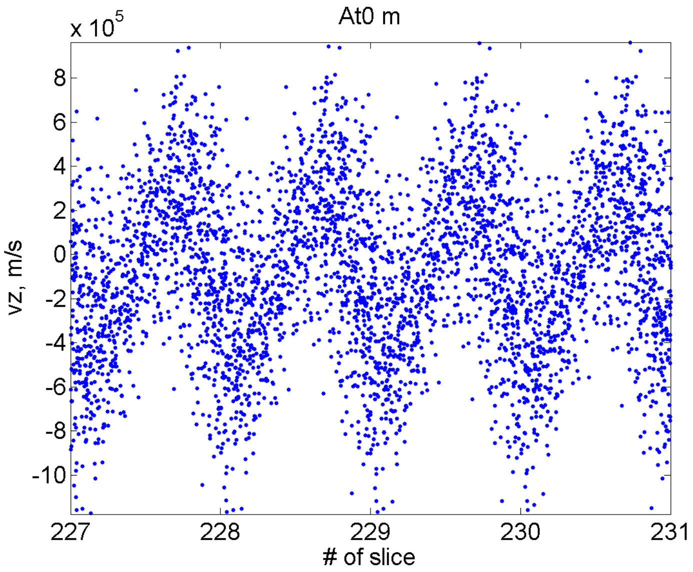


(b) Final

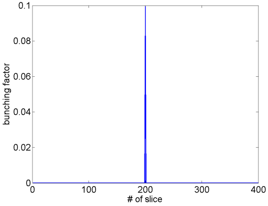


(c) Growth

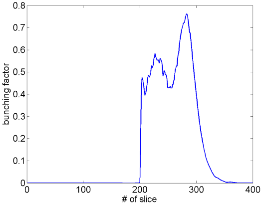
FEL gain, no saturation



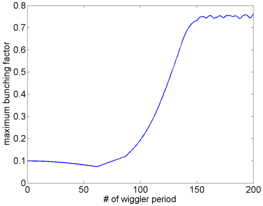
FEL gain, saturation



(a) Initial

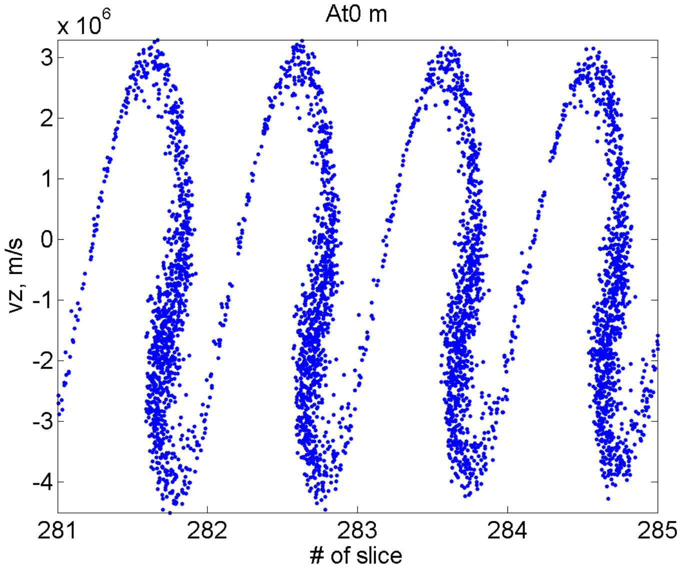


(b) Final

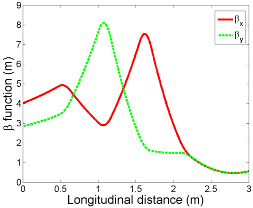


(c) Growth

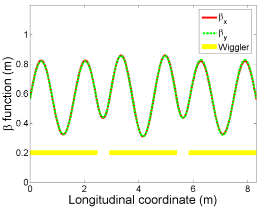
FEL gain, saturation



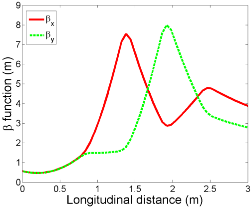
Beam envelope in FEL-based CeC



(a) Modulator

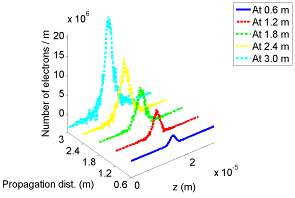


(b) FEL amplifier

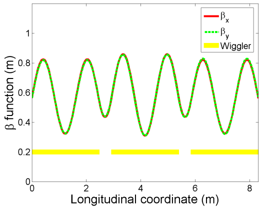


(c) Kicker

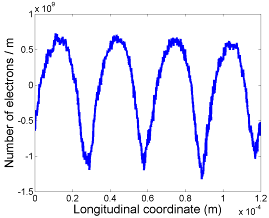
Density modulation in FEL-based CeC



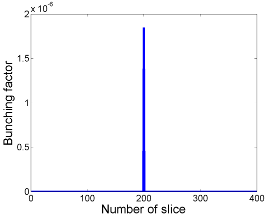
(a) Exit of modulator



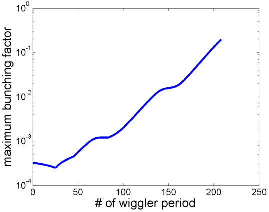
(b) FEL amplifier



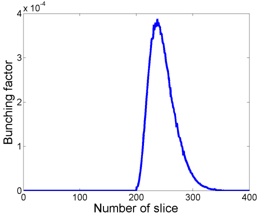
(c) Entrance of kicker



(d) Entrance of FEL

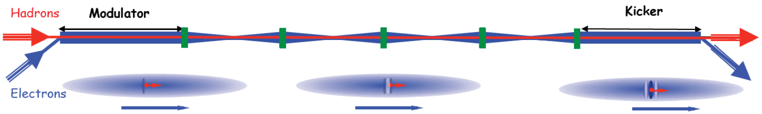


(e) FEL amplifier



(f) Exit of FEL

PCA-based CeC



Working principle of PCA is the plasma cascade instability (PCI). Litvinenko, Vladimir N., et al. Physical Review Accelerators and Beams 24.1 (2021): 014402.

$$\frac{\partial n}{\partial t} + \mathrm{div}(n\vec{v}) = 0.$$

$$\frac{\partial \vec{v}}{\partial t} = \frac{e\vec{E}}{m}; \qquad \mathrm{div}\vec{E} = 4\pi e\tilde{n} \to \mathrm{div}\frac{\partial \vec{v}}{\partial t} = \frac{4\pi e^2}{m}\tilde{n},$$

$$\frac{\partial^2 \tilde{n}}{\partial t^2} + 4\pi\frac{e^2 n_o}{m}\tilde{n} = \left(\frac{\partial \vec{v}}{\partial t}\cdot\vec{\nabla}\tilde{n} + \frac{\partial \tilde{n}}{\partial t}\cdot\mathrm{div}\vec{v}\right) = O\left(\left|\frac{\tilde{n}}{n_o}\right|^2\right);$$

$$\ddot{\tilde{n}} + \omega_p^2\tilde{n} = 0,$$

$$\tilde{n} = \delta n(\vec{r})\cdot\cos[\omega_p t + \varphi(\vec{r})];$$

$$\dot{\tilde{n}} = -\omega_p\delta n(\vec{r})\cdot\sin[\omega_p t + \varphi(\vec{r})],$$

Varying plasma frequency

$$\ddot{\tilde{n}} + \omega_p^2(t)\tilde{n} = 0$$

$$\begin{bmatrix} \tilde{n}(t_2) \\ \dot{\tilde{n}}(t_2) \end{bmatrix} = \mathbf{M}(t_1|t_2) \begin{bmatrix} \tilde{n}(t_1) \\ \dot{\tilde{n}}(t_1) \end{bmatrix},$$

$$\mathbf{M}(t_1|t_2) = \underset{\text{ordered}}{\exp}\left[\int_{t_1}^{t_2} \mathbf{D}(t)dt\right]; \quad \mathbf{D}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_p^2(t) & 0 \end{bmatrix}$$

$$\underset{\text{ordered}}{\exp}\left[\int_{t_1}^{t_2} \mathbf{D}(t)\right] dt = \lim_{N\to\infty} \prod_{\substack{n=1 \\ \text{ordered}}}^{N} M_n \equiv M_N \dots M_2 M_1;$$

$$\begin{aligned} M_n &= \exp[\mathbf{D}(t_n)\Delta t] \\ &= \begin{bmatrix} \cos\omega_p(t_n)\Delta t & \frac{\sin\omega_p(t_n)\Delta t}{\omega_p(t_n)} \\ -\omega_p(t_n)\sin\omega_p(t_n)\Delta t & \cos\omega_p(t_n)\Delta t \end{bmatrix}; \end{aligned}$$

$$\Delta t = \frac{t_2 - t_1}{N}; \qquad t_n \in \{t_1 + (n-1)\Delta t, t_1 + n\Delta t\}.$$

Stability

$$\det[\mathbf{M}-\lambda\mathbf{I}]=0; \quad \mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}; \quad \lambda_1\lambda_2=\det\mathbf{M}=1$$

Periodic system $\omega_p(t+T)=\omega_p(T)$

$$\mathbf{M}_c=\mathbf{M}(0|T)=\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}; \quad \mathbf{M}(0|nT)=\mathbf{M}_c{}^n;$$

Stable

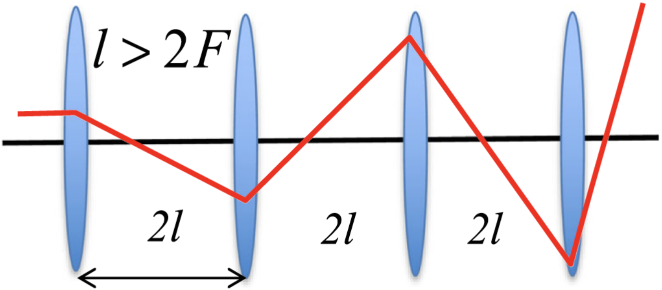
$$-2 < TraceM_c < 2$$

Stability

Unstable

$$|TraceM_c| > 2$$

$$\lambda_1 = \lambda_2{}^{-1} = \frac{\mathrm{Trace}\mathbf{M}_c}{2}\left(1+\sqrt{1-\frac{4}{(\mathrm{Trace}\mathbf{M}_c)^2}}\right); \quad |\lambda_1| > 1;$$



Instability

$$\tilde{n}(nT) = \tilde{n}(0)\frac{{\lambda_1}^n + {\lambda_1}^{-n}}{2} + \left(m_{12}\dot{\tilde{n}}(0) + \frac{m_{11} - m_{22}}{2}\tilde{n}(0)\right)\frac{{\lambda_1}^n - {\lambda_1}^{-n}}{\lambda_1 - {\lambda_1}^{-1}};$$

$$\dot{\tilde{n}}(nT) = \dot{\tilde{n}}(0)\frac{{\lambda_1}^n + {\lambda_1}^{-n}}{2} + \left(m_{21}\tilde{n}(0) + \frac{m_{22} - m_{11}}{2}\dot{\tilde{n}}(0)\right)\frac{{\lambda_1}^n - {\lambda_1}^{-n}}{\lambda_1 - {\lambda_1}^{-1}};$$

Instability

$$\tilde{n}(nT) \to \tilde{n}(0)\frac{{\lambda_1}^n}{2} + \left(m_{12}\dot{\tilde{n}}(0) + \frac{m_{11} - m_{22}}{2}\tilde{n}(0)\right)$$

$$\times \frac{{\lambda_1}^n}{\lambda_1 - {\lambda_1}^{-1}};$$

$$\dot{\tilde{n}}(nT) \to \dot{\tilde{n}}(0)\frac{{\lambda_1}^n}{2} + \left(m_{21}\tilde{n}(0) + \frac{m_{22} - m_{11}}{2}\dot{\tilde{n}}(0)\right)$$

$$\times \frac{{\lambda_1}^n}{\lambda_1 - {\lambda_1}^{-1}};$$

Note the alternating sign when $\lambda_1 < -1$

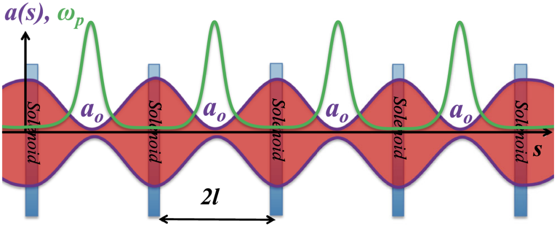
Transverse Kapchinsky-Vladimirsky (KV) distribution

$$f_{\perp}(x,x',y,y') = N \cdot \delta\left(\frac{x^2+y^2}{a(s)^2} + \frac{[a(s)x' - a'(s)x]^2 + [a(s)y' - a'(s)y]^2}{{\varepsilon_{KV}}^2} - 1\right);$$

$$a'(s) = \frac{da(s)}{ds},$$

$$\varepsilon_{\mathrm{KV}} = 4\varepsilon_{\mathrm{rms}}$$

Beam envelope



$$a'' + K(s)a - \frac{2}{\beta_o{}^3\gamma_o{}^3}\frac{I_o}{I_A}\frac{1}{a} - \frac{\varepsilon_{KV}{}^2}{a^3} = 0;$$

$$K(s) = \left(\frac{eB_{\text{sol}}(s)}{2p_o c}\right)^2 \equiv \left(\frac{eB_{\text{sol}}(s)}{2\beta_o\gamma_o mc^2}\right)^2,$$

Beam density

$$\rho(z) = \frac{I_o}{e_o c} \frac{1}{\pi a^2(s)},$$

(a) Lab frame

$$n_o(t) = \frac{I_o}{e\beta_o\gamma_o c} \frac{1}{\pi a^2(\gamma_o\beta_o c t)},$$

(b) Beam frame

Beam density

$$\frac{\partial}{\partial t}\tilde{f}_k + ikv\tilde{f}_k + \frac{eE_s}{m}\cdot\frac{\partial f_o}{\partial v} = 0;$$

$$ikE_s = 4\pi e n_o(t)\cdot\int_{-\infty}^{\infty}\tilde{f}_k(v,t)dv$$

$$f_o(v) = \sigma_v/\pi({\sigma_v}^2 + v^2)$$

$$\frac{d^2\tilde{g}_k}{dt^2} + \omega_p^2(t)\tilde{g}_k = 0; \qquad \tilde{g}_k = \tilde{f}_k e^{|k\sigma_v t|};$$

$$\rho_k(t) = \int_{-\infty}^{\infty}\tilde{f}_k(v,t)dv = \tilde{g}_k(t)\cdot\exp(-|k\sigma_v t|).$$

Periodic lattice

$$\frac{d^2\hat{a}}{d\hat{s}^2} - {k_{sc}}^2\hat{a}^{-1} - {k_\beta}^2\hat{a}^{-3} = 0;$$

$$\frac{d^2}{d\hat{s}^2}\tilde{g}_k + 2\frac{k_{sc}^2}{\hat{a}(\hat{s})^2}\cdot\tilde{g}_k = 0; \qquad \hat{a} = \frac{a}{a_o} \geq 1;$$

$$\hat{s} = \frac{s}{l}; \qquad \hat{s} \in \{-1, 1\},$$

$$k_{sc} = \sqrt{\frac{2}{{\beta_o}^3{\gamma_o}^3}\frac{I_o}{I_A}\frac{l^2}{{a_o}^2}}; \qquad k_\beta = \frac{\varepsilon l}{{a_o}^2}.$$

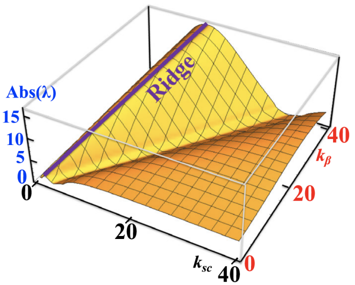
$$\ddot{\tilde{n}} + \omega_p^2(t)\tilde{n} = 0$$

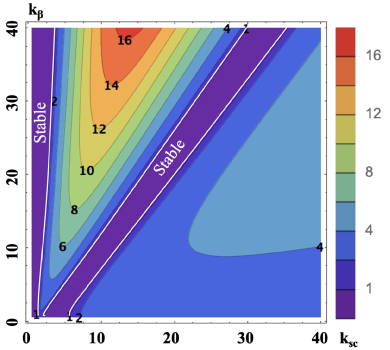
Cell

$$\mathbf{M}_c = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{11} \end{bmatrix}; \quad \lambda_1 = \lambda_2{}^{-1} = m_{11} - \sqrt{m_{11}^2 - 1};$$

$$\tilde{g}_k(mT) = \frac{\lambda_1^m + \lambda_1^{-m}}{2}\tilde{g}_k(0) - \frac{m_{12}}{\sqrt{m_{11}^2 - 1}}\frac{\lambda_1^m - \lambda_1^{-m}}{2}\dot{\tilde{g}}_k(0),$$

Cell

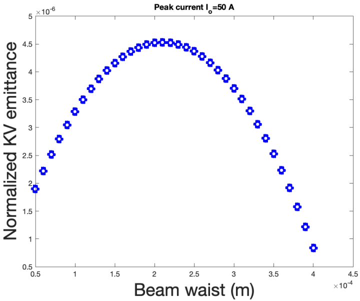


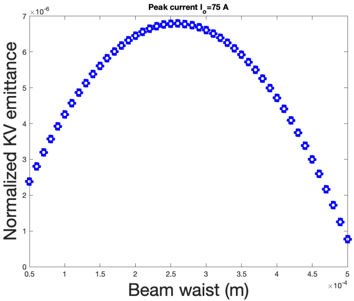


$$k_\beta \approx 3 \cdot (k_{sc} - 1.2) \quad \lambda \propto 1.25 k_{sc} \approx 1.5 + 0.413 k_\beta$$

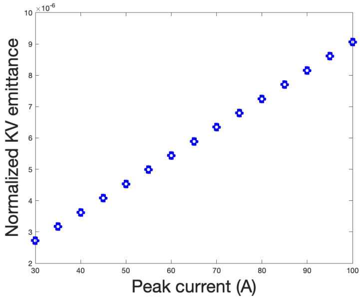
Cell

$$k_{sc} = \sqrt{\frac{2}{{\beta_o}^3{\gamma_o}^3}\frac{I_o}{I_A}\frac{l^2}{{a_o}^2}}; \qquad k_\beta = \frac{\epsilon l}{{a_o}^2}.$$

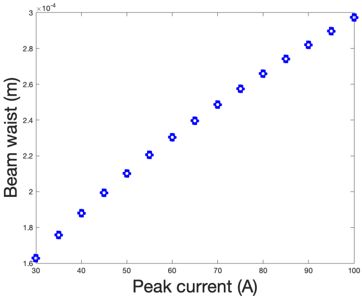




Cell



×10^-6
10
9
8
7
6
5
4
3
2
Normalized KV emittance
30
40
50
60
70
80
90
100
Peak current (A)



×10^-4
3
2.8
2.6
2.4
2.2
2
1.8
1.6
Beam waist (m)
30
40
50
60
70
80
90
100
Peak current (A)

Landau damping

$$k_{\max} = \frac{\ln \lambda}{T\sigma_v}; \qquad \omega_{\max} = \frac{v_o}{2l} \cdot \frac{{\gamma_o}^3}{\sigma_\gamma} \ln \lambda,$$

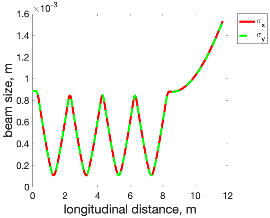
PCI, general 3D case

$$\tilde{\rho}_{\vec{k}}(t) = e\int e^{-i\vec{k}\cdot\overleftrightarrow{\mathbf{B}}(t)\cdot\vec{P}}\breve{f}_{\vec{k}(0)}(P,0)dP^3 - 4\pi e^2 n_o\beta_o(t)\int_o^t \frac{\tilde{\rho}_{\vec{k}(\tau)}(\tau)}{\det\mathbf{A}(\tau)}\frac{d\tau}{\beta_o{}^2(\tau)}\frac{\vec{k}\cdot\overleftrightarrow{\mathbf{U}}(t)\cdot\vec{k}-\vec{k}(\tau)\cdot\overleftrightarrow{\mathbf{U}}(\tau)\cdot\vec{k}(\tau)}{\gamma_o(\tau)^2[\vec{k}^2(\tau)-k_z{}^2(\tau)\beta_o{}^2(\tau)]}$$
$$\times\int e^{i(\vec{k}(\tau)\cdot\overleftrightarrow{\mathbf{B}}(\tau)-\vec{k}\cdot\overleftrightarrow{\mathbf{B}}(t))\cdot\vec{P}}F_o(P)dP^3,$$

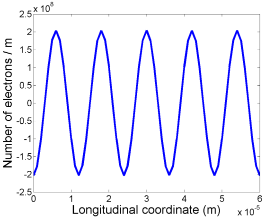
where $\mathbf{U} = \mathbf{U}^T = \mathbf{A}^{-1}\mathbf{B}^T$ and

$$k \equiv \{k_x, k_y, k_z\} \equiv \{k_1, k_2, k_3\}; \quad k(t) = k(0)\mathbf{A}^{-1}(t)$$

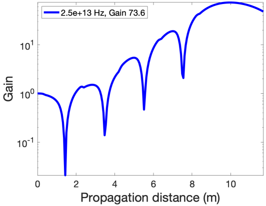
Periodic PCA



(a) Beam size

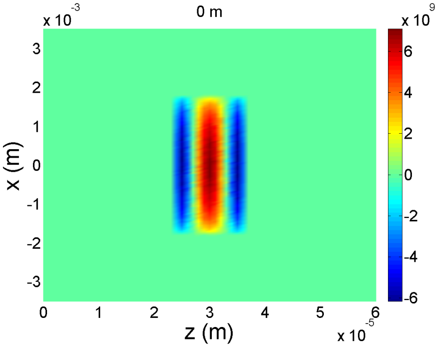


(b) Initial signal at 2.5e+13 Hz

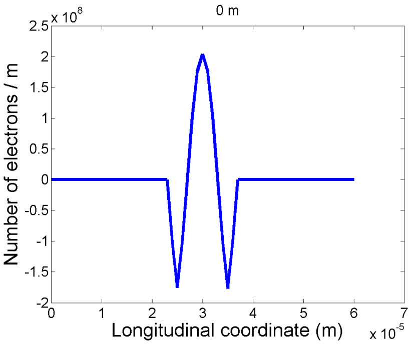


(c) Gain of 2.5e+13 Hz signal

Evolution of density modulation

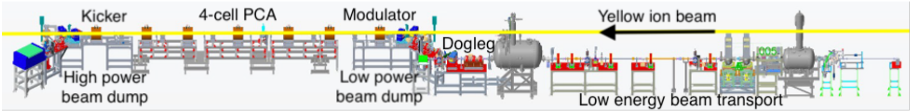


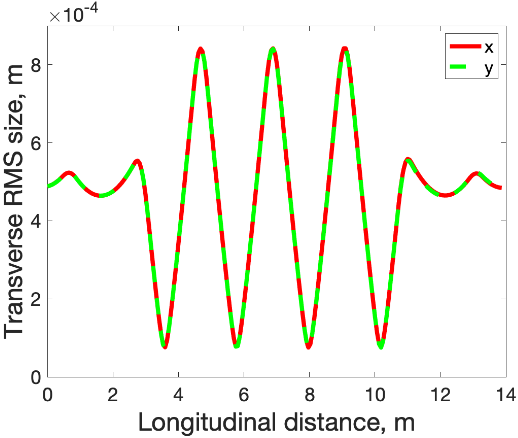
(a) z-x plot



(b) z plot

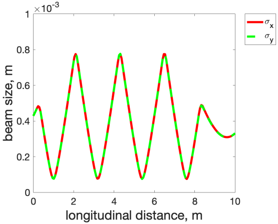
PCA-based CeC experiment



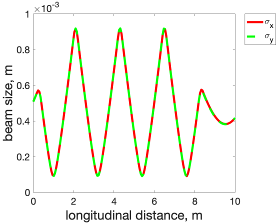


CeC PCA lattice design

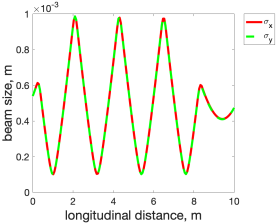
- I=50 A, a_{min}=1.5e-4 m, $\varepsilon_{n,KV}$=5 μm
- I=75 A, a_{min}=1.8e-4 m, $\varepsilon_{n,KV}$=7 μm
- I=100 A, a_{min}=2e-4 m, $\varepsilon_{n,KV}$=8 μm



(a) 50 A

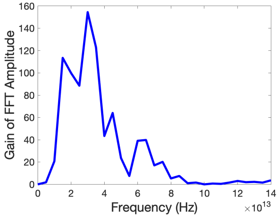


(b) 75 A

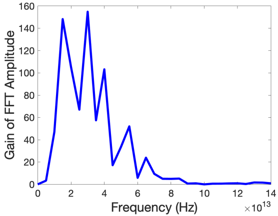


(c) 100 A

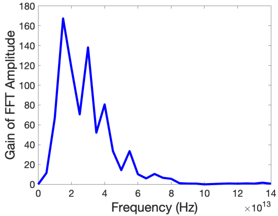
CeC PCA gain



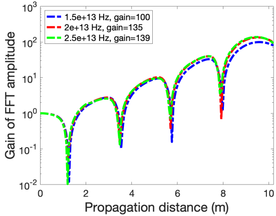
(a) 50 A



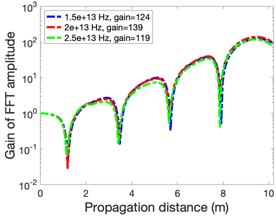
(b) 75 A



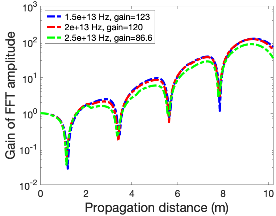
(c) 100 A



(d) 50 A

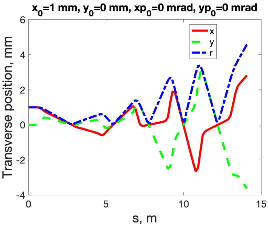


(e) 75 A

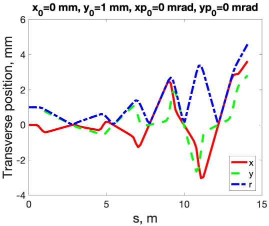


(f) 100 A

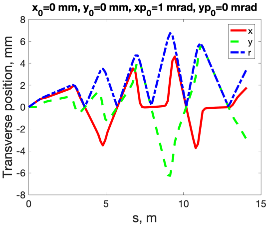
Single particle orbit with initial offsets



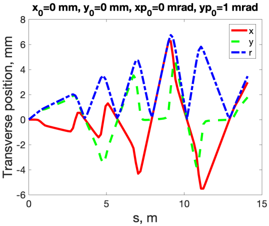
(a) x_0=1 mm



(b) y_0=1 mm

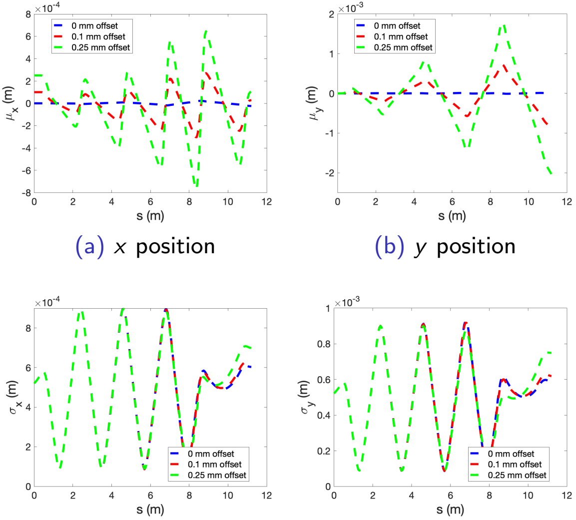


(c) x_0'=1 mrad



(d) y_0'=1 mrad

Transverse beam position and size with initial offsets



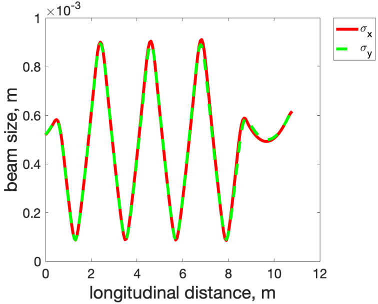
(a) x position

(b) y position

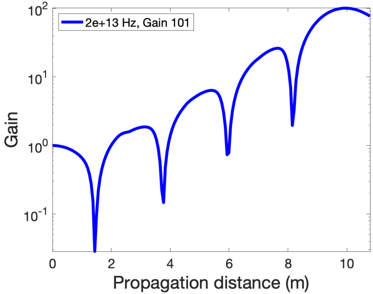
(c) x size

(d) y size

Sensitivity study of orbit, no offset

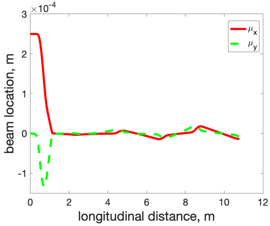


(a) Beam size

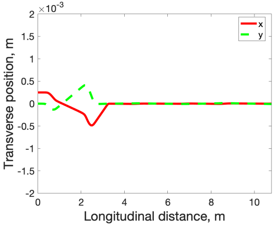


(b) PCA gain

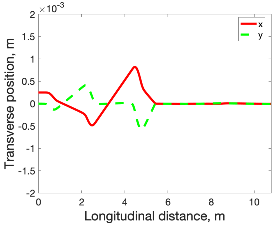
Sensitivity study of orbit, with initial offset



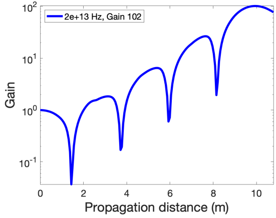
(a) Position



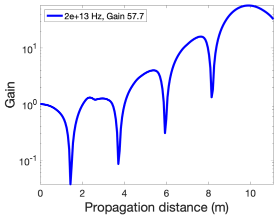
(b) Position



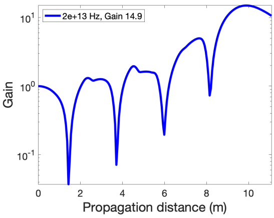
(c) Position



(d) PCA gain

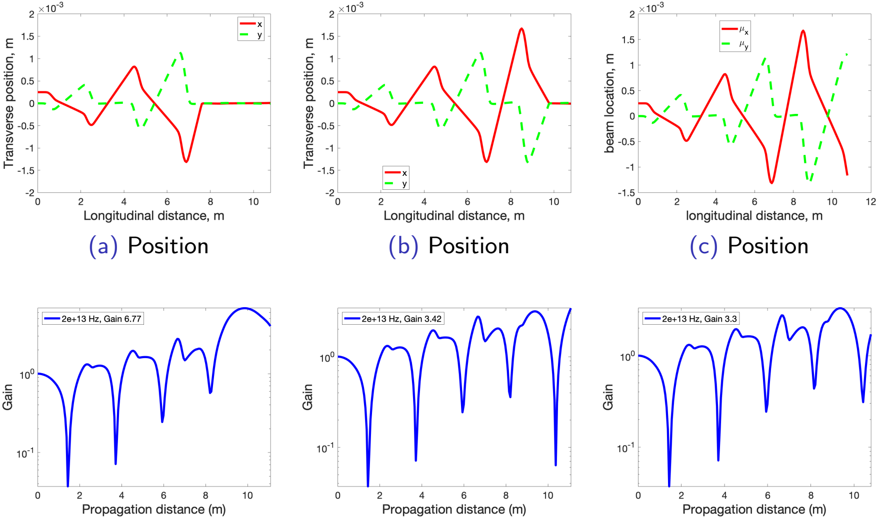


(e) PCA gain



(f) PCA gain

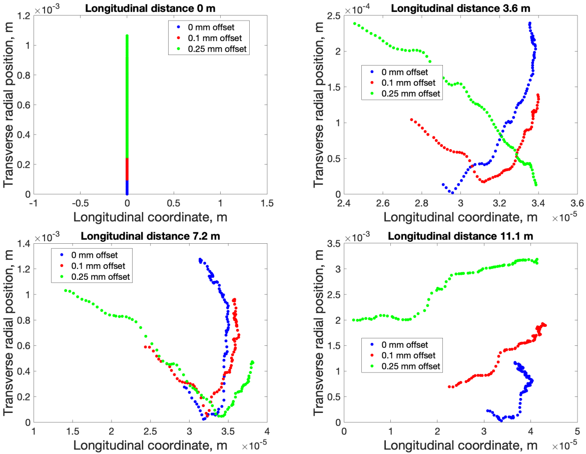
Sensitivity study of orbit, with initial offset



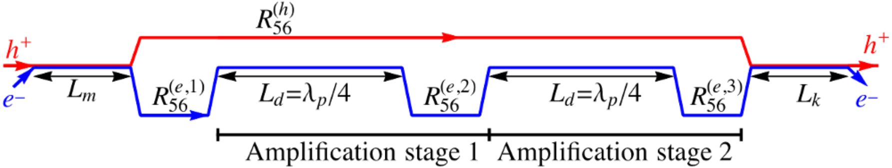
(a) Position (b) Position (c) Position

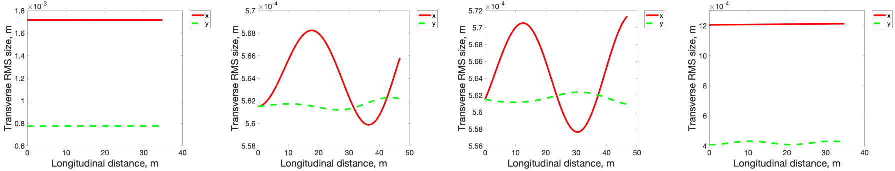
(d) PCA gain (e) PCA gain (f) PCA gain

Track a line of particles



MBEC





(a) Modulator (b) First stage (c) Second stage (d) Kicker

G. Stupakov, and P. Baxevanis. Physical Review Accelerators and Beams 22.3 (2019): 034401.

Interaction of charged Gaussian disk

$$f_z(z) = -\frac{Ze^2}{\Sigma^2}\Phi\left(\frac{z\gamma}{\Sigma}\right),$$

$$F_z(z) = \frac{e^2}{\Sigma_p^2}\Phi\left(\frac{\gamma z}{\Sigma_p}\right).$$

$$\Phi(x) = \frac{1}{2}\left[\frac{x}{|x|} - \frac{x\sqrt{\pi}}{2}\exp\left(\frac{1}{4}x^2\right)\text{erfc}\left(\frac{1}{2}|x|\right)\right],$$

Interaction of charged Gaussian disk

$$f_z(z) = -\frac{Ze^2}{\Sigma^2}\Phi\left(\frac{z\gamma}{\Sigma}\right),$$

$$F_z(z) = \frac{e^2}{\Sigma_p^2}\Phi\left(\frac{\gamma z}{\Sigma_p}\right).$$

$$\Delta\eta(z) = -\frac{Zr_eL_m}{\gamma\Sigma^2}\Phi\left(\frac{z\gamma}{\Sigma}\right),$$

Interaction of charged Gaussian disk

$$\Phi(x) = \frac{1}{2}\left[\frac{x}{|x|} - \frac{x\sqrt{\pi}}{2}\exp\left(\frac{1}{4}x^2\right)\operatorname{erfc}\left(\frac{1}{2}|x|\right)\right],$$

$$H(\varkappa) = \frac{i}{2}\int_{-\infty}^{\infty} dx\Phi(x)e^{-i\varkappa x} = \int_{0}^{\infty} dx\Phi(x)\sin(\varkappa x).$$

Plasma oscillation

$$\frac{\partial \delta f}{\partial t} + \frac{c\eta}{\gamma^2}\frac{\partial \delta f}{\partial z} + \dot{\eta} n_0 F_0'(\eta) = 0,$$

$$\dot{\eta} = \frac{1}{\gamma m_e c}\int_{-\infty}^{\infty} dz' \delta n(z', t) F_z(z - z'),$$

$$\delta n(z, t) = \int_{-\infty}^{\infty} d\eta \, \delta f(z, \eta, t)$$

Plasma oscillation

$$\delta\hat{f}_k(\eta, t) = \int_{-\infty}^{\infty} dz\, e^{-ikz}\delta f(z, \eta, t),$$

$$\delta\hat{n}_k(t) = \int_{-\infty}^{\infty} dz\, e^{-ikz}\delta n(z, t),$$

$$\frac{\partial\delta\hat{f}_k}{\partial t} + \frac{ikc\eta}{\gamma^2}\delta\hat{f}_k + \zeta(k)\delta\hat{n}_k n_0 F_0'(\eta) = 0,$$

$$\zeta(k) = \frac{1}{\gamma mc}\int_{-\infty}^{\infty} d\xi e^{-ik\xi}F_z(\xi) = -\frac{2ie^2}{\Sigma_p\gamma^2 mc}H\left(\frac{k\Sigma_p}{\gamma}\right)$$

Plasma oscillation

$$\frac{\partial \delta \hat{f}_k}{\partial t} + \frac{ikc\eta}{\gamma^2}\delta \hat{f}_k + \zeta(k)\delta \hat{n}_k n_0 F_0'(\eta) = 0,$$

Neglect second term

$$\delta \hat{f}_k(\eta, t) = \delta \hat{f}_k(\eta, 0) - \zeta(k) n_0 F_0'(\eta) \int_0^t dt' \delta \hat{n}_k(t').$$

Plasma oscillation

$$\frac{\partial \delta \hat{f}_k}{\partial t} + \frac{ikc\eta}{\gamma^2} \delta \hat{f}_k + \zeta(k) \delta \hat{n}_k n_0 F_0'(\eta) = 0,$$

Integrate over η

$$\frac{d\delta \hat{n}_k}{dt} + \frac{ikc}{\gamma^2} \delta \hat{q}_k = 0,$$

$$\delta \hat{q}_k = \int_{-\infty}^{\infty} d\eta \, \eta \delta \hat{f}_k$$

$$\frac{d\delta \hat{q}_k}{dt} - \zeta(k) \delta \hat{n}_k n_0 = 0,$$

Plasma oscillation

$$\frac{d^2\delta\hat{n}_k}{dt^2} + \frac{ikc}{\gamma^2}\zeta(k)n_0\delta\hat{n}_k = 0,$$

$$\omega_p^2 = \frac{ikcn_0}{\gamma^2}\zeta(k) = \frac{2kn_0e^2}{\Sigma_p\gamma^4 m}H\left(\frac{k\Sigma_p}{\gamma}\right) = 2\Omega^2\varkappa_p H(\varkappa_p),$$

$$\Omega^2 = \frac{n_0e^2}{m\Sigma_p^2\gamma^3}$$

MBEC gain

$$\delta\hat{n}_k = \delta\hat{n}_k(0)\cos(\omega_p t) - \frac{ikc}{\gamma^2\omega_p}\delta\hat{q}_k(0)\sin(\omega_p t),$$

$$\frac{c\sigma_e}{\gamma\Omega\Sigma_p} \sim \sigma_e\sqrt{\frac{\gamma I_A}{I_e}} \ll 1,$$

$$\delta\hat{n}_k \approx \delta\hat{n}_k(0)\cos(\omega_p t).$$

MBEC gain

$$\delta\hat{f}_k(\eta,t) = \delta\hat{f}_k(\eta,0) - \frac{1}{\omega_p}\zeta(k)n_0 F_0'(\eta)\delta\hat{n}_k(0)\sin(\omega_p t).$$

$$\delta\hat{n}_k^{(2)} = \int_{-\infty}^{\infty} d\eta \delta\hat{f}_k e^{-ikR_{56}^{(e,2)}\eta}$$
$$= -\frac{1}{\omega_p}\zeta(k)n_0\delta\hat{n}_k(0)g(k)\sin\left(\frac{\omega_p L_d}{c}\right),$$

$$g(k) = \int_{-\infty}^{\infty} d\eta F_0'(\eta)e^{-ikR_{56}^{(e,2)}\eta} = ikR_{56}^{(e,2)}e^{-k^2(R_{56}^{(e,2)})^2\sigma_e^2/2}.$$

MBEC gain

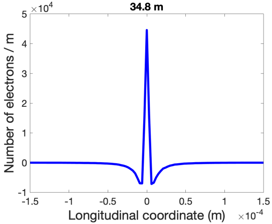
$$G = \delta\hat{n}_k^{(2)} / \delta\hat{n}_k(0)$$

$$G = -\frac{1}{\sigma_e}\sqrt{\frac{2I_e}{\gamma I_A}}\sqrt{\varkappa_p H(\varkappa_p)}q_p e^{-\varkappa_p^2 q_p^2/2} \sin\left(\frac{\omega_p L_d}{c}\right),$$

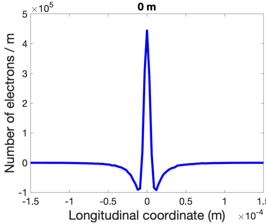
$$G_{\max} = -\frac{1}{\sigma_e}\sqrt{\frac{2I_e}{\gamma I_A}}\sqrt{\frac{2H(\varkappa_p)}{e\varkappa_p}} \sin\left(\frac{\omega_p L_d}{c}\right),$$

$$q_p = 1/\varkappa_p$$

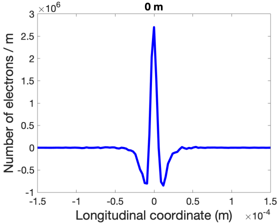
MBEC gain



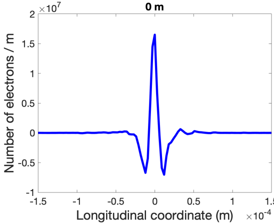
(a) Exit of modulator



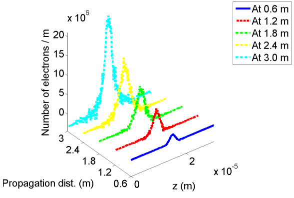
(b) After 1st chicane



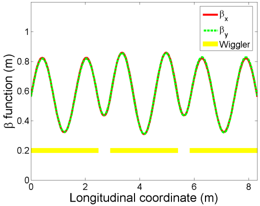
(c) After 2nd chicane



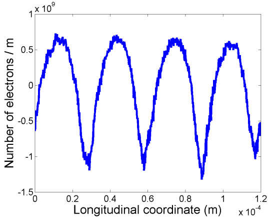
(d) After 3rd chicane



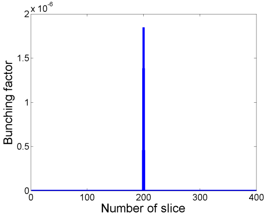
(a) Exit of modulator



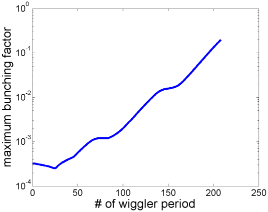
(b) FEL amplifier



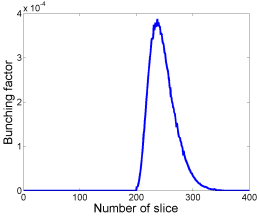
(c) Entrance of kicker



(d) Entrance of FEL

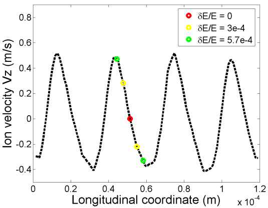


(e) FEL amplifier

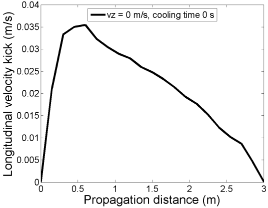


(f) Exit of FEL

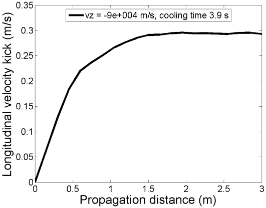
Cooling force in FEL-based CeC



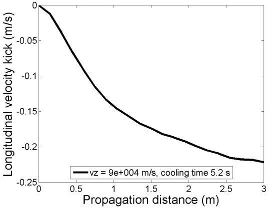
(a) Ions at different locations



(b) Ion with reference energy

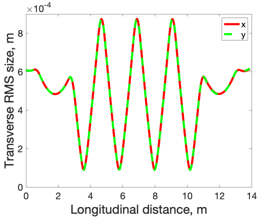


(c) Ion with lower energy

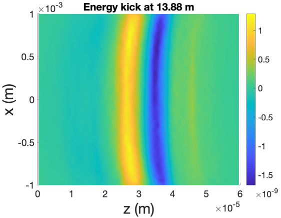


(d) Ion with higher energy

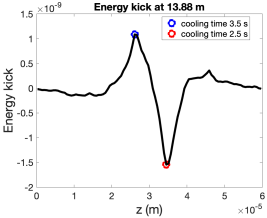
PCA-based CeC, beam current 75 A



(a) Beam size

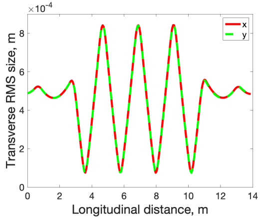


(b) Cooing force

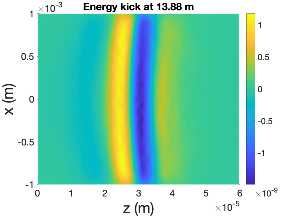


(c) Cooing force

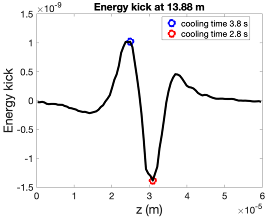
PCA-based CeC, beam current 50 A



(a) Beam size

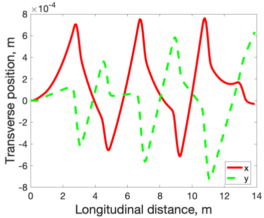


(b) Cooing force

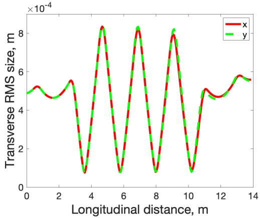


(c) Cooing force

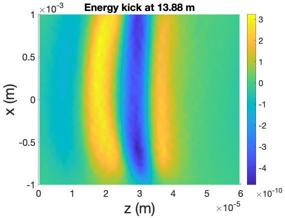
PCA-based CeC, earth field



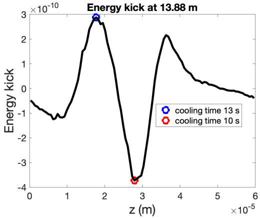
(a) Beam position



(b) Beam size

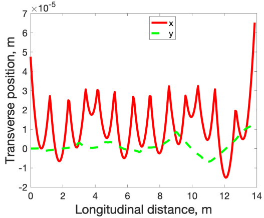


(c) Cooing force

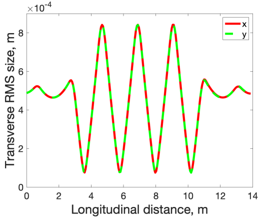


(d) Cooing force

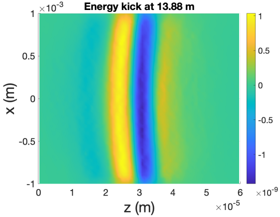
PCA-based CeC, earth field, orbit correction



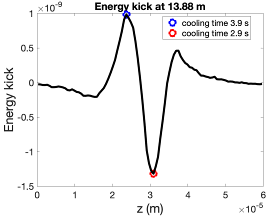
(a) Beam position



(b) Beam size



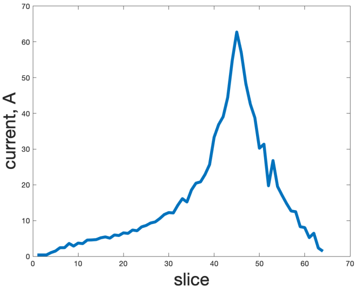
(c) Cooing force



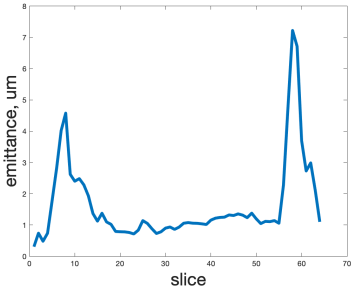
(d) Cooing force

PCA-based CeC

- Beam dynamics simulations propagate beam starting from the gun.
- Take slice parameters from beam dynamics simulations at the entrance of modulator.

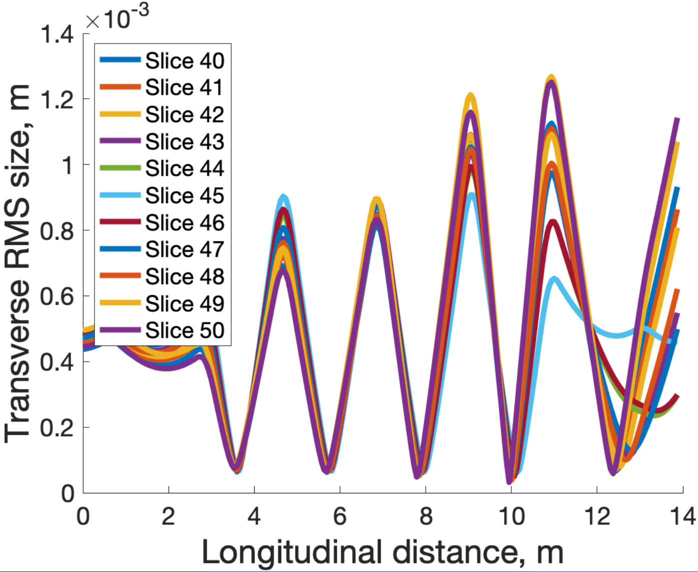

current, A
slice
0
10
20
30
40
50
60
70

(a)

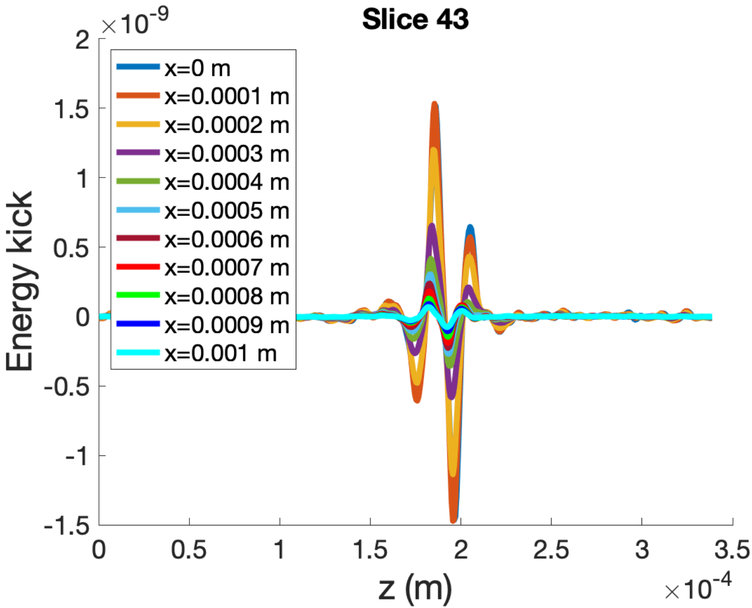

emittance, um
slice
0
10
20
30
40
50
60
70

(b)

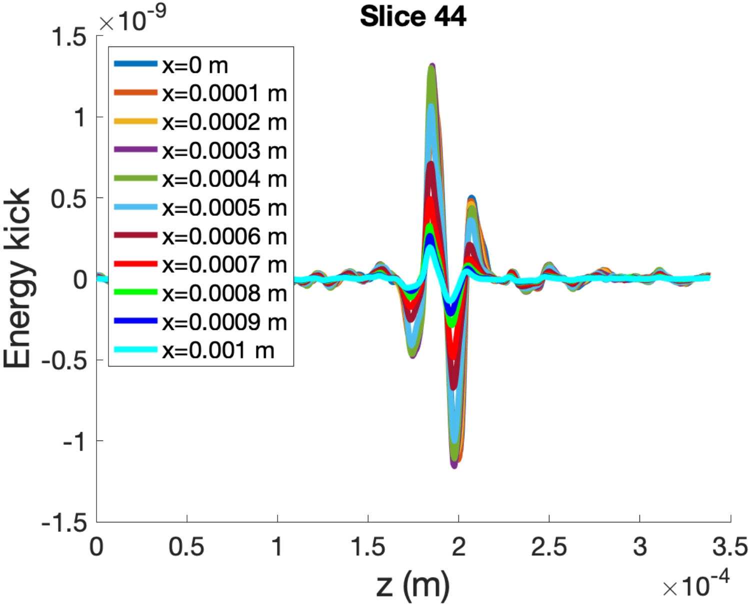
Beam size



$\times 10^{-3}$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Transverse RMS size, m
0
2
4
6
8
10
12
14
Longitudinal distance, m
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50

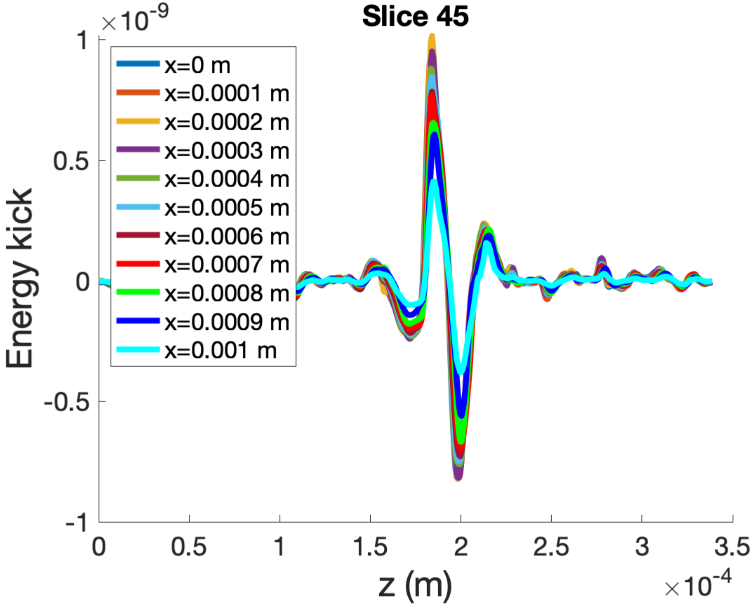

Slice 43
×10^-9
2
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

(a)

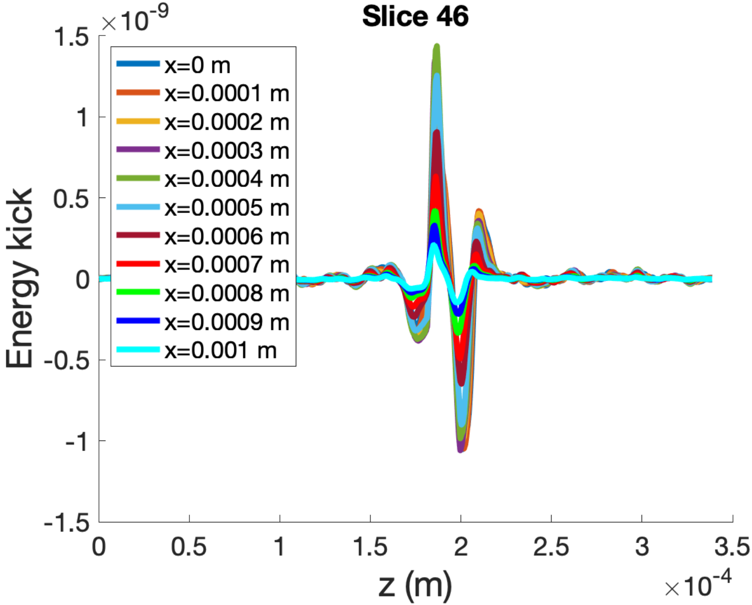


Slice 44
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
Energy kick
0
0.5
1
1.5
2
2.5
3
3.5
×10^-4
z (m)

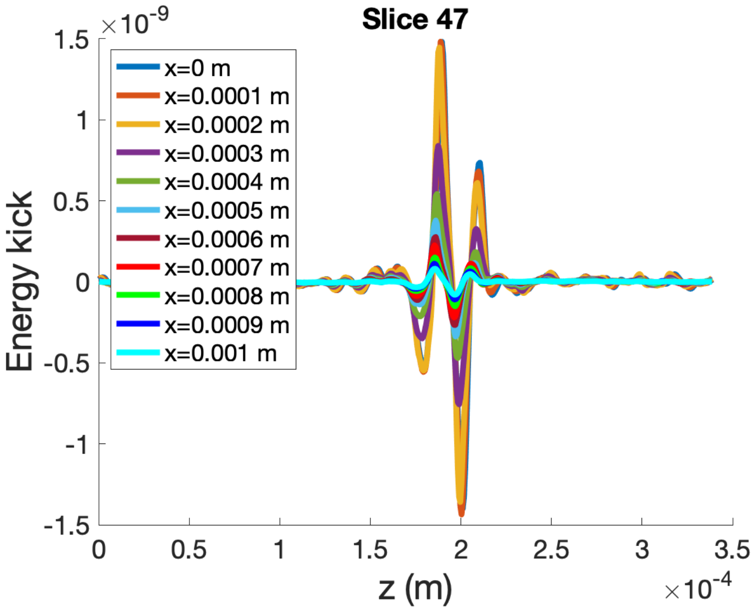
(a)


Slice 45
×10^-9
1
0.5
0
-0.5
-1
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

(a)

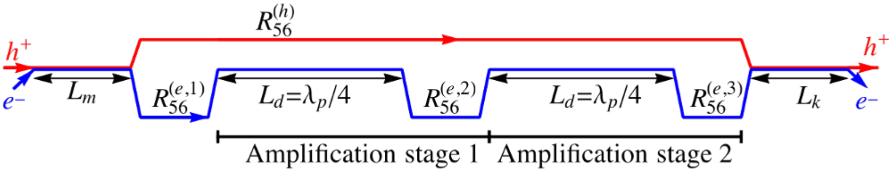

Slice 46
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

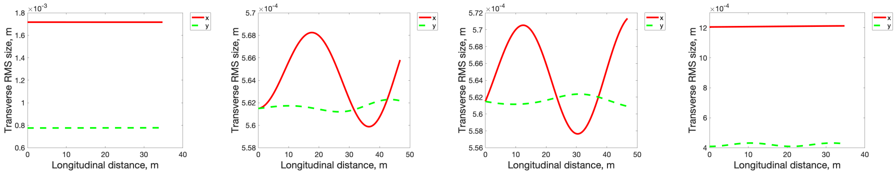
(a)



Slice 47
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

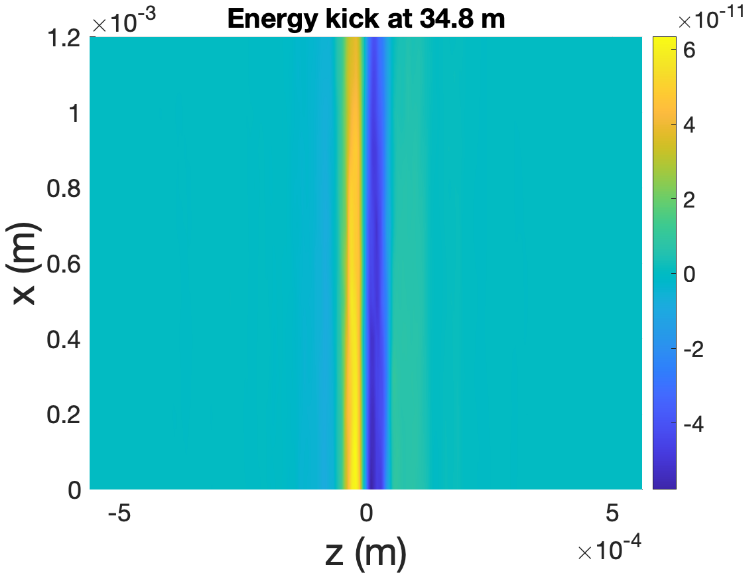
(a)



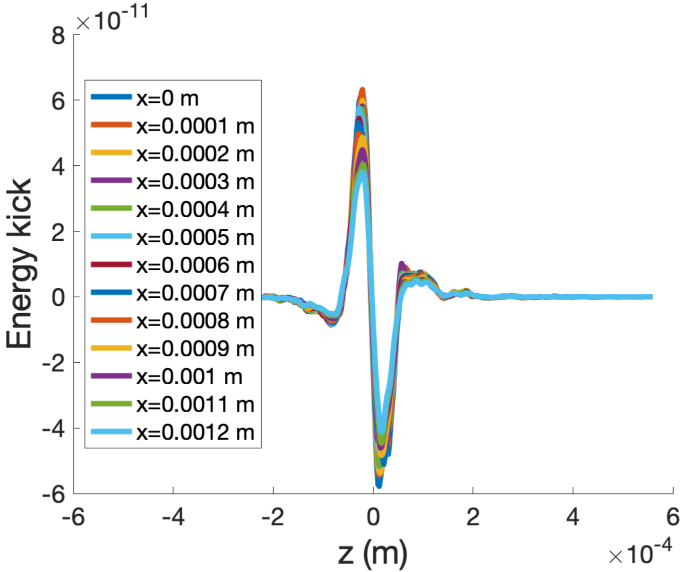


(a) Modulator (b) First stage (c) Second stage (d) Kicker

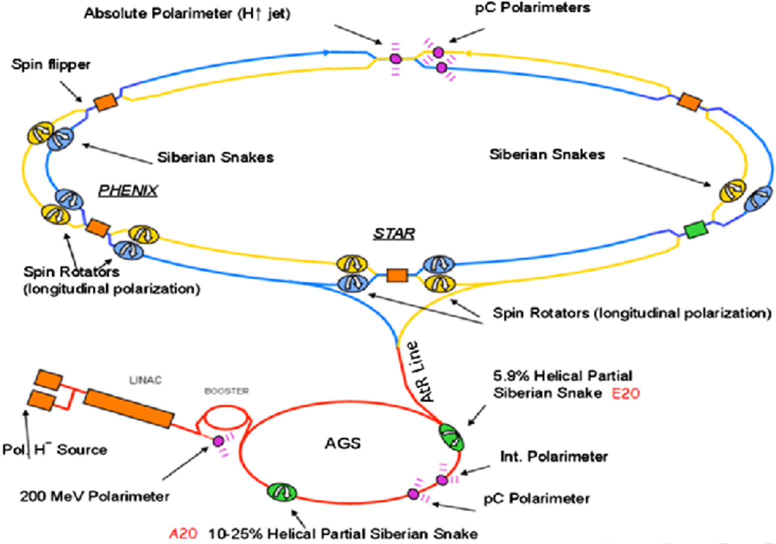
MBEC cooling force



MBEC cooling force



The Relativistic Heavy Ion Collider at the Brookhaven National Laboratory



Cooling time

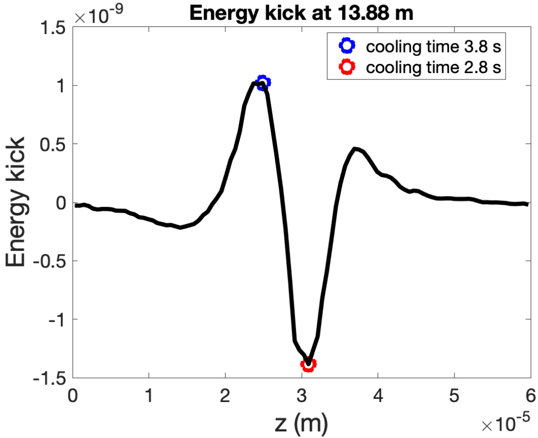
$$\frac{1}{\tau_{v_i}} \equiv -\frac{1}{v_i}\frac{dv_i}{dt}$$

If τ is independent of t

$$v_i(t) = v_i(0)\exp\left(-\frac{t}{\tau_{v_i}}\right)$$

τ is cooling time, $1/\tau$ is cooling rate.

PCA-based CeC, beam current 50 A

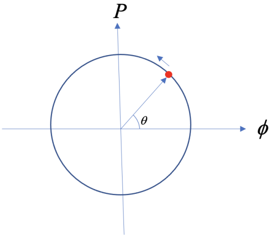


(a) Cooing force

x-axis $z = R_{56} * \delta\gamma/\gamma$
y-axis $\Delta\delta\gamma = f(\delta\gamma) \approx k_0 \delta\gamma$

$$
\begin{aligned}
\frac{1}{\tau} &= -\frac{1}{\delta\gamma}\frac{d\delta\gamma}{dt} \\
&= -\frac{1}{\delta\gamma}\frac{\Delta\delta\gamma}{T_{rev}} \\
&\approx -\frac{1}{\delta\gamma}\frac{k_0\delta\gamma}{T_{rev}} \\
&= -\frac{k_0}{T_{rev}}
\end{aligned}
$$

Averaging longitudinal cooling over synchrotron oscillation (linear cooling force)



Action-angle variable: $P \equiv -h\frac{|\eta|}{\nu_s}\frac{\Delta p}{p} = \sqrt{2I}\sin\theta \qquad \phi \equiv \omega_{rf}\tau = \sqrt{2I}\cos\theta$

Reduction of action due to cooling: $\Delta I_c = \frac{1}{2}\Delta\left(P^2+\phi^2\right) = P\Delta P_c \qquad \Delta P_c = \frac{h|\eta|}{\nu_s\gamma}\Delta\delta\gamma_c$

Assuming cooling force is linear, $\Delta\delta\gamma_c = -\zeta_0 T_{rev}\delta\gamma \qquad \Delta P_c = -\zeta_0 T_{rev} P$

the action reduction becomes $\Delta I_c = -\zeta_0 T_{rev} P^2 = -2I\zeta_0 T_{rev}\sin^2\theta$

The average cooling rate is given by

$$\zeta(I) = -\frac{1}{I}\left\langle\frac{\Delta I_c}{T_{rev}}\right\rangle_{T_s} = \zeta_0\bar{\zeta}(I)$$

$$\bar{\zeta}(I) = \frac{1}{2\pi}\int_0^{2\pi}\sin^2\theta d\theta = \frac{1}{2}$$

For the linear cooling force, synchrotron motion reduces the averaged cooling rate by a factor of 2.

Ion bunch profile evolution with cooling

G. Wang. Physical Review Accelerators and Beams 22.11 (2019): 111002
1D Fokker-Planck equation

$$\frac{\partial}{\partial t}F(I,t)-\frac{\partial}{\partial I}[\zeta(I)\cdot I\cdot F(I,t)]-\frac{\partial}{\partial I}\left(I\cdot D(I)\cdot\frac{\partial F(I,t)}{\partial I}\right)$$
$$=0,$$

$$\zeta(I)=-\frac{1}{I}\left\langle\frac{\Delta I_c}{T_{\mathrm{rev}}}\right\rangle_{T_s}\quad D(I)=\frac{1}{I}\frac{\langle\Delta I_d^2\rangle_{T_s}}{2T_{\mathrm{rev}}}$$

$F(I,t)$ is longitudinal phase space density, $\zeta(I)$ is cooling rate, $D(I)$ is diffusion coefficient.

Fokker-Planck equation

$$\phi = \sqrt{2I}\sin w,$$

$$P \equiv -h\frac{|\eta|}{\nu_s}\frac{\Delta p}{p}$$

$$H_0 = \frac{1}{2}\omega_0\nu_s P^2 + \omega_0\nu_s\frac{1}{2}\phi^2 = \omega_0\nu_s I,$$

Fokker-Planck equation

$$\int_0^\infty F(I,t)dI = \frac{N}{2\pi},$$

$$\int_{-\infty}^{\infty} F\left(\frac{P^2+\phi^2}{2},t\right)dPd\phi = \int_{-\infty}^{\infty} F\left(\frac{P^2+\phi^2}{2},t\right)\frac{\partial(P,\phi)}{\partial(I,w)}dIdw$$

$$= \int_0^{2\pi}\int_0^\infty F(I,t)dIdw$$

$$= N,$$

$$K(P,\phi,t) = F\left(\frac{1}{2}P^2+\frac{1}{2}\phi^2,t\right).$$

$$F_{\mathrm{eq}}(I) = A\exp\left\{-\int\frac{\zeta(I)}{D(I)}dI\right\}.$$

Zero diffusion

$$\frac{\partial}{\partial t}\tilde{F}(I,t) - \alpha(I)\frac{\partial}{\partial I}\tilde{F}(I,t) = 0,$$

$$\tilde{F}(I,t) \equiv \zeta(I)\cdot I\cdot F(I,t),$$

$$\alpha(I) \equiv I\cdot\zeta(I).$$

$$F(I,t) = \frac{h^{-1}(C)\zeta[h^{-1}(C)]F_0[h^{-1}(C)]}{\zeta(I)\cdot I},$$

$$C = t - t_0 + \int\frac{dI}{\alpha(I)},$$

$$h(I) \equiv \int\frac{dI}{\alpha(I)}.$$

Zero diffusion

$$\zeta(I) = \zeta_0 \frac{I_e}{I + I_e},$$

$$h(I) = \frac{1}{\zeta_0} \int \frac{1 + \frac{I}{I_e}}{I} dI = \frac{1}{\zeta_0} \ln\left[\frac{I}{I_e} \exp\left(\frac{I}{I_e}\right)\right],$$

$$h(I) = \frac{1}{\zeta_0} \ln\left[\frac{I}{I_e} \exp\left(\frac{I}{I_e}\right)\right] = C.$$

$$I = h^{-1}(C) = I_e P_{\log}[\exp(\zeta_0 C)],$$

$P_{log}(x) = w^{-1}(x)$ is the inverse function of $w(x) = xe^x$

Zero diffusion

$$F(I,t) = \left(1 + \frac{I_e}{I}\right) \cdot \frac{P_{\log}[\exp(\zeta_0 C)] F_0\{I_e P_{\log}[\exp(\zeta_0 C)]\}}{1 + P_{\log}[\exp(\zeta_0 C)]}.$$

$$F_0(I) = \frac{N}{2\pi I_{\text{ion}}} \exp\left(-\frac{I}{I_{\text{ion}}}\right),$$

$$C = t + \frac{1}{\zeta_0} \ln\left[\frac{I}{I_e} \exp\left(\frac{I}{I_e}\right)\right].$$

$$F(I,t) = \frac{N}{2\pi I_{\text{ion}}} g\left(\frac{I}{I_e}\right),$$

$$g(\eta,t) = \left(1 + \frac{1}{\eta}\right) \frac{P_{\log}[\eta \exp(\zeta_0 t + \eta)]}{1 + P_{\log}[\eta \exp(\zeta_0 t + \eta)]} \times \exp\left(-\frac{I_e}{I_{\text{ion}}} P_{\log}[\eta \exp(\zeta_0 t + \eta)]\right).$$

Zero diffusion

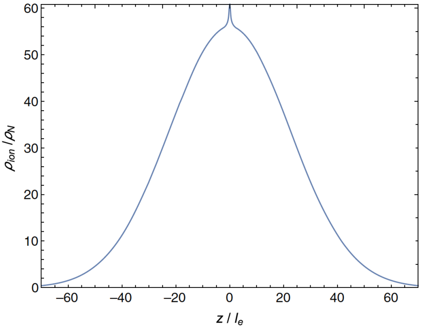
$$\rho_{\text{ion}}(t,z) = k_{rf} \int_{-\infty}^{\infty} F\left(\frac{1}{2}k_{rf}^2 z^2 + \frac{1}{2}P^2, t\right) dP$$

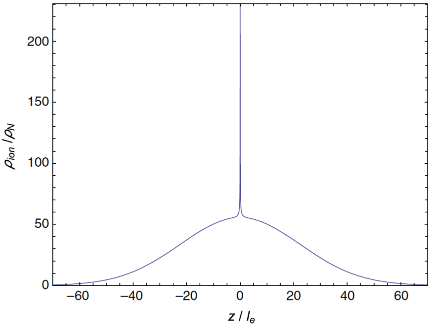
$$= \rho_N \int_{-\infty}^{\infty} g\left(\frac{z^2}{l_e^2} + y^2, t\right) dy,$$

$$k_{rf} = 2\pi/\lambda_{rf}$$

$$\rho_N \equiv \frac{N}{2\pi} \frac{l_e}{\sigma_i^2}.$$

Zero diffusion





Ion profile after $t = 2\zeta_0^{-1}$ (left) and $t = 10\zeta_0^{-1}$ (right).

Finite diffusion, numerical solution

$$
\begin{aligned}
\bar{D}(r) &= D(r^2 I_e)/D(0) \\
\bar{\zeta} &= \zeta(r^2 I_e)\zeta_0 \\
\bar{D}_0 &= D(0)/(\zeta_0 I_e) \\
\bar{t} &= t\zeta_0 \\
R(r,\bar{t}) &= \frac{2\pi I_{ion}}{N} F(r^2 I_e, \bar{t}\zeta_0^{-1}) \\
r &= \sqrt{I/I_e}
\end{aligned}
$$

Finite diffusion, numerical solution

$$r\frac{\partial R(r,\bar{t})}{\partial \bar{t}} + \alpha(r)\frac{\partial R(r,\bar{t})}{\partial r} + \beta(r)\frac{\partial^2 R(r,\bar{t})}{\partial r^2}$$
$$+\gamma(r)R(r,\bar{t}) = 0,$$

$$\alpha(r) = -\frac{r^2}{2}\bar{\zeta}(r) - \frac{\bar{D}_0}{4}\bar{D}(r) - \frac{\bar{D}_0 r}{4}\frac{d\bar{D}(r)}{dr},$$

$$\beta(r) = -\frac{\bar{D}_0 r}{4}\bar{D}(r),$$

$$\gamma(r) = -\frac{r^2}{2}\frac{d\bar{\zeta}(r)}{dr} - r\bar{\zeta}(r).$$

Finite diffusion, numerical solution

$2 \le j < N$:

$$\frac{\beta_j}{\Delta r^2} R_{j-1}^{n+1} + \left(\frac{r_j}{\Delta t} - \frac{\alpha_j}{\Delta r} - 2\frac{\beta_j}{\Delta r^2} + \gamma_j \right) R_j^{n+1} + \left(\frac{\alpha_j}{\Delta r} + \frac{\beta_j}{\Delta r^2} \right) R_{j+1}^{n+1} = \frac{r_j}{\Delta t} R_j^n$$

$j = 1$:

$$-\frac{\alpha_j}{\Delta r} R_1^{n+1} + \frac{\alpha_j}{\Delta r} R_2^{n+1} = 0$$

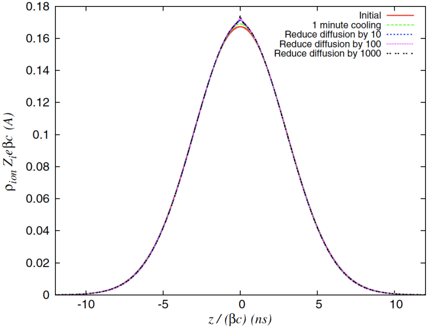
$j = N$:

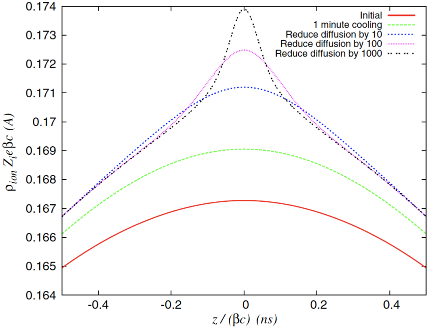
$$\frac{\beta_N}{\Delta r^2} R_{N-1}^{n+1} + \left(\gamma_N + \frac{r_N}{\Delta t} - \frac{\alpha_N}{\Delta r} - 2\frac{\beta_N}{\Delta r^2} \right) R_N^{n+1} = \frac{r_N}{\Delta t} R_N^n$$

Finite diffusion, numerical solution

$$\rho(z,t) = \rho_N \int_{-\infty}^{\infty} R\left(\sqrt{y^2 + \frac{z^2}{l_e^2}}, t\zeta_0\right) dy.$$

FEL-based CeC





Ion profile after 1 min of cooling.

Backup Slides

- V. N. Litvinenko, and Y. S. Derbenev. "Coherent electron cooling." Physical Review Letters 102.11 (2009): 114801.
- G. Wang, and M. Blaskiewicz. "Dynamics of ion shielding in an anisotropic electron plasma." Physical Review E 78.2 (2008): 026413.
- G. Wang, V. N. Litvinenko, and M. Blaskiewicz. "Energy Modulation in Coherent Electron Cooling." Proceedings of IPAC (2013).
- J. Ma, et al. "Simulation studies of modulator for coherent electron cooling." Physical Review Accelerators and Beams 21.11 (2018): 111001.
- V. N. Litvinenko, et al. "Plasma-cascade instability." Physical Review Accelerators and Beams 24.1 (2021): 014402.
- G. Stupakov, and P. Baxevanis. "Microbunched electron cooling with amplification cascades." Physical Review Accelerators and Beams 22.3 (2019): 034401.
- G. Wang. "Evolution of ion bunch profile in the presence of longitudinal coherent electron cooling." Physical Review Accelerators and Beams 22.11 (2019): 111002.

In the frame of the moving ion, the velocity distribution of the electrons is

$$f_0(\vec{v}) = n_0 \delta^3(\vec{v} + \vec{v}_0). \quad (1)$$

Inserting eq. (1) into the definition of $g(\vec{u})$ in lecture slide 3 leads to

$$g(\vec{u}) \equiv \frac{1}{n_0} \int f_0(\vec{v}) e^{-i\vec{u}\cdot\vec{v}} d^3v = e^{i\vec{u}\cdot\vec{v}_0}, \quad (2)$$

and hence the integral equation for the electron density modulation becomes

$$\tilde{n}_1(\vec{k},t) = \omega_p^2 \int_0^t \left[\tilde{n}_1(\vec{k},t_1) - Z_i \right] (t_1 - t) e^{i\vec{k}(t-t_1)\cdot\vec{v}_0} dt_1. \quad (3)$$

Multiplying both sides of eq. (3) by $e^{-i\vec{k}\cdot\vec{v}_0 t}$ yields

$$\tilde{H}_1\left(\vec{k},t\right)=\omega_p^2\int_0^t\left[\tilde{H}_1\left(\vec{k},t_1\right)-Z_i e^{-i\vec{k}\cdot\vec{v}_0 t_1}\right]\left(t_1-t\right)dt_1, \tag{5}$$

with

$$\tilde{H}_1\left(\vec{k},t\right)\equiv\tilde{n}_1\left(\vec{k},t\right)e^{-i\vec{k}\cdot\vec{v}_0 t}. \tag{6}$$

Taking the time derivative of eq. (5) gives

$$\frac{d}{dt}\tilde{H}_1\left(\vec{k},t\right)=-\omega_p^2\int_0^t\left[\tilde{H}_1\left(\vec{k},t_1\right)-Z_i e^{-i\vec{k}\cdot\vec{v}_0 t_1}\right]dt_1, \tag{7}$$

and taking the time derivative of eq. (7) leads to

$$\frac{d^2}{dt^2}\tilde{H}_1\left(\vec{k},t\right)+\omega_p^2\tilde{H}_1\left(\vec{k},t\right)=\omega_p^2 Z_i e^{-i\vec{k}\cdot\vec{v}_0 t}. \tag{8}$$

Assuming the inhomogeneous solution of eq. (8) takes the form

$$\tilde{H}_{1,inh}\left(\vec{k},t\right)=C\cdot e^{-i\vec{k}\cdot\vec{v}_0 t}, \tag{9}$$

and inserting it into eq. (8) gives

$$C=\frac{\omega_p^2 Z_i}{\omega_p^2-\left(\vec{k}\cdot\vec{v}_0\right)^2}. \tag{10}$$

Thus, the solution of eq. (8) reads

$$\tilde{H}_1\left(\vec{k},t\right)=A\cos\left(\omega_p t\right)+B\sin\left(\omega_p t\right)+\frac{\omega_p^2 Z_i}{\omega_p^2-\left(\vec{k}\cdot\vec{v}_0\right)^2}e^{-i\vec{k}\cdot\vec{v}_0 t}. \tag{11}$$

Applying eq. (6) into eq. (11) yields

$$\tilde{n}_1\left(\vec{k},t\right)=Ae^{i\vec{k}\cdot\vec{v}_0 t}\cos\left(\omega_p t\right)+Be^{i\vec{k}\cdot\vec{v}_0 t}\sin\left(\omega_p t\right)+\frac{\omega_p^2 Z_i}{\omega_p^2-\left(\vec{k}\cdot\vec{v}_0\right)^2}. \tag{12}$$

Applying initial condition $\tilde{n}_1\left(\vec{k},0\right)=0$ and $\frac{d}{dt}\tilde{n}_1\left(\vec{k},0\right)=0$ yields

$$A=-\frac{\omega_p^2 Z_i}{\omega_p^2-\left(\vec{k}\cdot\vec{v}_0\right)^2}, \tag{13}$$

and

$$B=\frac{i\vec{k}\cdot\vec{v}_0\omega_p Z_i}{\omega_p^2-\left(\vec{k}\cdot\vec{v}_0\right)^2}. \tag{14}$$

Inserting eq. (13) and eq. (14) into eq. (12) leads to

$$\tilde{n}_1\left(\vec{k},t\right)=\frac{\omega_p^2 Z_i}{\omega_p^2-\left(\vec{k}\cdot\vec{v}_0\right)^2}\left[1-e^{i\vec{k}\cdot\vec{v}_0 t}\left(\cos\left(\omega_p t\right)-\frac{i\vec{k}\cdot\vec{v}_0}{\omega_p}\sin\left(\omega_p t\right)\right)\right]. \tag{15}$$

Taking time derivative of eq. (15) yields

$$\frac{d}{dt}\tilde{n}_1\left(\vec{k},t\right)=\dot{\tilde{n}}_1\left(\vec{k},t\right)=\omega_p Z_i e^{i\vec{k}\cdot\vec{v}_0 t}\sin\left(\omega_p t\right). \tag{16}$$

Taking 3-D inverse Fourier transformation of eq. (16) gives

$$\dot{\tilde{n}}_1\left(\vec{x},t\right)=\omega_p Z_i\sin\left(\omega_p t\right)\frac{1}{\left(2\pi\right)^3}\int_{-\infty}^{\infty}e^{i\vec{k}\cdot\vec{v}_0 t}e^{i\vec{k}\cdot\vec{x}}d^3k=Z_i\omega_p\sin\left(\omega_p t\right)\delta^3\left(\vec{x}+\vec{v}_0 t\right). \tag{17}$$

Integrating eq. (17) yields the electron density perturbation in the frame of the moving ion

$$\tilde{n}_1\left(\vec{x},t\right)=Z_i\omega_p\int_0^t\sin\left(\omega_p t_1\right)\delta^3\left(\vec{x}+\vec{v}_0 t_1\right)dt_1. \tag{18}$$

Longitudinal equation of motion

In the presence of the radiation field, the longitudinal equation of motion of an electron read

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \cos(\psi) \qquad \psi = k_w z + k(z - ct)$$

$\mathcal{E}_0$ is the average energy of the beam.

$$\frac{d}{dz}\psi = k_w + k - \frac{\omega}{v_z(\mathcal{E})}$$

$$\approx k_w + k - \omega\left[\frac{1}{v_z(\mathcal{E}_0)} + (\mathcal{E} - \mathcal{E}_0)\frac{d}{d\mathcal{E}}\frac{1}{v_z}\right] \Leftarrow$$

$$\approx k_w + k - \frac{\omega}{v_z(\mathcal{E}_0)} + \frac{\omega}{\gamma_z^2 c}\frac{(\mathcal{E} - \mathcal{E}_0)}{\mathcal{E}_0}$$

$$\frac{d}{d\mathcal{E}}\frac{1}{v_z} = \frac{1}{mc^3}\frac{d}{d\gamma}\frac{1}{\beta_z} = \frac{1}{mc^3}\frac{d\gamma_z}{d\gamma}\frac{d}{d\gamma_z}\frac{1}{\beta_z}$$

$$\gamma_z^2 = \frac{\gamma^2}{(1 + K^2)} \qquad \frac{d\gamma_z}{d\gamma} = \frac{\gamma}{\gamma_z(1 + K^2)}$$

$$\frac{d}{d\gamma_z}\frac{1}{\beta_z} = -\frac{1}{2\beta_z^3}\frac{d}{d\gamma_z}\left(1 - \frac{1}{\gamma_z^2}\right) = -\frac{1}{\beta_z^3\gamma_z^3}$$

$$\Rightarrow \begin{cases} \dfrac{dP}{dz} = -eE\theta_s \cos(\psi) \\ \dfrac{d}{dz}\psi \approx C + \dfrac{\omega}{\gamma_z^2 c\mathcal{E}_0} P \end{cases}$$

Energy deviation: $P \equiv \mathcal{E} - \mathcal{E}_0$

Detuning parameter: $C \equiv k_w + k - \dfrac{\omega}{v_z(\mathcal{E}_0)}$

Low Gain Regime: Pendulum Equation

$$\left.\begin{aligned}\frac{dP}{dz} &= -eE\theta_s\cos(\psi)\\ \frac{d}{dz}\psi &= C + \frac{\omega}{\gamma_z^2 c\mathcal{E}_0}P\end{aligned}\right\} \Rightarrow \qquad \frac{d^2}{dz^2}\psi + \frac{eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0}\cos(\psi) = 0$$

We assume that the change of the amplitude of the radiation field, E, is negligible and treat it as a constant over the whole interaction.

$$\frac{d^2}{d\hat{z}^2}\psi + \hat{u}\cos(\psi) = 0 \qquad \hat{u} = \frac{l_w^2 eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0} \qquad \hat{z} = \frac{z}{l_w}$$

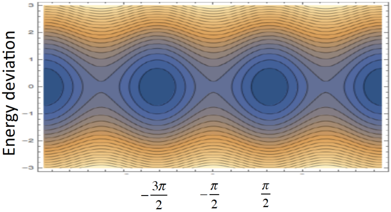
Pendulum equation:

$$\boxed{\frac{d^2}{d\hat{z}^2}\left(\psi + \frac{\pi}{2}\right) + \hat{u}\sin\left(\psi + \frac{\pi}{2}\right) = 0}$$

Low Gain Regime: Similarity to Synchrotron Oscillation

FEL

ψ is the angle between the transverse velocity vector and the radiation field vector and hence there is no energy kick for $\psi = \pi/2$



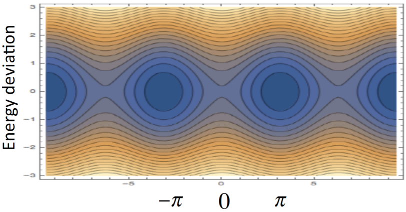
Pondermotive phase, ψ

$$\frac{d^2}{d\hat{z}^2}\left(\psi + \frac{\pi}{2}\right) + \hat{u}\sin\left(\psi + \frac{\pi}{2}\right) = 0$$

$$\hat{u} = \frac{l_w^2 eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0} \qquad \psi = k_u z + k(z - ct)$$

Synchrotron Oscillation

$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau; \quad \frac{d\pi_\tau}{ds} = \frac{1}{C}\frac{eV_{RF}}{p_o c}\sin(k_o h_{rf}\tau);$$

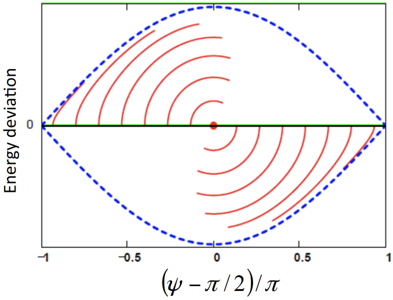


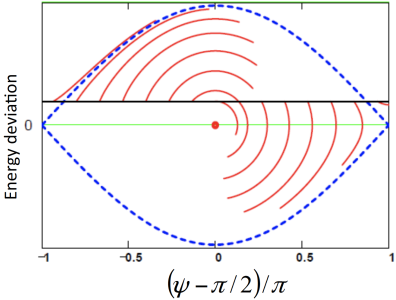
RF phase, ϕ_{rf}

$$\frac{d^2\phi_{rf}}{ds^2} = u_{rf}\sin\phi_{rf}$$

$$u_{rf} = \eta\frac{1}{C}\frac{eV_{RF}k_0 h_{rf}}{p_0 c} \qquad \phi_{rf} = k_0 h_{rf}\tau$$

Low Gain Regime: Qualitative Observation





The average energy of the electrons is right at resonant energy:

$$\lambda_0 \approx \frac{\lambda_w \left(1 + K^2\right)}{2\gamma^2} \Rightarrow \gamma = \gamma_0 = \sqrt{\frac{\lambda_w \left(1 + K^2\right)}{2\lambda_0}}$$

*Plots are taken from talk slides by Peter Schmuser.

The average energy of the electrons is slightly above the resonant energy:

$$\gamma = \gamma_0 + \Delta\gamma$$

With positive detuning, there is net energy loss by electrons.

Low Gain Regime: Derivation of FEL Gain

Change in radiation power density (energy gain per seconds per unit area):

$$\Delta\Pi_r = c\varepsilon_0\left(E_{ext} + \Delta E\right)^2 - c\varepsilon_0 E_{ext}^2 \approx 2c\varepsilon_0 E_{ext}\Delta E$$

Average change rate in electrons' energy per unit beam area:

$$\Delta\Pi_e = \frac{j_0\langle P\rangle}{e}$$

*The average, <…>, is over all electrons in the beam.

Energy deviation at entrance

Pondermotive phase at entrance

$$\langle P(z)\rangle = \int_{-\infty}^{\infty} dP_0 \int_0^{2\pi} d\psi_0 f(P_0,\psi_0) P(P_0,\psi_0,z)$$

Assuming radiation has the same cross section area as the electron beam, we obtain the change in electric field amplitude:

$$\Delta\Pi_r + \Delta\Pi_e = 0 \Rightarrow \boxed{\Delta E = -\frac{j_0\langle P\rangle}{2c\varepsilon_0 E_{ext} e}}$$

$$\left.\begin{aligned} \frac{dP}{dz} &= -eE\theta_s\cos(\psi) \\ \frac{d}{dz}\psi &= C + \frac{\omega}{\gamma_z^2 c\varepsilon_0}P \end{aligned}\right\} \Rightarrow \langle P\rangle = -eE\theta_s\left\langle \int_0^1 \cos[\psi(\hat{z})]d\hat{z} \right\rangle$$

Low Gain Regime: Derivation of FEL Gain

$$\frac{d^2}{d\hat{z}^2}\psi + \hat{u}\cos\psi = 0$$

$$\psi(\hat{z}) = \psi(0) + \psi'(0)\hat{z} - \hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\psi(\hat{z}_2) d\hat{z}_2 \qquad (1)$$

Assuming that all electrons have the same energy and uniformly distributed in the Pondermotive phase at the entrance of FEL: $P_0 = 0$ and $f(\psi_0) = \frac{1}{2\pi}$.

The zeroth order solution for phase evolution is given by ignoring the effects from FEL interaction:

$$\left.\begin{array}{l} \frac{dP}{dz} = -eE\theta_s \cos(\psi) \\ \frac{d}{dz}\psi = C + \frac{\omega}{\gamma_z^2 c \mathcal{E}_0} P \end{array}\right\} \Rightarrow \frac{d}{d\hat{z}}\psi = \hat{C} \Rightarrow \left\{\begin{array}{l} \psi(\hat{z}) = \psi_0 + \hat{C}\hat{z} \\ \psi'(0) = \hat{C} \end{array}\right. \qquad \hat{C} \equiv C l_w$$

Inserting the zeroth order solution back into eq. (1) yields the 1st order solution:

$$\psi(\hat{z}) = \psi_0 + \hat{C}\hat{z} + \Delta\psi(\psi_0, \hat{z}) \qquad \Delta\psi(\psi_0, \hat{z}) \equiv -\hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\left[\psi_0 + \hat{C}\hat{z}_2\right] d\hat{z}_2$$

Low Energy Regime: Derivation of FEL Gain

$$\Delta\psi(\psi_0,\hat{z}) \equiv -\hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\left[\psi_0 + \hat{C}\hat{z}_2\right] d\hat{z}_2$$

$$= -\frac{\hat{u}}{\hat{C}^2}\left\{\int_0^{\hat{C}\hat{z}} \sin(\psi_0 + x_1)dx_1 - \hat{C}\hat{z}\sin\psi_0\right\} = \frac{\hat{u}}{\hat{C}^2}\left[\cos(\psi_0 + \hat{C}\hat{z}) - \cos\psi_0 + \hat{C}\hat{z}\sin\psi_0\right]$$

$$\langle P\rangle = -eEl_w\theta_s\left\langle\int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z} + \Delta\psi(\psi_0,\hat{z})\right]d\hat{z}\right\rangle$$

← Average energy loss of electrons

$$= eE\theta_s l_w\left\langle\int_0^1 \sin\left[\psi_0 + \hat{C}\hat{z}\right]\sin(\Delta\psi(\psi_0,\hat{z}))d\hat{z}\right\rangle - eE\theta_s l_w\left\langle\int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z}\right]\cos(\Delta\psi(\psi_0,\hat{z}))d\hat{z}\right\rangle$$

$$\approx eE\theta_s l_w\left\langle\int_0^1 \Delta\psi(\psi_0,\hat{z})\sin\left[\psi_0 + \hat{C}\hat{z}\right]d\hat{z}\right\rangle - \frac{eE\theta_s l_w}{2\pi}\int_0^1 d\hat{z}\int_0^{2\pi}\cos\left[\psi_0 + \hat{C}\hat{z}\right]d\psi_0$$

$$\left\langle\int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z}\right]\sin\left[\psi_0 + \hat{C}\hat{z}\right]d\hat{z}\right\rangle = 0$$

$$= \frac{eE\theta_s l_w}{2\pi}\frac{\hat{u}}{\hat{C}^2}\int_0^1 d\hat{z}\left\{\hat{C}\hat{z}\cos(\hat{C}\hat{z})\int_0^{2\pi}\sin^2\psi_0 d\psi_0 - \sin(\hat{C}\hat{z})\int_0^{2\pi}\cos^2\psi_0 d\psi_0\right\}$$

$$= -eE\theta_s l_w\frac{\hat{u}}{\hat{C}^3}\left(1 - \frac{\hat{C}}{2}\sin\hat{C} - \cos\hat{C}\right)$$

Low Energy Regime: Derivation of FEL Gain

Growth in the amplitude of radiation field:

$$\Delta E = -\frac{j_0\langle P\rangle}{2c\varepsilon_0 E_{ext} e} = \frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2\gamma}\frac{l_w^3 E_{ext}}{I_A}\frac{2}{\hat{C}^3}\left(1-\frac{\hat{C}}{2}\sin\hat{C}-\cos\hat{C}\right)$$

$$\hat{u} = \frac{l_w^2 e E_{ext}\theta_s\omega}{\gamma_z^2 c\gamma mc^2}$$

$$I_A = \frac{4\pi\varepsilon_0 mc^3}{e}$$

The gain is defined as the relative growth in radiation power:

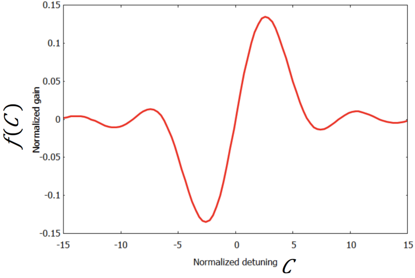
$$g_s = \frac{(E_{ext}+\Delta E)^2 - E_{ext}^2}{E_{ext}^2} \approx \frac{2\Delta E}{E_{ext}} = \tau\cdot f(\hat{C})$$

$$\tau \equiv \frac{2\pi j_0\theta_s^2\omega}{c\gamma_z^2\gamma}\frac{l_w^3}{I_A}$$

↗ Cubic in FEL length

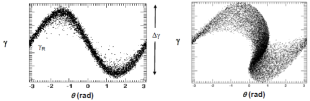
$$f(\hat{C}) = \frac{2}{\hat{C}^3}\left(1-\cos\hat{C}-\frac{\hat{C}}{2}\sin\hat{C}\right) = -2\frac{d}{d\hat{C}}\frac{\sin^2(\hat{C}/2)}{\hat{C}^2}$$

As observed earlier, there is no gain if the electrons has resonant energy.



High Gain Regime: Concept

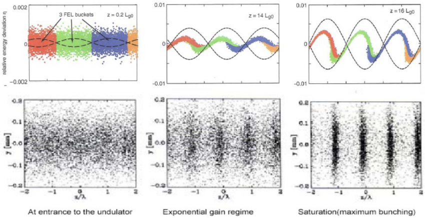
1. Energy kick from radiation field + dispersion/drift -> electron density bunching;

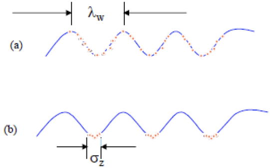


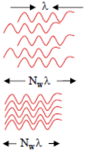
*The plots are for illustration only. The right plot actually shows somewhere close to saturation.

2. Electron density bunching makes more electrons radiates coherently -> higher radiation field;

3. Higher radiation fields leads to more density bunching through 1 and hence closes the positive feedback loop -> FEL instability.







$$|E| \propto \sqrt{N_e}$$
$$I_{incoherent} \propto N_e$$

$$|E| \propto N_e$$
$$I_{coherent} \propto N_e^2$$

The positive feedback loop between radiation field and electron density bunching is the underlying mechanism of high gain FEL regime.

1-D Model for cold beam without detuning

$$B(z) = \langle e^{-i\psi} \rangle = \sum_{j=1}^{N} e^{-i\psi_j} \qquad D(z) = \langle P e^{-i\psi} \rangle = \sum_{j=1}^{N} P_j e^{-i\psi_j}$$

$$\frac{d}{dz} B(z) = -i \left\langle e^{-i\psi} \frac{d}{dz}\psi \right\rangle = -i \frac{\omega}{c\gamma_z^2 \mathrm{E}_0} \langle e^{-i\psi} P \rangle = -i \frac{\omega}{c\gamma_z^2 \mathrm{E}_0} D(z)$$

$$\frac{dP}{dz} = -e\theta_s E(z)\cos(\psi)$$

$$\frac{d}{dz} D = \left\langle e^{-i\psi} \frac{d}{dz} P \right\rangle - i \left\langle e^{-i\psi} P \frac{d}{dz}\psi \right\rangle \approx \left\langle e^{-i\psi} \frac{d}{dz} P \right\rangle = -\langle e^{-i\psi} eE\theta_s \cos(\psi) \rangle \approx -\frac{1}{2} e\theta_s E$$

Wave Equation

$$\psi = k_w z + k(z - ct)$$

1-D theory and hence $\partial / \partial x = 0$ and $\partial / \partial y = 0$

Wave equation for transverse vector potential: $$\frac{\partial^2 \vec{A}_\perp}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \vec{A}_\perp}{\partial t^2} = -\mu_0 \vec{j}_\perp \qquad (1)$$

Transverse current perturbation: $$j_x + i j_y = \frac{1}{v_z}\left(v_x + i v_y\right) j_z = -\theta_s e^{-ik_w z} j_z \qquad (2)$$

We seek the solution for vector potential of the form:

$$A_{x,y}(z,t) = \widetilde{A}_{x,y}(z) e^{i\omega(z/c-t)} + \widetilde{A}_{x,y}{}^*(z) e^{-i\omega(z/c-t)} \qquad (3)$$

Inserting eq. (2) and (3) into eq. (1) yields

$$e^{i\omega(z/c-t)}\left\{\frac{2i\omega}{c}\frac{\partial}{\partial z}\begin{pmatrix}\widetilde{A}_x \\ \widetilde{A}_y\end{pmatrix} + \frac{\partial^2}{\partial z^2}\begin{pmatrix}\widetilde{A}_x \\ \widetilde{A}_y\end{pmatrix}\right\} + C.C. = -\mu_0 \theta_s \begin{pmatrix}\cos(k_w z) \\ -\sin(k_w z)\end{pmatrix} j_z$$

$$\left\{\frac{2i\omega}{c}\frac{\partial}{\partial z}\begin{pmatrix}\widetilde{A}_{tot,x} \\ \widetilde{A}_{tot,y}\end{pmatrix} + \frac{\partial^2}{\partial z^2}\begin{pmatrix}\widetilde{A}_{tot,x} \\ \widetilde{A}_{tot,y}\end{pmatrix}\right\} = -\frac{\mu_0 \theta_s}{2}\begin{pmatrix} e^{ik_w z} + e^{-ik_w z} \\ ie^{ik_w z} - ie^{-ik_w z}\end{pmatrix}\left\langle j_z e^{-i\psi}\right\rangle e^{ik_w z}$$

1. Ignoring fast oscillating term $\sim e^{2ik_w z}$

2. Ignoring second derivative by assuming that the variation of $\widetilde{A}_x{}'$ is negligible over the optical wave length.

Wave Equation

After neglecting the fast oscillation terms, we get the following relation between the current perturbation and the vector potential of the radiation field:

$$\frac{\partial}{\partial z}\tilde{A}_x = -\frac{c\mu_0\theta_s}{4i\omega}\left\langle j_z e^{-i\psi}\right\rangle \qquad \frac{\partial}{\partial z}\tilde{A}_y = \frac{\mu_0 c\theta_s}{4\omega}\left\langle j_z e^{-i\psi}\right\rangle$$

In order to relate the vector potential to the electric field, we use the Maxwell equation:

$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \nabla\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0 \Rightarrow \left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = \vec{\nabla}\varphi \Rightarrow E_{x,y} = -\frac{\partial A_{x,y}}{\partial t}$$

$$\Rightarrow Ee^{i\omega(z/c-t)} = E_x + iE_y = -\frac{\partial}{\partial t}\left[\left(\tilde{A}_x + i\tilde{A}_y\right)e^{i\omega(z/c-t)}\right]$$

$$\Rightarrow E = i\omega\left(\tilde{A}_x + i\tilde{A}_y\right)$$

Finally, the relation between the radiation field and the current modulation is obtained:

$$\boxed{\frac{d}{dz}E = i\omega\left(\frac{\partial}{\partial z}\tilde{A}_x + i\frac{\partial}{\partial z}\tilde{A}_y\right) = -\frac{c\mu_0\theta_s}{2}\left\langle j_z e^{-i\psi}\right\rangle = \frac{ec^2 n\mu_0\theta_s}{2}B}$$

$$\left\langle j_z e^{-i\psi}\right\rangle = -ec\sum_{k=1}^{N} e^{-i\psi_k} = -ecnB$$

1-D High Gain FEL Equation for Cold Beam and Zero Detuning

$$\frac{d}{dz}B(z) = -i\frac{\omega}{c\gamma_z^2 \mathrm{E}_0}D(z)$$

$$\frac{d}{dz}D = -\frac{1}{2}e\theta_s E$$

$$\frac{d}{dz}E = \frac{ec^2 n\mu_0\theta_s}{2}B$$

$$\Rightarrow \quad \frac{d^3}{d\hat{z}^3}E = iE$$

$\hat{z} \equiv \Gamma z$ is normalized longitudinal location along wiggler,

$\Gamma \equiv \left[\frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma I_A}\right]^{1/3}$ is the 1-D Gain rate parameter

$I_A = \frac{4\pi\varepsilon_0 mc^3}{e}$ is called Alfven current

$$E(\hat{z}) = \sum_{k=1}^{3} B_k e^{\lambda_k \hat{z}}$$

$\lambda^3 = i \Rightarrow$

$\lambda_1 = e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ ← Growing mode

$\lambda_2 = e^{i\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$ ← Damping mode

$\lambda_3 = e^{-i\frac{\pi}{2}} = -i$ ← Oscillating mode

1D Gain Length

- At high gain limit, the radiation field is given by

$$E(\hat{z}) \approx B_1 e^{\lambda_k \hat{z}} = B_1 \exp\left[\frac{\sqrt{3}}{2}\Gamma z\right] \exp\left[i\frac{1}{2}\Gamma z\right]$$

and the radiation power is A: cross section of the radiation field

$$P(\hat{z}) = \varepsilon_0 c \left|E(\hat{z})^2\right| A = \varepsilon_0 c |B_1|^2 \exp\left(\sqrt{3}\Gamma z\right) = \varepsilon_0 c |B_1|^2 A \exp\left(\frac{z}{L_G}\right)$$

and the 1-D power gain length is

$$L_G \equiv \frac{1}{\sqrt{3}\Gamma} = \frac{\lambda_w}{4\pi\sqrt{3}\rho}$$

Pierce Parameter

$$\rho \equiv \frac{\gamma_z^2 \Gamma c}{\omega} = \frac{\Gamma}{2k_w}$$

1-D amplitude gain length is $L_{GA} = 2L_G \equiv \frac{2}{\sqrt{3}\Gamma} = \frac{\lambda_w}{2\pi\sqrt{3}\rho}$

Solution for Cold Beam with Nonzero Detuning

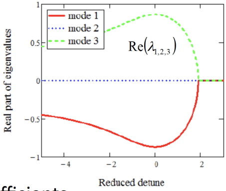
For non-vanishing detuning, the differential equation becomes

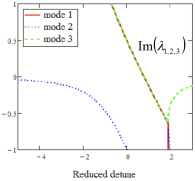
$$\frac{d^3}{d\hat{z}^3}E(\hat{z})+2i\hat{C}\frac{d^2}{d\hat{z}^2}E(\hat{z})-\hat{C}^2\frac{d}{d\hat{z}}E(\hat{z})=iE(\hat{z})$$

The general solution of the ODE reads:

$$E(\hat{z})=\sum_{k=1}^{3}B_k e^{\lambda_k \hat{z}}$$

$$\lambda^3+2i\hat{C}\lambda^2-\hat{C}^2\lambda=i$$





Applying initial condition to get the coefficients

$$\begin{pmatrix} E(0) \\ E'(0) \\ E''(0) \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}\Rightarrow \qquad \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}^{-1}\begin{pmatrix} E(0) \\ E'(0) \\ E''(0) \end{pmatrix}$$

For $E(0)=E_{ext}$ and $E'(0)=E''(0)=0$, the solution can be explicitly written as

$$E(\hat{z})=E_{ext}\left[\frac{\lambda_2\lambda_3 e^{\lambda_1\hat{z}}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{\lambda_1\lambda_3 e^{\lambda_2\hat{z}}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}+\frac{\lambda_1\lambda_2 e^{\lambda_3\hat{z}}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right]$$

Low Gain Limit of High Gain Solution

Can we reproduce the previously obtained low gain solution by taking the proper limit of the high gain solution?

$$g_l = \frac{(E_{ext}+\Delta E)^2 - E_{ext}^2}{E_{ext}^2} \approx \frac{2\Delta E}{E_{ext}} = \tau \cdot f(\hat{C}_l) = 2\Gamma^3 l_w^3 f_l(\hat{C}_l)$$

$$f_l(\hat{C}_l) = \frac{2}{\hat{C}_l^3}\left(1-\cos\hat{C}_l - \frac{\hat{C}_l}{2}\sin\hat{C}_l\right)$$

$$\tau \equiv \frac{2\pi j_0 \theta_s^2 \omega}{c\gamma_z^2\gamma}\frac{l_w^3}{I_A} = 2\Gamma^3 l_w^3$$

$$\hat{C}_l = C l_w$$

$$g_h(\hat{C}_l) = \frac{\tilde{E}^2 - E_{ext}^2}{E_{ext}^2} = \left|\frac{\lambda_2\lambda_3 e^{\lambda_1 \hat{l}_w}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2 \hat{l}_w}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3 \hat{l}_w}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right|^2 - 1$$

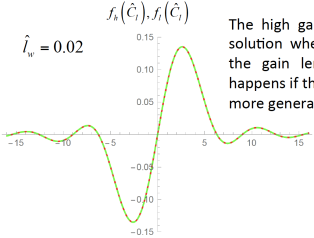
$$= 2\Gamma^3 l_w^3 f_h(\hat{C}_l) \qquad \hat{l}_w = l_w \Gamma$$

$$f_h(\hat{C}_l) = \frac{1}{2\hat{l}_w^3}\left\{\left|\frac{\lambda_2\lambda_3 e^{\lambda_1 \hat{l}_w}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2 \hat{l}_w}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3 \hat{l}_w}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right|^2 - 1\right\}$$

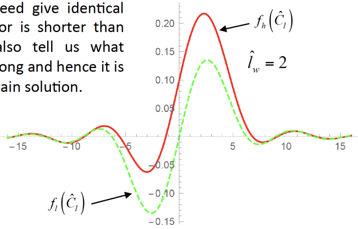
The normalization factor for high gain is different from that of low gain:

$$\hat{C}_h = C/\Gamma = C l_w / \hat{l}_w = \hat{C}_l / \hat{l}_w$$

$$\lambda^3 + 2i\frac{\hat{C}_l}{\hat{l}_w}\lambda^2 - \left(\frac{\hat{C}_l}{\hat{l}_w}\right)^2 \lambda = i$$



The high gain solution indeed give identical solution when the undulator is shorter than the gain length. But it also tell us what happens if the undulator is long and hence it is more general than the low gain solution.



PCI, general 3D case

$$X(t) = \mathbf{M}(t)X(0); \quad \mathbf{M}(0) = \mathbf{I}_{6\times 6}; \quad \mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{M}\mathbf{S}\mathbf{M}^T = \mathbf{S};$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{3\times 3} \\ -\mathbf{I}_{3\times 3} & \mathbf{0} \end{bmatrix} \rightarrow \det \mathbf{M} = 1; \quad \mathbf{M}^{-1} = -\mathbf{S}\mathbf{M}^T\mathbf{S};$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}; \quad \mathbf{M}^{-1} = -\mathbf{S}\mathbf{M}^T\mathbf{S} = \begin{bmatrix} \mathbf{D}^T & -\mathbf{B}^T \\ -\mathbf{C}^T & \mathbf{A}^T \end{bmatrix}, \quad (29)$$

$$X^T = \{x, y, z, P_x, P_y, P_z\},$$

PCI, general 3D case

$$\vec{k}_{\text{lab}} = \vec{k}_{\perp} + \hat{z}k_{//}; \qquad k_{\text{eff}} = \sqrt{\vec{k}_{\perp}^{\,2} + \frac{k_{//}^{\,2}}{\gamma^2}};$$

$$k_{\text{eff}}a_x \gg 1; \qquad k_{\text{eff}}a_y \gg 1; \qquad \gamma k_{\text{eff}}a_z \gg 1,$$

$$f(q,p,t) = F_o[q\cdot\mathbf{D}(t) - p\cdot\mathbf{B}(t), t] + \tilde{f}(q,p,t);$$

$$\tilde{\rho}_{\vec{k}}(t) = e\iint e^{-i\vec{k}\vec{q}}dq^3dp^3\tilde{f}(q,p,t);$$

$$q = (x,y,z); \qquad p = (P_x, P_y, P_z);$$

PCI, general 3D case

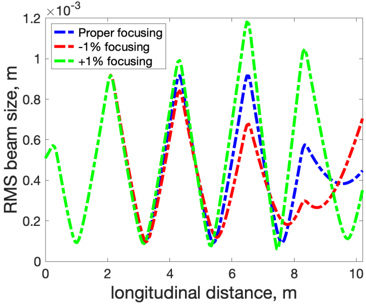
$$\tilde{\rho}_{\vec{k}}(t) = e\int e^{-i\vec{k}\cdot\overleftrightarrow{\mathbf{B}}(t)\cdot\vec{P}}\breve{f}_{\vec{k}(0)}(P,0)dP^3 - 4\pi e^2 n_o \beta_o(t)\int_o^t \frac{\tilde{\rho}_{\vec{k}(\tau)}(\tau)}{\det\mathbf{A}(\tau)}\frac{d\tau}{{\beta_o}^2(\tau)}\frac{\vec{k}\cdot\overleftrightarrow{\mathbf{U}}(t)\cdot\vec{k}-\vec{k}(\tau)\cdot\overleftrightarrow{\mathbf{U}}(\tau)\cdot\vec{k}(\tau)}{\gamma_o(\tau)^2[\vec{k}^2(\tau)-{k_z}^2(\tau){\beta_o}^2(\tau)]}$$

$$\times\int e^{i(\vec{k}(\tau)\cdot\overleftrightarrow{\mathbf{B}}(\tau)-\vec{k}\cdot\overleftrightarrow{\mathbf{B}}(t))\cdot\vec{P}}F_o(P)dP^3,$$

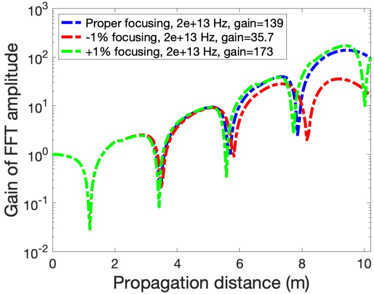
where $\mathbf{U} = \mathbf{U}^T = \mathbf{A}^{-1}\mathbf{B}^T$ and

$$k \equiv \{k_x, k_y, k_z\} \equiv \{k_1, k_2, k_3\}; \quad k(t) = k(0)\mathbf{A}^{-1}(t)$$

75 A, sensitivity study of focusing

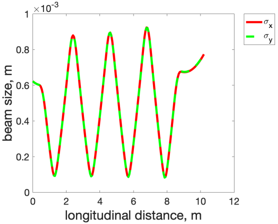


(a) Beam size

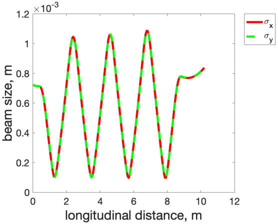


(b) PCA gain

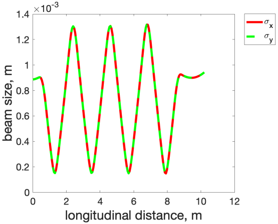
75 A, sensitivity study of $\varepsilon_{n,KV}$



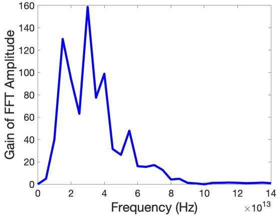
(a) $\varepsilon_{n,KV} = 7\mu m$



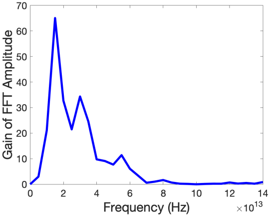
(b) $\varepsilon_{n,KV} = 10\mu m$



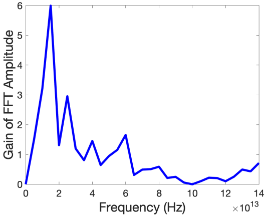
(c) $\varepsilon_{n,KV} = 20\mu m$



(d) $\varepsilon_{n,KV} = 7\mu m$

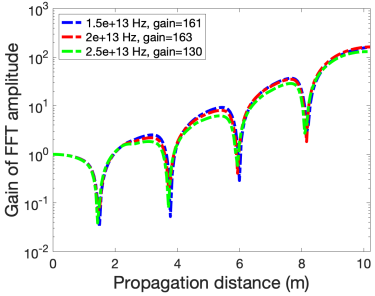


(e) $\varepsilon_{n,KV} = 10\mu m$

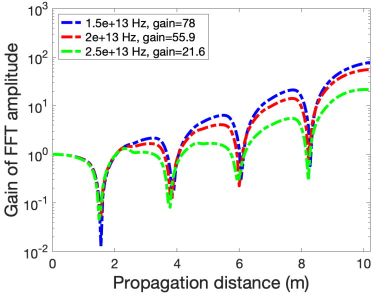


(f) $\varepsilon_{n,KV} = 20\mu m$

75 A, sensitivity study of $\varepsilon_{n,KV}$

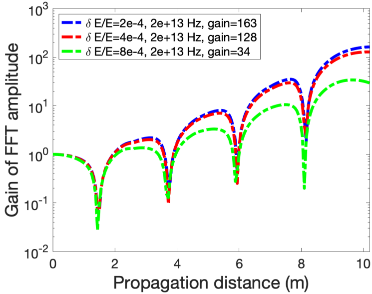


(a) $\varepsilon_{n,KV} = 7\mu m$

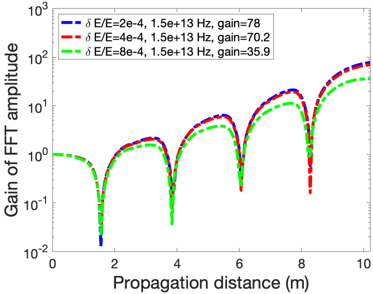


(b) $\varepsilon_{n,KV} = 10\mu m$

75 A, sensitivity study of $\delta E/E$

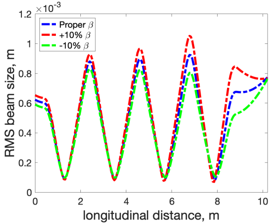


(a) $\varepsilon_{n,KV} = 7\mu m$

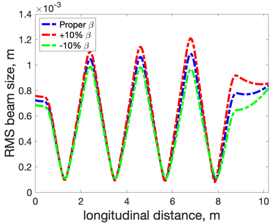


(b) $\varepsilon_{n,KV} = 10\mu m$

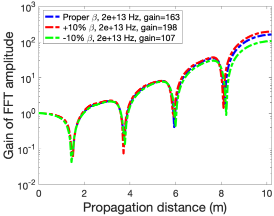
75 A, sensitivity study of β



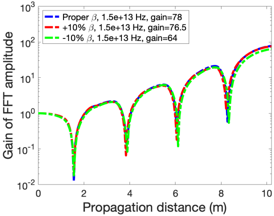
(a) $\varepsilon_{n,KV} = 7\mu m$



(b) $\varepsilon_{n,KV} = 10\mu m$

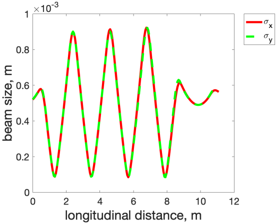


(c) $\varepsilon_{n,KV} = 7\mu m$

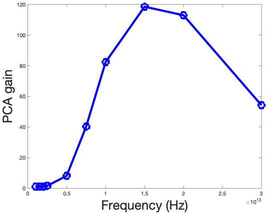


(d) $\varepsilon_{n,KV} = 10\mu m$

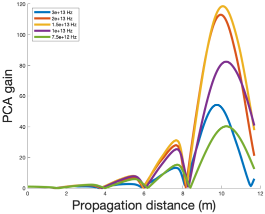
CeC PCA gain



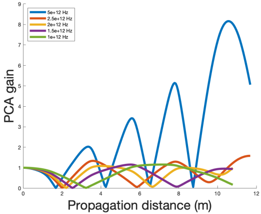
(a) Beam size



(b) PCA gain

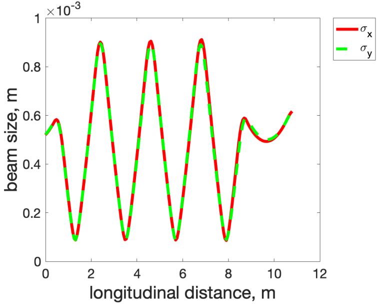


(c) PCA gain

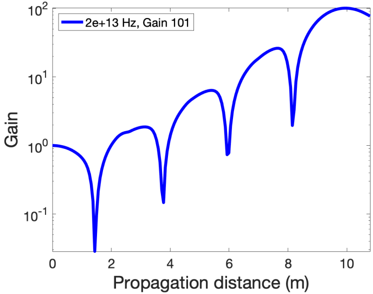


(d) PCA gain

Transverse symmetry

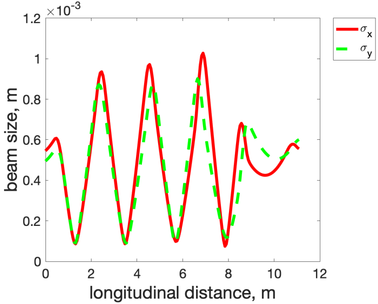


(a) Beam size

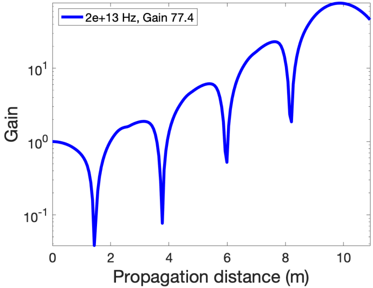


(b) PCA gain

ε_x+10%, ε_y-10%

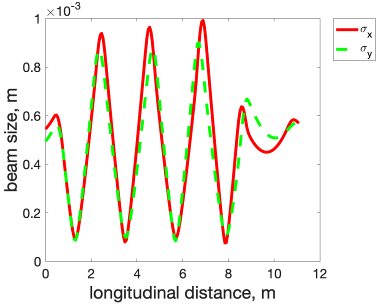


(a) Beam size

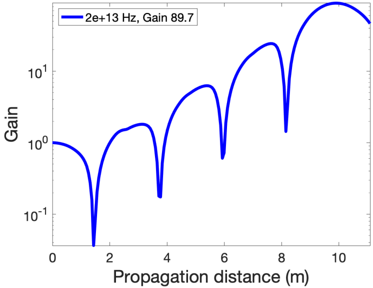


(b) PCA gain

β_x+10%, β_y-10%

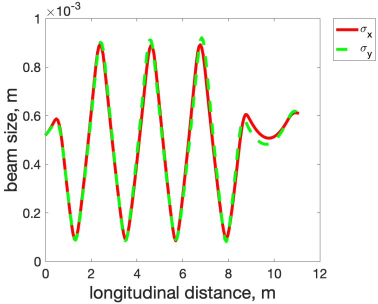


(a) Beam size

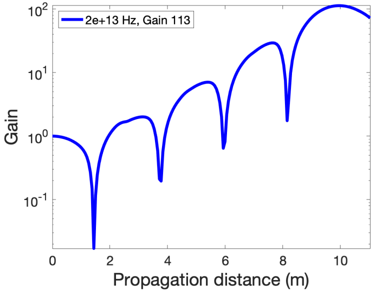


(b) PCA gain

α_x+10%, α_y-10%



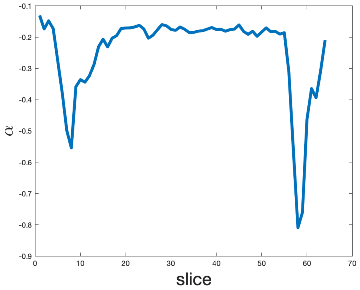
(a) Beam size



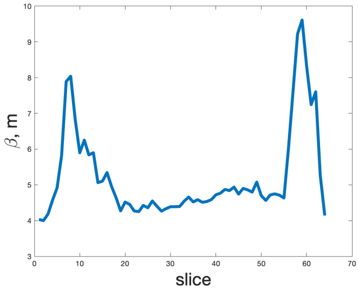
(b) PCA gain

PCA-based CeC

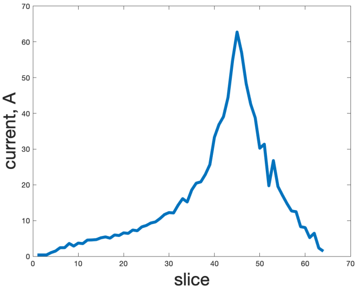
- Beam dynamics simulations propagate beam starting from the gun.
- Take slice parameters from beam dynamics simulations at the entrance of modulator.


α
slice
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
0
10
20
30
40
50
60
70

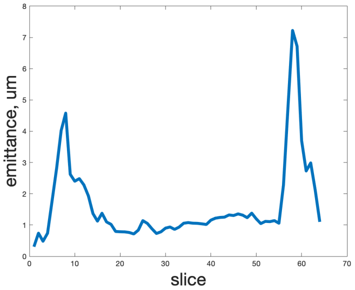
(a)


β, m
slice
10
9
8
7
6
5
4
3
0
10
20
30
40
50
60
70

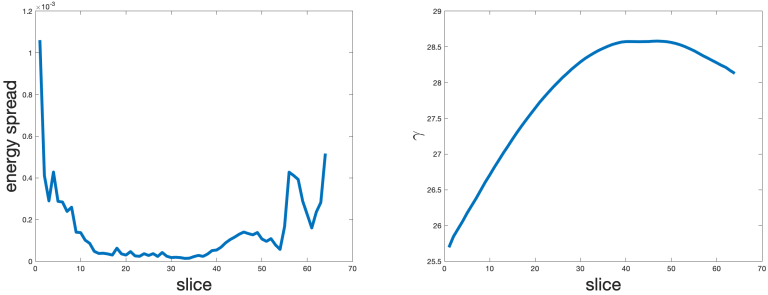
(b)



(a)

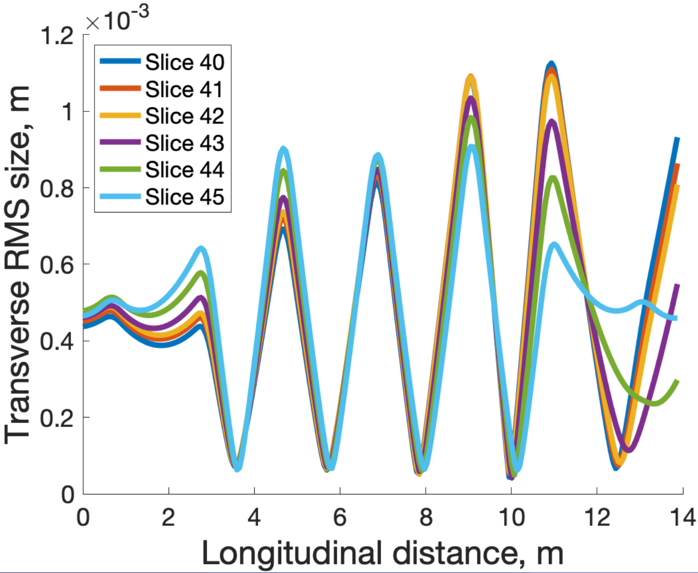


(b)



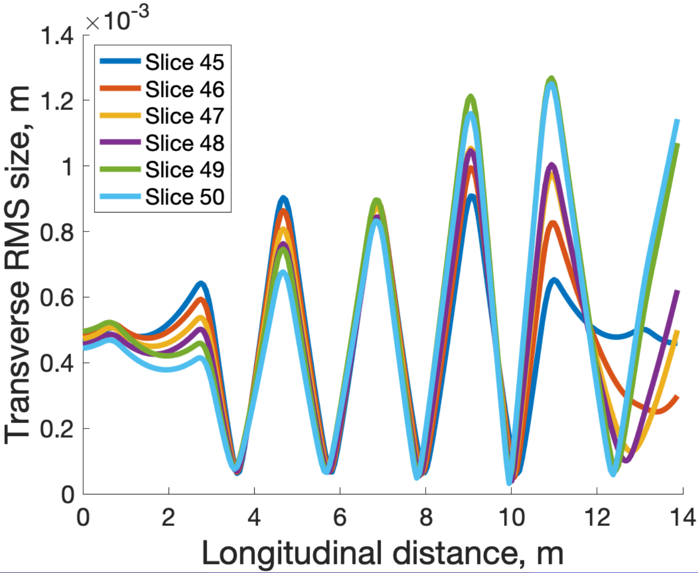
(a) (b)

Beam size



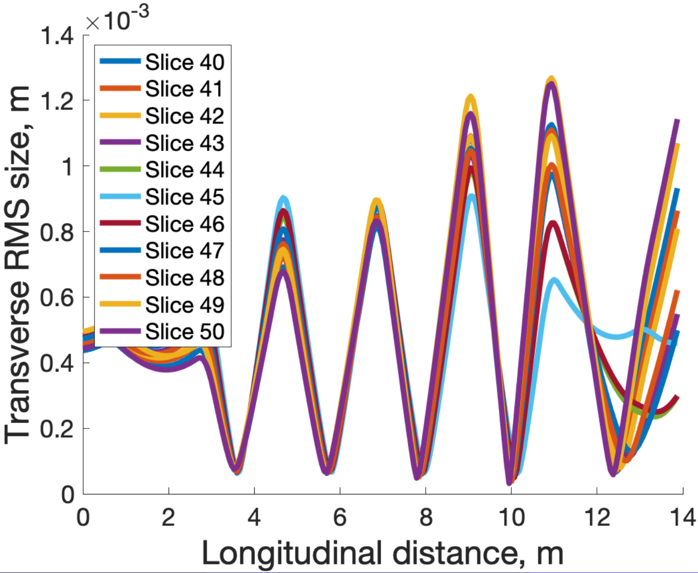
×10^-3
1.2
1
0.8
0.6
0.4
0.2
0
Transverse RMS size, m
0
2
4
6
8
10
12
14
Longitudinal distance, m
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45

Beam size

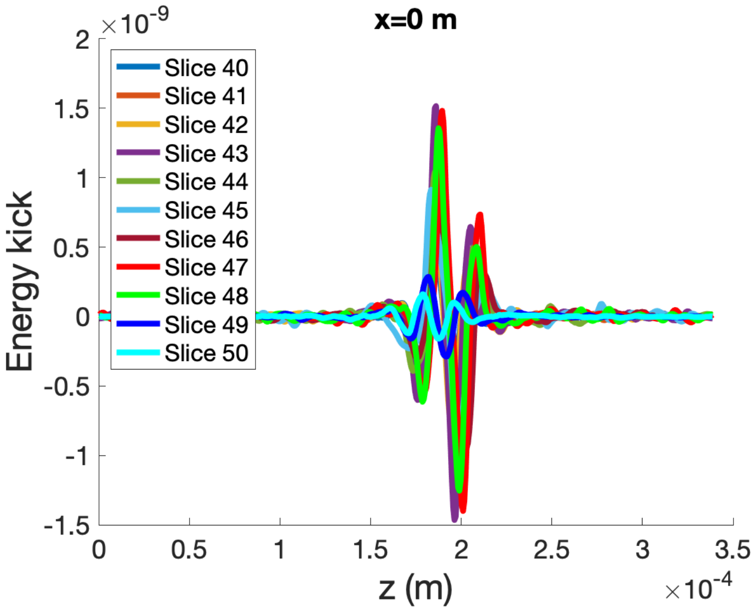


×10^-3
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Transverse RMS size, m
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
0
2
4
6
8
10
12
14
Longitudinal distance, m

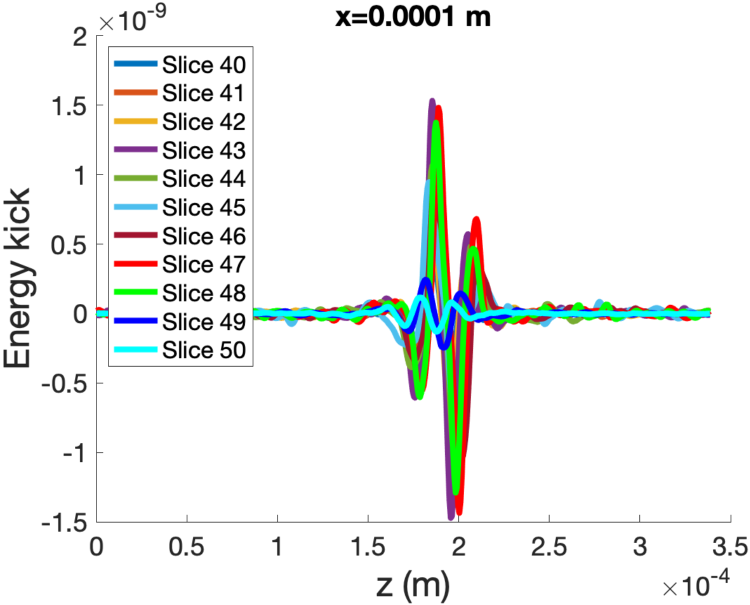
Beam size



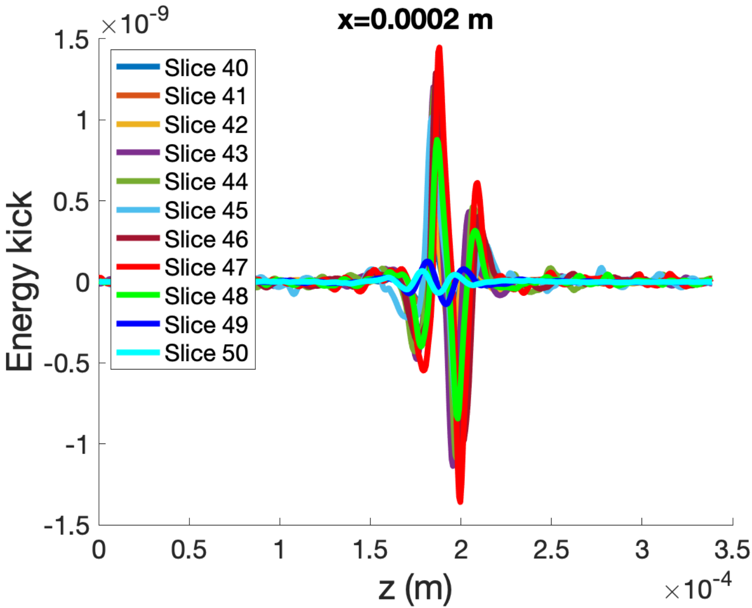
×10^-3
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Transverse RMS size, m
0
2
4
6
8
10
12
14
Longitudinal distance, m
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50


x=0 m
$\times 10^{-9}$
2
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
$\times 10^{-4}$

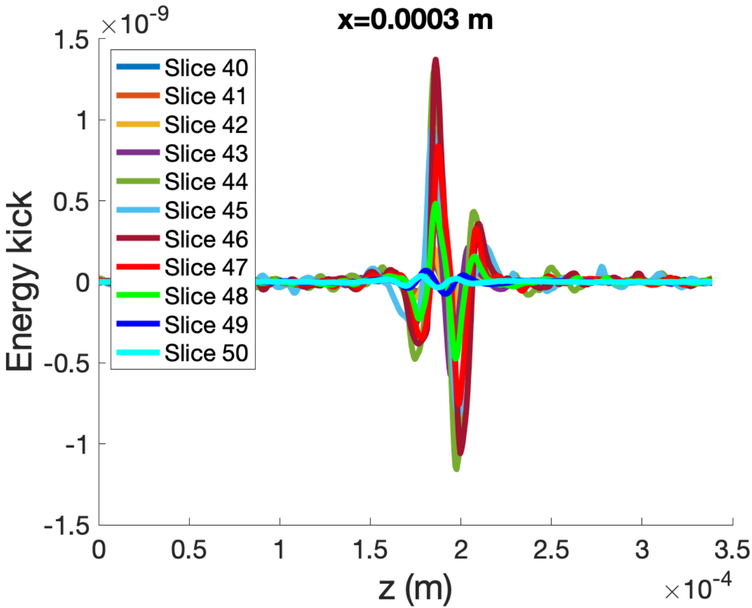
(a)


x=0.0001 m
×10^-9
2
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

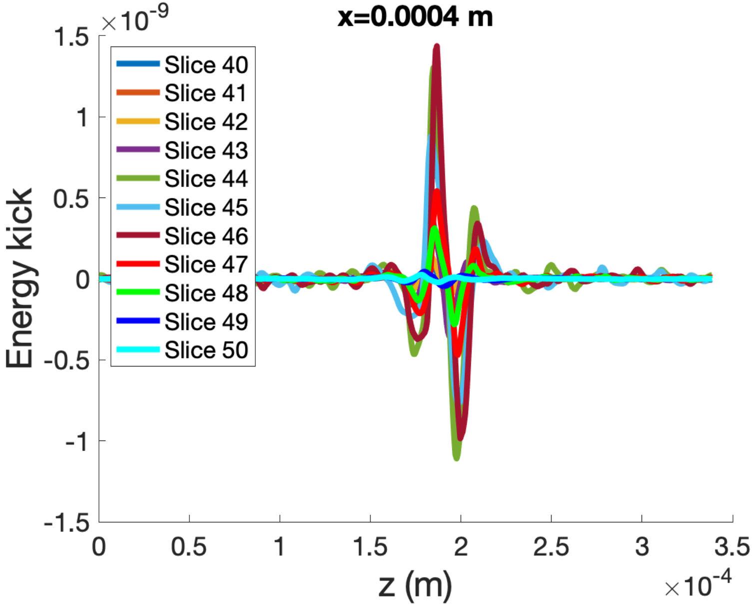
(a)


x=0.0002 m
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

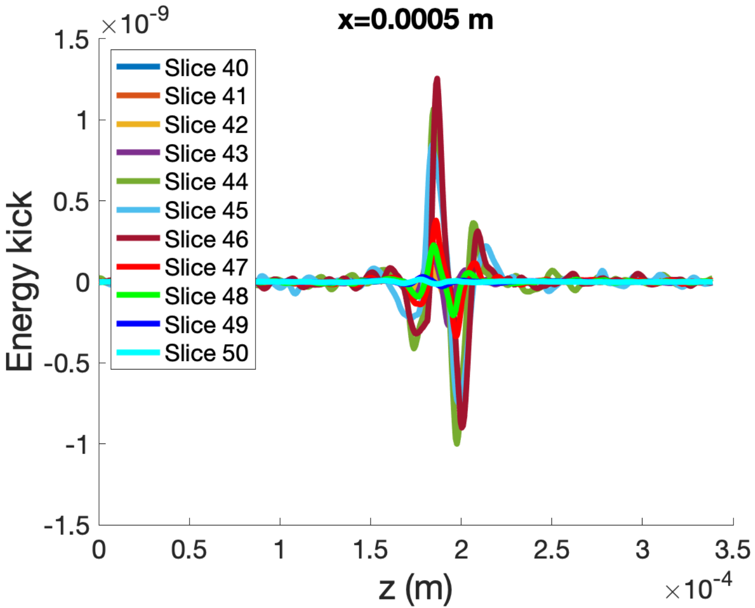
(a)


x=0.0003 m
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

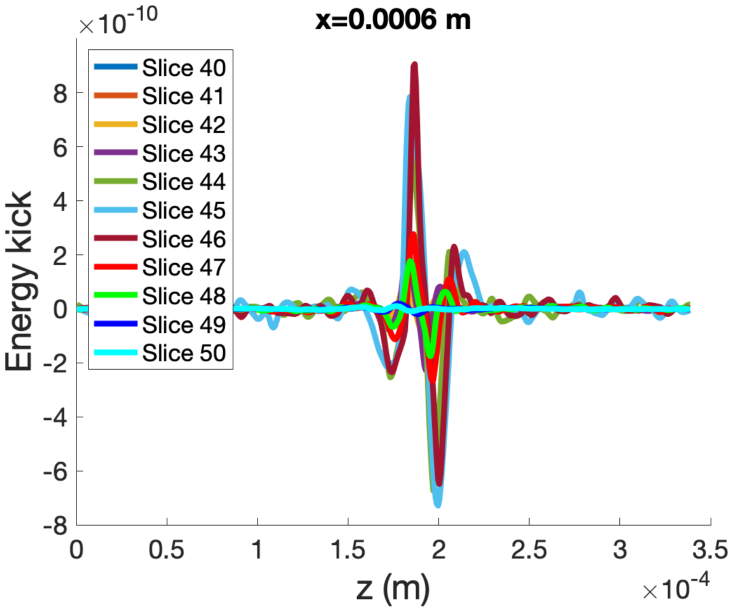
(a)


x=0.0004 m
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

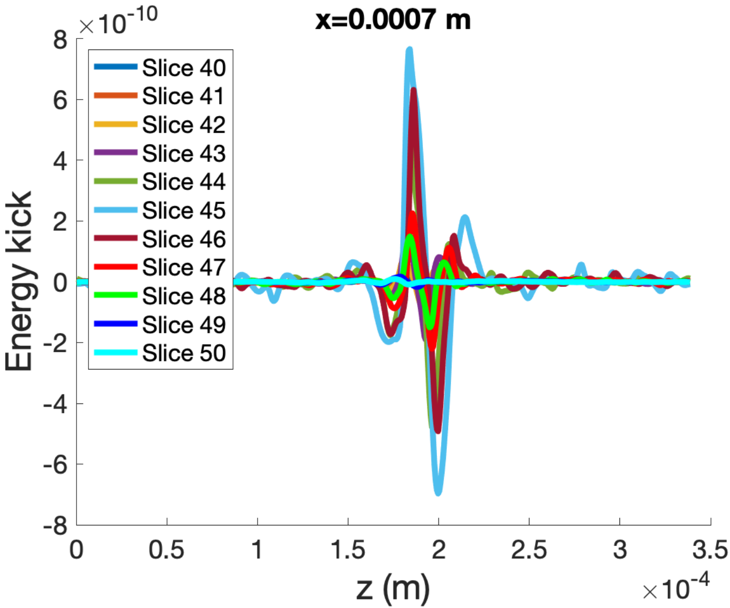
(a)


x=0.0005 m
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

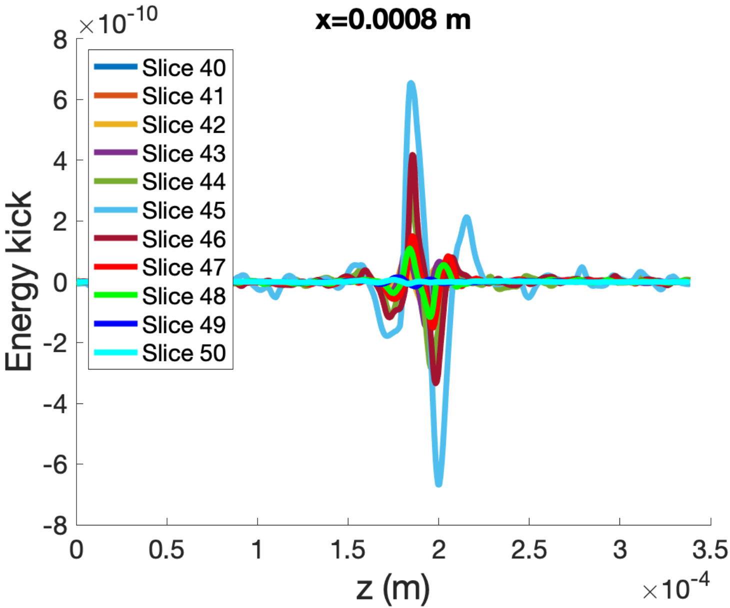
(a)


x=0.0006 m
×10^-10
Energy kick
8
6
4
2
0
-2
-4
-6
-8
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

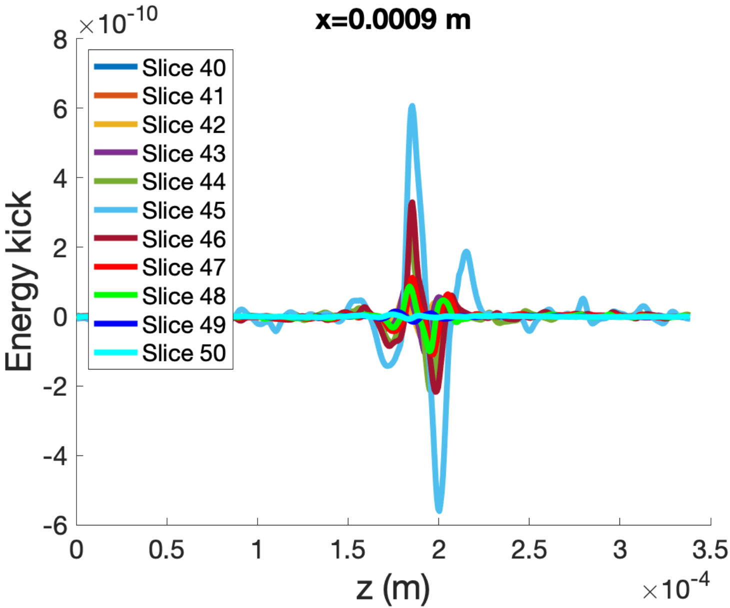
(a)


x=0.0007 m
$\times 10^{-10}$
8
6
4
2
0
-2
-4
-6
-8
Energy kick
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
$\times 10^{-4}$

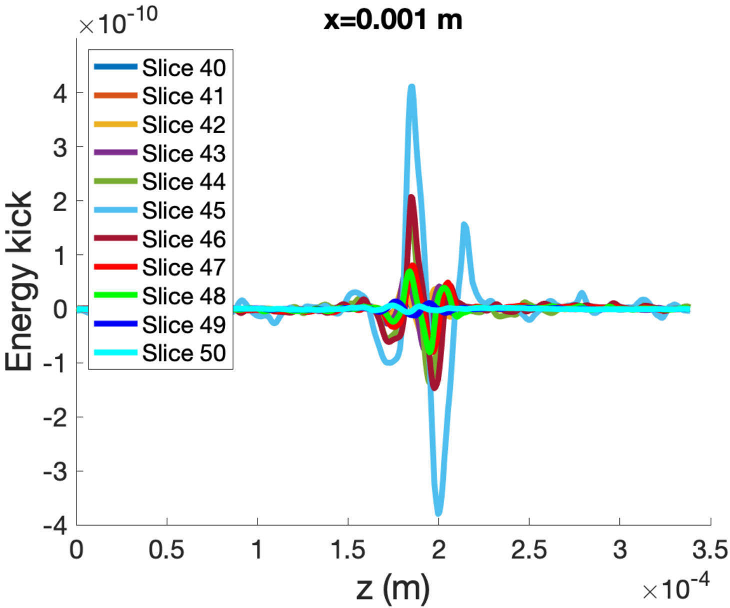
(a)


x=0.0008 m
×10^-10
8
6
4
2
0
-2
-4
-6
-8
Energy kick
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

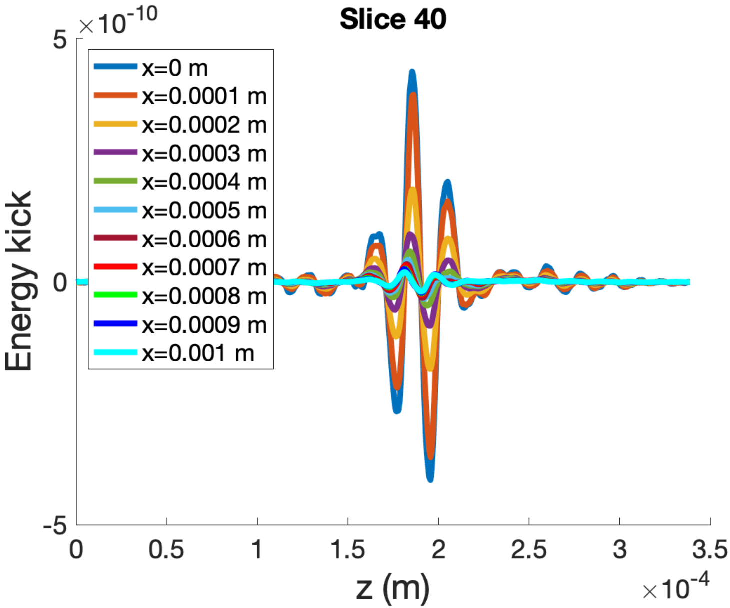
(a)


x=0.0009 m
$\times 10^{-10}$
Energy kick
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
z (m)
$\times 10^{-4}$

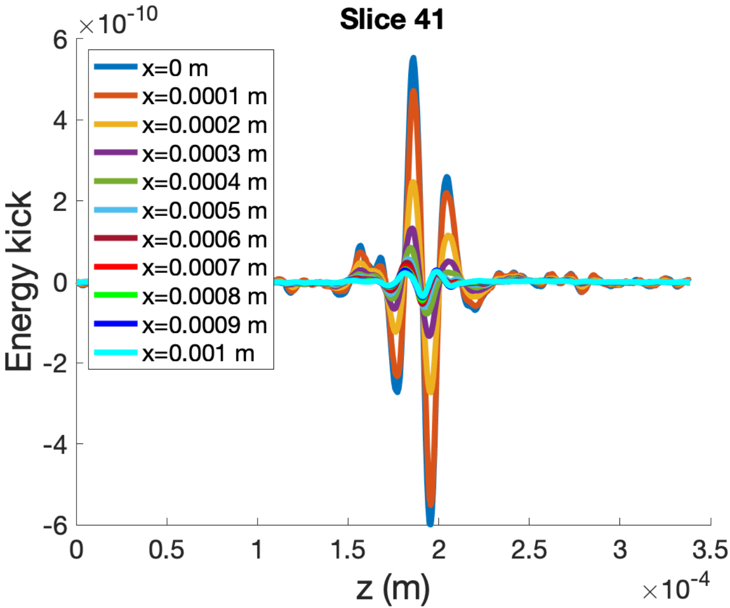
(a)


x=0.001 m
×10^-10
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
Energy kick
4
3
2
1
0
-1
-2
-3
-4
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

(a)

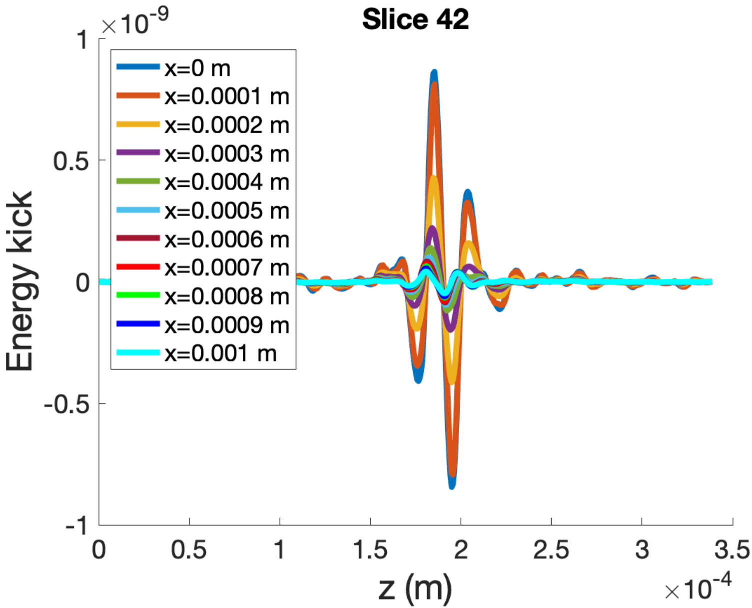

Slice 40
×10^-10
5
0
-5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

(a)

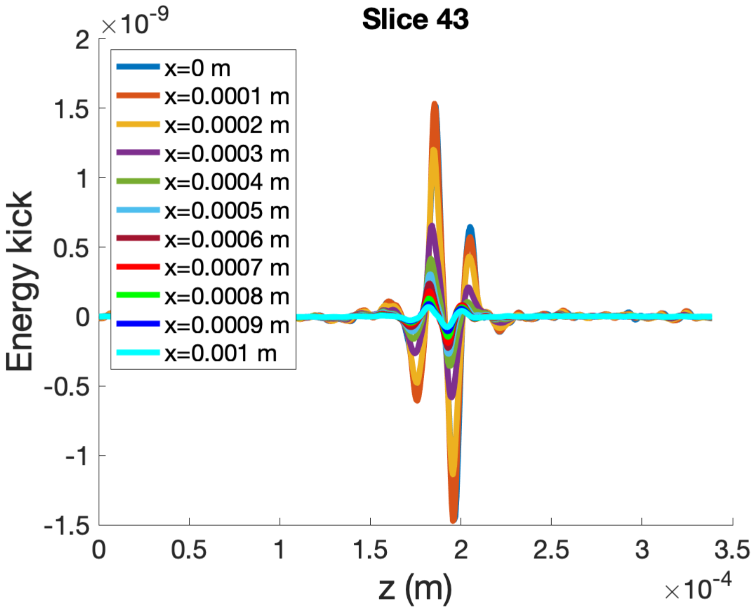


Slice 41
×10^-10
6
4
2
0
-2
-4
-6
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

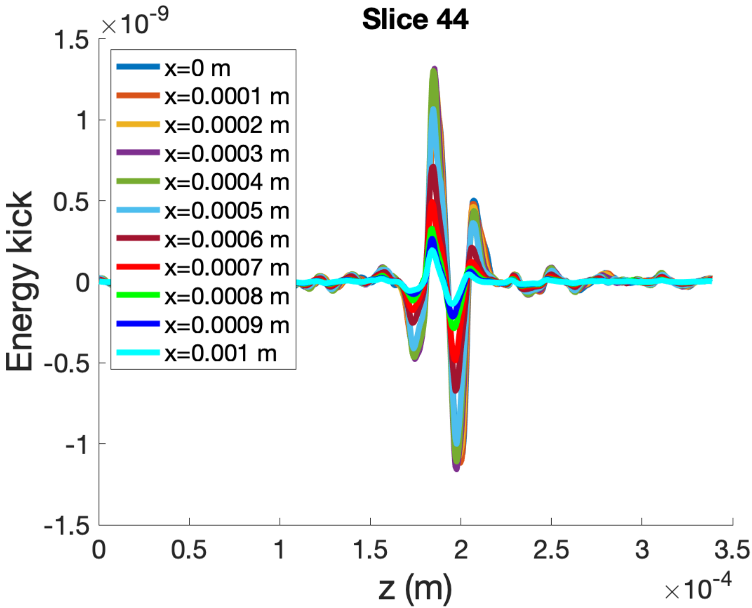
(a)


Slice 42
×10^-9
1
0.5
0
-0.5
-1
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

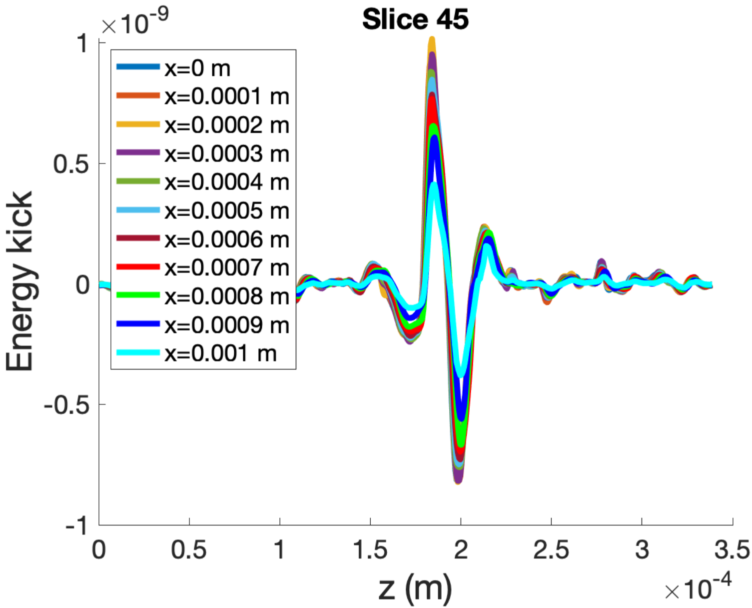
(a)


Slice 43
×10^-9
2
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

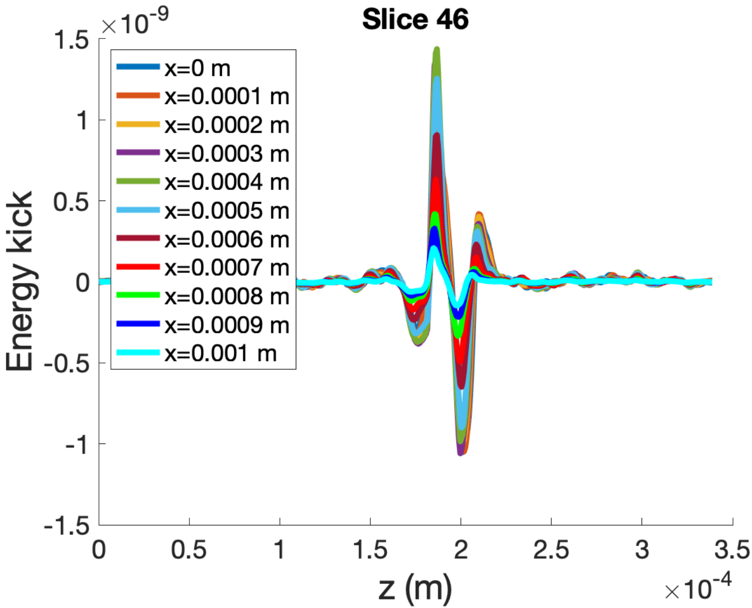
(a)


Slice 44
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

(a)

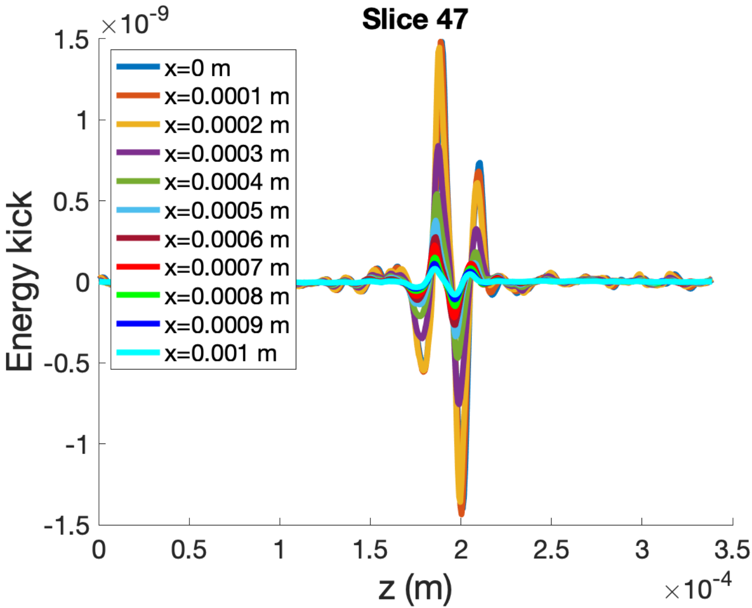

Slice 45
×10^-9
1
0.5
0
-0.5
-1
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

(a)



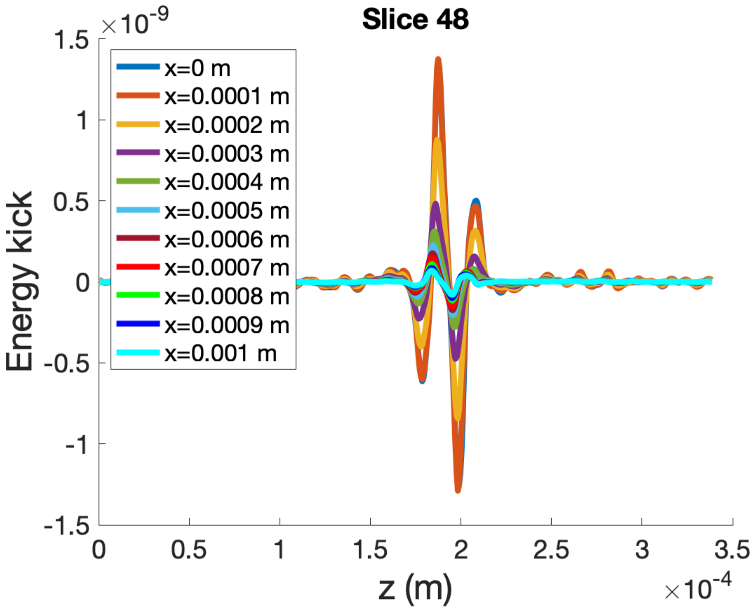
Slice 46
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

(a)

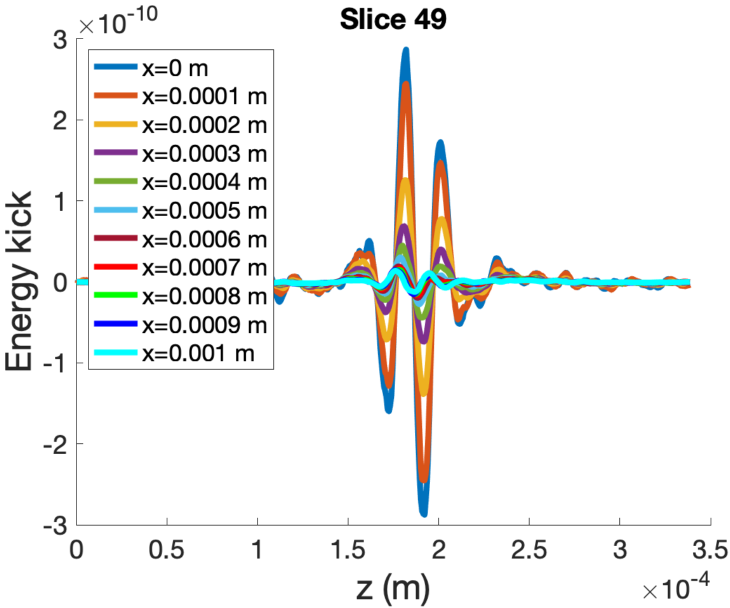


Slice 47
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

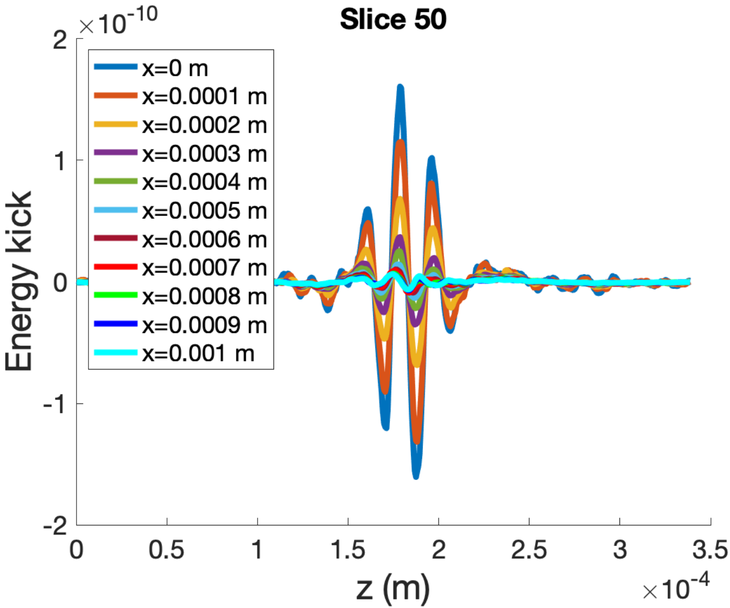
(a)


Slice 48
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

(a)

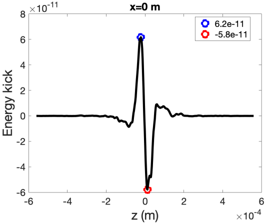

Slice 49
×10^-10
3
2
1
0
-1
-2
-3
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

(a)

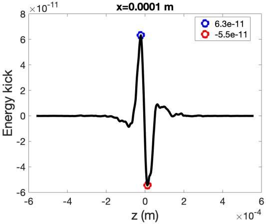

Slice 50
×10^-10
2
1
0
-1
-2
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

(a)

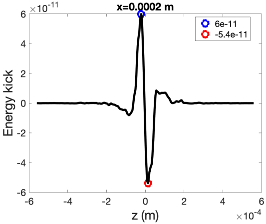
MBEC cooling force



(a)

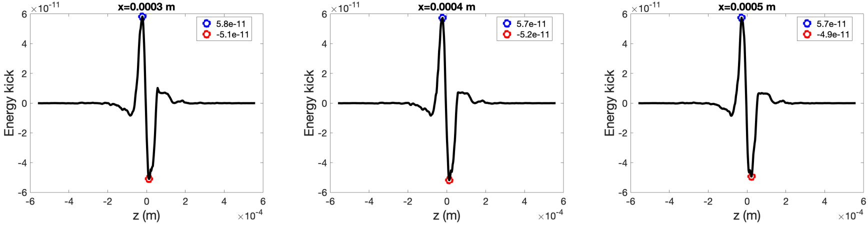


(b)



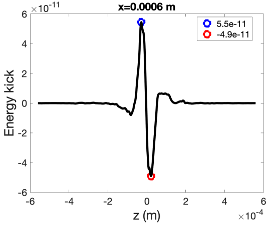
(c)

MBEC cooling force

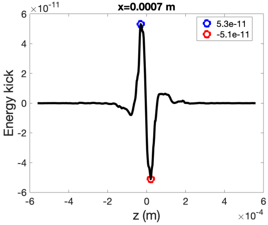


(a) (b) (c)

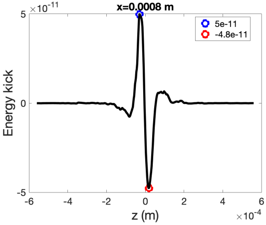
MBEC cooling force



(a)

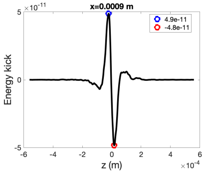


(b)

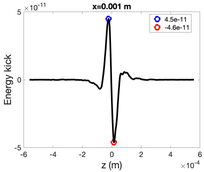


(c)

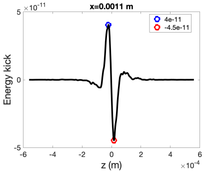
MBEC cooling force



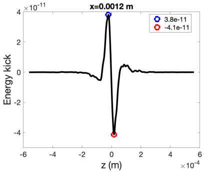
(a)



(b)



(c)



(d)