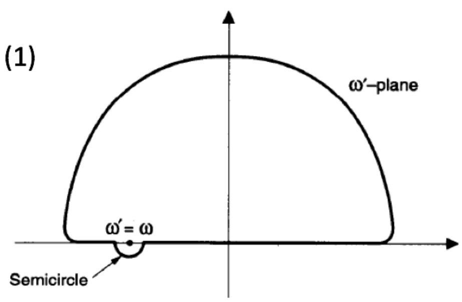
Homework 17. Due November 20

Problem 1. 20 points.

Perform a contour integral of $\frac{Z_{//}(\omega')}{\omega'-\omega}$ in the complex ω'-plane over the upper half plane along the contour shown in the figure. Show that if $Z_{//}(\omega')$ converges sufficiently fast as $|\omega'| \to \infty$

$$Z_{//}(\omega) = -\frac{i}{\pi} P.V.\int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' \quad (1)$$



Show that eq. (1) leads to Kramers-Kronig relations

$$\mathrm{Re}\left[Z_{//}(\omega)\right] = \frac{1}{\pi} P.V.\int_{-\infty}^{\infty} \frac{\mathrm{Im}\left[Z_{//}(\omega')\right]}{\omega'-\omega} d\omega'$$

$$\mathrm{Im}\left[Z_{//}(\omega)\right] = -\frac{1}{\pi} P.V.\int_{-\infty}^{\infty} \frac{\mathrm{Re}\left[Z_{//}(\omega')\right]}{\omega'-\omega} d\omega'$$

About Principal Value Integral:

The trick of P.V. is to utilize the property that the divergences on the side $\omega' < \omega$ and the side $\omega' > \omega$ are of opposite signs and, if the integration is taken *symmetrically* about the singularity so that the divergences on the two sides cancel each other, the integral is actually well defined. Algebraically, this leads to

$$\mathrm{P.V.}\int_{-\infty}^{\infty} dx \frac{f(x)}{x-a} = \int_{0}^{\infty} du \frac{f(a+u) - f(a-u)}{u}, \quad (2.95)$$

where the expression on the right is well behaved at $u = 0$.