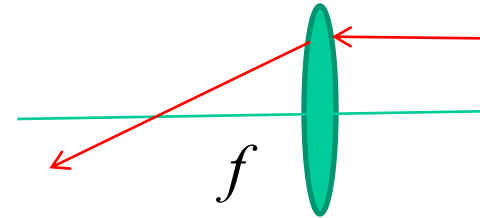


$$y'' + K(s)y = 0, \quad K_x(s) = \frac{1}{\rho^2} \mp \frac{1}{B\rho} \frac{\partial B_z}{\partial x}, \quad K_z(s) = \pm \frac{1}{B\rho} \frac{\partial B_z}{\partial x},$$

Thin lens approximation: Let $|K|\ell \rightarrow 1/f$ as $\ell \rightarrow 0$.

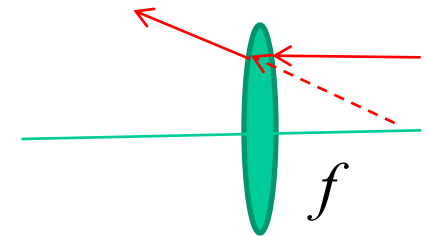
1. focusing quadrupole:

$$M(s, s_0) = \begin{pmatrix} \cos \sqrt{K} \ell & \frac{1}{\sqrt{K}} \sin \sqrt{K} \ell \\ -\sqrt{K} \sin \sqrt{K} \ell & \cos \sqrt{K} \ell \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$



2. de-focusing quadrupole:

$$M(s, s_0) = \begin{pmatrix} \cosh \sqrt{|K|} \ell & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K|} \ell \\ \sqrt{|K|} \sinh \sqrt{|K|} \ell & \cosh \sqrt{|K|} \ell \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$



3. Dipole: $K_x(s) = 1/\rho^2$. $M(s, s_0) = \begin{pmatrix} \cos \frac{\ell}{\rho} & \rho \sin \frac{\ell}{\rho} \\ -\frac{1}{\rho} \sin \frac{\ell}{\rho} & \cos \frac{\ell}{\rho} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$

4. Drift space: $K=0$ $M(s, s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$

The Floquet Theorem

We consider the linear Hill's equation of motion $y'' + K(s)y = 0$, where y and y' are conjugate phase space coordinates, $K(s)$ is the focusing function, and the prime is the derivative with respect to the independent variable s . In many accelerator applications, $K(s)$ is a periodic function of s with period L , i.e. $K(s+L)=K(s)$. It is advantageous to make the Floquet transformation and express the solution in amplitude and phase functions. The Floquet theorem states that the amplitude and phase functions satisfy a periodic boundary condition similar to that of the potential function $K(s)$, i.e.

$$y(s) = w(s)e^{j\psi(s)}, \quad w(s) = w(s+L), \quad \psi(s+L) - \psi(s) = 2\pi\mu,$$

where the phase advance μ in one period is independent of s . Although the periodic boundary condition is not necessary, it would simplify the solution of the differential equation. Using the Floquet transformation, we get the differential equation

$$2w'\psi' + w\psi'' = 0, \quad \psi' = \frac{1}{w^2}, \quad \psi = \int_{s_0}^s \frac{dt}{w^2},$$
$$w'' + K(s)w - w\psi'^2 = 0, \quad w'' + K(s)w - \frac{1}{w^3} = 0,$$

The Courant-Snyder parameterization

The most general representation of the matrix $\mathbf{M}(s)$ with **unit modulus** is given by the Courant-Snyder parameterization.

$$\mathbf{M}(s) = \begin{pmatrix} \cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\ -\gamma \sin \Phi & \cos \Phi - \alpha \sin \Phi \end{pmatrix} = I \cos \Phi + J \sin \Phi$$
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad J^2 = -I, \quad \text{or} \quad \beta\gamma = 1 + \alpha^2$$

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The ambiguity in the sign of $\sin$ can be resolved by requiring β to be a positive definite number if $|\text{Trace}(\mathbf{M})| \leq 2$, and by requiring $\text{Im}(\sin \Phi) > 0$ if $|\text{Trace}(\mathbf{M})| > 2$. The definition of the phase factor is still ambiguous up to an integral multiple of 2π . This ambiguity will be resolved when the matrix is tracked along the accelerator elements. Using the property of matrix J , we obtain the De Moivre's theorem:

$$\mathbf{M}^k = (I \cos \Phi + J \sin \Phi)^k = I \cos k\Phi + J \sin k\Phi,$$
$$\mathbf{M}^{-1} = I \cos \Phi - J \sin \Phi.$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1,$$

1. The evolution of the betatron amplitude function in a drift space is

$$\begin{aligned} \beta_2 &= \frac{1}{\gamma_1} + \gamma_1 \left(s - \frac{\alpha_1}{\gamma_1} \right)^2 = \beta^* + \frac{(s - s^*)^2}{\beta^*}, \\ \alpha_2 &= \alpha_1 - \gamma_1 s = -\frac{(s - s^*)}{\beta^*}, \quad \gamma_2 = \gamma_1 = \frac{1}{\beta^*} \end{aligned}$$

Note that γ is constant in a drift space, and $s^* = \alpha_1/\gamma_1$ is the location for an extremum of the betatron amplitude function with $\alpha(s^*) = 0$.

2. Passing through a thin-lens quadrupole, the evolution of betatron function is

$$\beta_2 = \beta_1, \quad \alpha_2 = \alpha_1 + \frac{\beta_1}{f}, \quad \gamma_2 = \gamma_1 + \frac{2\alpha_1}{f} + \frac{\beta_1}{f^2},$$

where f is the focal length of the quadrupole. Thus a thin-lens quadrupole gives rise to an angular kick to the betatron amplitude function without changing its magnitude.

Applying Floquet transformation to transport matrix

$$\boxed{y = aw(s)e^{j\psi(s)}} \quad y'' + K(s)y = 0, \quad w'' + K(s)w - \frac{1}{w^3} = 0 \quad \psi' = \frac{1}{w^2}$$

Floquet theorem: If $K(s)=K(s+L)$, we can choose the solution with the properties: $w(s)=w(s+L)$, and $\psi(s+L)=\psi(s)+\Phi$, where Φ is the phase advance in one period. The mapping matrix is

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = M(s, s_0) \begin{pmatrix} y(s_0) \\ y'(s_0) \end{pmatrix}$$

$$M(s_2, s_1) = \begin{pmatrix} \frac{w_2}{w_1} \cos \psi - w_2 w_1' \sin \psi & w_1 w_2 \sin \psi \\ -\frac{1+w_1 w_1' w_2 w_2'}{w_1 w_2} \sin \psi - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1}\right) \cos \psi & \frac{w_1}{w_2} \cos \psi + w_1 w_2' \sin \psi \end{pmatrix}$$

$$w_1 = w_2 = w, \quad w_1' = w_2' = w', \quad \psi(s_1 + L) - \psi(s_1) = \Phi.$$

$$M(s) = \begin{pmatrix} \cos \Phi - ww' \sin \Phi & w^2 \sin \Phi \\ -\gamma \sin \Phi & \cos \Phi + ww' \sin \Phi \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\ -\gamma \sin \Phi & \cos \Phi - \alpha \sin \Phi \end{pmatrix}$$

$$\beta(s) = w^2, \quad \alpha = -\frac{1}{2}\beta', \quad \gamma = \frac{1+\alpha^2}{\beta}, \quad w(s) = \sqrt{\beta(s)}, \quad \psi(s) = \int_{s_0}^s \frac{1}{\beta} ds$$

$$\frac{1}{2}\beta''' + K\beta - \frac{1}{\beta} \left[1 + \left(\frac{\beta'}{2}\right)^2 \right] = 0, \quad \text{or} \quad \alpha' = K\beta - \frac{1}{\beta} (1 + \alpha^2).$$

$$M(s_2, s_1) = \begin{pmatrix} \frac{w_2}{w_1} \cos \psi - w_2 w_1' \sin \psi & w_1 w_2 \sin \psi \\ -\frac{1+w_1 w_1' w_2 w_2'}{w_1 w_2} \sin \psi - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1}\right) \cos \psi & \frac{w_1}{w_2} \cos \psi + w_1 w_2' \sin \psi \end{pmatrix}$$

$w_1 = w(s_1), w_2 = w(s_2), \psi = \psi(s_2) - \psi(s_1), w_1' = w'(s_1), w_2' = w'(s_2),$

The transfer matrix from s_1 to s_2 in any beam transport line becomes

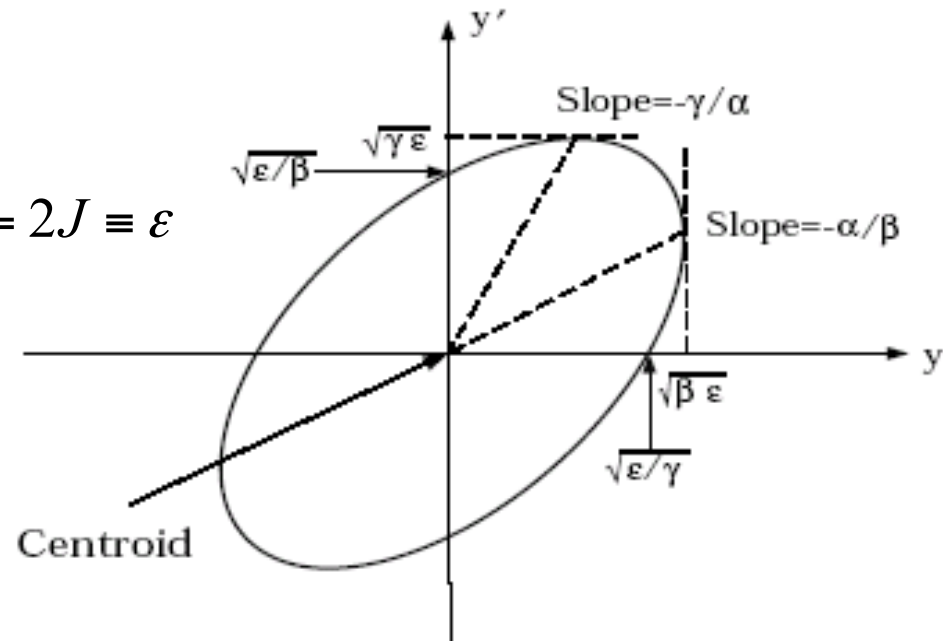
$$\begin{aligned} M(s_2|s_1) &= \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos \psi + \alpha_1 \sin \psi) & \sqrt{\beta_1 \beta_2} \sin \psi \\ -\frac{1+\alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \psi + \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}} \cos \psi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos \psi - \alpha_2 \sin \psi) \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix} \\ &\equiv \mathbf{B}(s_2) \begin{pmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix} \mathbf{B}^{-1}(s_1), \\ \mathbf{B}(s) &= \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha(s)}{\sqrt{\beta(s)}} & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix} \quad \text{and} \quad \mathbf{B}^{-1}(s) = \begin{pmatrix} \frac{1}{\sqrt{\beta(s)}} & 0 \\ \frac{\alpha(s)}{\sqrt{\beta(s)}} & \sqrt{\beta(s)} \end{pmatrix} \end{aligned}$$

In a typical tracking code, one can use Courant-Snyder parameters in transporting particle phase space coordinates (x, x') .

The phase advance in one revolution is $2\pi\nu$, where ν is the betatron tune.

Courant-Snyder Invariant

$$\gamma y^2 + 2\alpha y y' + \beta y'^2 = \frac{1}{\beta} [y^2 + (\alpha y + \beta y')^2] = 2J \equiv \varepsilon$$



Emittance of a beam

Given a normalized distribution function $\rho(y, y')$ with $\int \rho(y, y') dy dy' = 1$, the moments of the beam distribution are

$$\langle y \rangle = \int y \rho(y, y') dy dy', \quad \langle y' \rangle = \int y' \rho(y, y') dy dy',$$

$$\sigma_y^2 = \int (y - \langle y \rangle)^2 \rho(y, y') dy dy', \quad \sigma_{y'}^2 = \int (y' - \langle y' \rangle)^2 \rho(y, y') dy dy',$$

$$\sigma_{yy'} = \int (y - \langle y \rangle)(y' - \langle y' \rangle) \rho(y, y') dy dy' = r \sigma_y \sigma_{y'},$$

where σ_y and $\sigma_{y'}$ are the rms beam widths, $\sigma_{yy'}$ is the correlation, and r is the correlation coefficient. The rms beam emittance is then defined as

$$\epsilon_{\text{rms}} = \sqrt{\sigma_y^2 \sigma_{y'}^2 - \sigma_{yy'}^2} = \sigma_y \sigma_{y'} \sqrt{1 - r^2}.$$

The rms emittance is invariant in linear transport:

$$\epsilon^2 = \sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2$$

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2, \quad \sigma_{x'}^2 = \langle x'^2 \rangle - \langle x' \rangle^2 \quad \sigma_{xx'} = \langle xx' \rangle - \langle x \rangle \langle x' \rangle,$$

we find

$$\frac{d\sigma_x^2}{ds} = 2 \langle xx' \rangle - 2 \langle x \rangle \langle x' \rangle$$

$$\frac{d\sigma_{x'}^2}{ds} = 2 \langle x'x'' \rangle - 2 \langle x' \rangle \langle x'' \rangle$$

$$\frac{d\sigma_{xx'}}{ds} = \langle x'^2 \rangle - \langle x' \rangle^2 - \langle x \rangle \langle x'' \rangle + \langle xx'' \rangle$$

$$\begin{aligned} \frac{d\epsilon^2}{ds} &= \sigma_x^2 \frac{d\sigma_{x'}^2}{ds} + \sigma_{x'}^2 \frac{d\sigma_x^2}{ds} - 2\sigma_{xx'} \frac{d\sigma_{xx'}}{ds} & x'' &= -\frac{\partial H}{\partial x} \\ &= -2\sigma_x^2 \left(\left\langle x' \frac{\partial H}{\partial x} \right\rangle - \langle x' \rangle \left\langle \frac{\partial H}{\partial x} \right\rangle \right) + 2\sigma_{xx'} \left(\left\langle x \frac{\partial H}{\partial x} \right\rangle - \langle x \rangle \left\langle \frac{\partial H}{\partial x} \right\rangle \right) \end{aligned}$$

If $\partial H / \partial x = Kx$, then $\frac{d\epsilon^2}{ds} = -2\sigma_x^2 \sigma_{xx'} + 2\sigma_{xx'} \sigma_x^2 = 0$

The Gaussian distribution function

The equilibrium beam distribution in the linearized betatron phase space may be any function of the invariant action. However, the Gaussian distribution function

$$\rho(y, y') = \mathcal{N} \exp \left(\frac{-1}{2 \det \sigma} (\sigma_{22} y^2 - 2 \sigma_{12} y y' + \sigma_{11} y'^2) \right)$$

is commonly used to evaluate the beam properties. Expressing the normalized Gaussian distribution in the normalized phase space, we obtain

$$\rho(y, \mathcal{P}_y) = \frac{1}{2\pi\sigma_y^2} e^{-(y^2 + \mathcal{P}_y^2)/2\sigma_y^2}$$

where $\langle y^2 \rangle = \langle p_y^2 \rangle = \sigma_y^2 = \beta_y \epsilon_{\text{rms}}$ with an rms emittance ϵ_{rms} . Transforming $(y, \mathcal{P}_y)$ into the action-angle variables (J, ψ) with

$$y = \sqrt{2\beta_y J} \cos \psi, \quad \mathcal{P}_y = -\sqrt{2\beta_y J} \sin \psi$$

The Jacobian of the transformation is β_y , and the distribution function becomes

$$\rho(J) = \frac{1}{\epsilon_{\text{rms}}} e^{-J/\epsilon_{\text{rms}}}, \quad \rho(\epsilon) = \frac{1}{2\epsilon_{\text{rms}}} e^{-\epsilon/2\epsilon_{\text{rms}}},$$

The percentage of particles contained within $\epsilon = n\epsilon_{\text{rms}}$ is $1 - e^{-n/2}$

$\epsilon/\epsilon_{\text{rms}}$	2	4	6	8
Percentage in 1D [%]	63	86	95	98
Percentage in 2D [%]	40	74	90	96

The maximum phase-space area that particles can survive in an accelerator is called the *admittance*, or the *dynamic aperture*. The admittance is determined by the vacuum chamber size, the kicker aperture, and nonlinear magnetic fields.

Effects of Linear Magnetic field Error

$$x'' + [K_x(s) + k(s)]x = \frac{b_0}{\rho}, \quad y'' + [K_y(s) - k(s)]y = -\frac{a_0}{\rho}$$

For a localized dipole field error:

$$\theta = \Delta B \ell / B \rho$$

Normalized closed orbit can distinguish errors more easily.

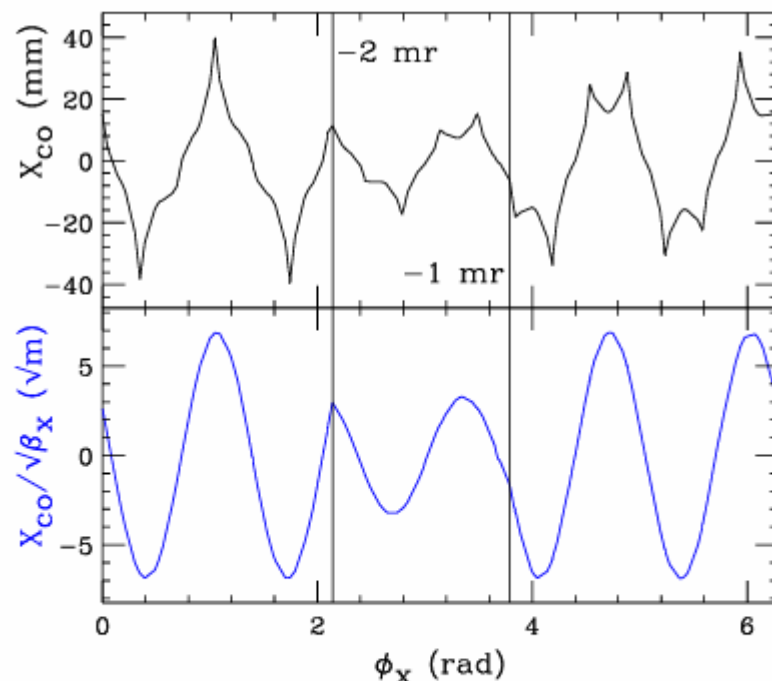
$$X'' + K_x(s)X = \theta \delta(s - s_0)$$

$$X_0 = \frac{\beta_0 \theta}{2 \sin \pi \nu} \cos \pi \nu,$$

$$X_0' = \frac{\theta}{2 \sin \pi \nu} (\sin \pi \nu - \alpha_0 \cos \pi \nu)$$

$$X_{co}(s) = G(s, s_0) \theta$$

$$G(s, s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2 \sin \pi \nu} \cos[\pi \nu - |\psi(s) - \psi(s_0)|]$$



Superposition works again here

$$x = x_{\beta} + x_{\text{co}} = x_{\beta} + D\delta$$

$$D'' + \left(\frac{1}{\rho^2} - K(s) \right) D = \frac{1}{\rho}, \quad \begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1) \begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

For a pure dipole ($K=0$):

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

Dispersion can be transported in a similar fashion as x and x prime.

$$M = \begin{pmatrix} \cos\theta & \rho \sin\theta & \rho(1 - \cos\theta) \\ -\frac{1}{\rho} \sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

When $\theta \ll 1$ i.e. $L \ll \rho$

For quadrupoles:

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

$$M(s, s_0) = \begin{pmatrix} \cos \sqrt{K} \ell & \frac{1}{\sqrt{K}} \sin \sqrt{K} \ell & 0 \\ -\sqrt{K} \sin \sqrt{K} \ell & \cos \sqrt{K} \ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M(s, s_0) = \begin{pmatrix} \cosh \sqrt{|K|} \ell & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K|} \ell & 0 \\ \sqrt{|K|} \sinh \sqrt{|K|} \ell & \cosh \sqrt{|K|} \ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For combined function magnets:

$$\bar{d} = \begin{pmatrix} \frac{1}{\rho K_x} (1 - \cos \sqrt{K_x} \ell) \\ \frac{1}{\rho \sqrt{K_x}} \sin \sqrt{K_x} \ell \end{pmatrix}$$

Synchrotron motion

Longitudinal field by RF cavity: $\mathcal{E} = \mathcal{E}_0 \sin(\phi_{\text{rf}}(t) + \phi_s), \quad \phi_{\text{rf}} = h\omega_0 t$

$$\Delta E = e\mathcal{E}_0\beta c \int_{-g/2\beta_0 c}^{g/2\beta_0 c} \sin(h\omega_0 t + \phi_s) dt = e\mathcal{E}_0 g T \sin \phi_s, \quad T = \frac{\sin(hg/2R_0)}{(hg/2R_0)}$$

Equations of motion:

$$\begin{aligned} \dot{\delta} &= \frac{\omega_0}{2\pi\beta^2 E} eV (\sin \phi - \sin \phi_s) && \text{Effect of the RF cavity} \\ \dot{\phi} &= h\omega_0 \eta \delta = \frac{h\omega_0^2 \eta}{\beta^2 E} \left(\frac{\Delta E}{\omega_0} \right) && \text{Effect of the accelerator lattice} \end{aligned}$$

$$\begin{aligned} \alpha_c &= \frac{1}{R_0} \frac{dR}{d\delta} \quad \frac{\Delta\omega}{\omega_0} = -\eta(\delta)\delta \\ \eta_0 &= \left(\alpha_0 - \frac{1}{\gamma_0^2} \right) \end{aligned}$$

Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2} \frac{h\eta\omega_0^2}{\beta^2 E} \left(\frac{\Delta E}{\omega_0} \right)^2 + \frac{eV}{2\pi} [\cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s] \\ &= \frac{1}{2} h\omega_0 \eta_0 \delta^2 + \frac{\omega_0 eV}{2\pi\beta^2 E} [\cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s] \end{aligned}$$

Synchrotron motion

For small deviation:

$$\frac{d^2}{dt^2}(\phi - \phi_s) = \frac{h\omega_0^2 eV \eta_0 \cos \phi_s}{2\pi\beta^2 E}(\phi - \phi_s)$$

The stability condition for synchrotron oscillation is $\eta_0 \cos \phi_s < 0$

The essential of the stability of small amplitude synchrotron motion:

Particle with momentum **larger** than the nominal value sees **less** RF accelerating voltage.

Particle with momentum **smaller** than the nominal value sees **more** RF accelerating voltage.

$$\omega_s = \omega_0 \sqrt{\frac{heV|\eta_0 \cos \phi_s|}{2\pi\beta^2 E}} = \frac{c}{R} \sqrt{\frac{heV|\eta \cos \phi_s|}{2\pi E}} \quad Q_s = \frac{\omega_s}{\omega_0} = \sqrt{\frac{heV|\eta_0 \cos \phi_s|}{2\pi\beta^2 E}}$$

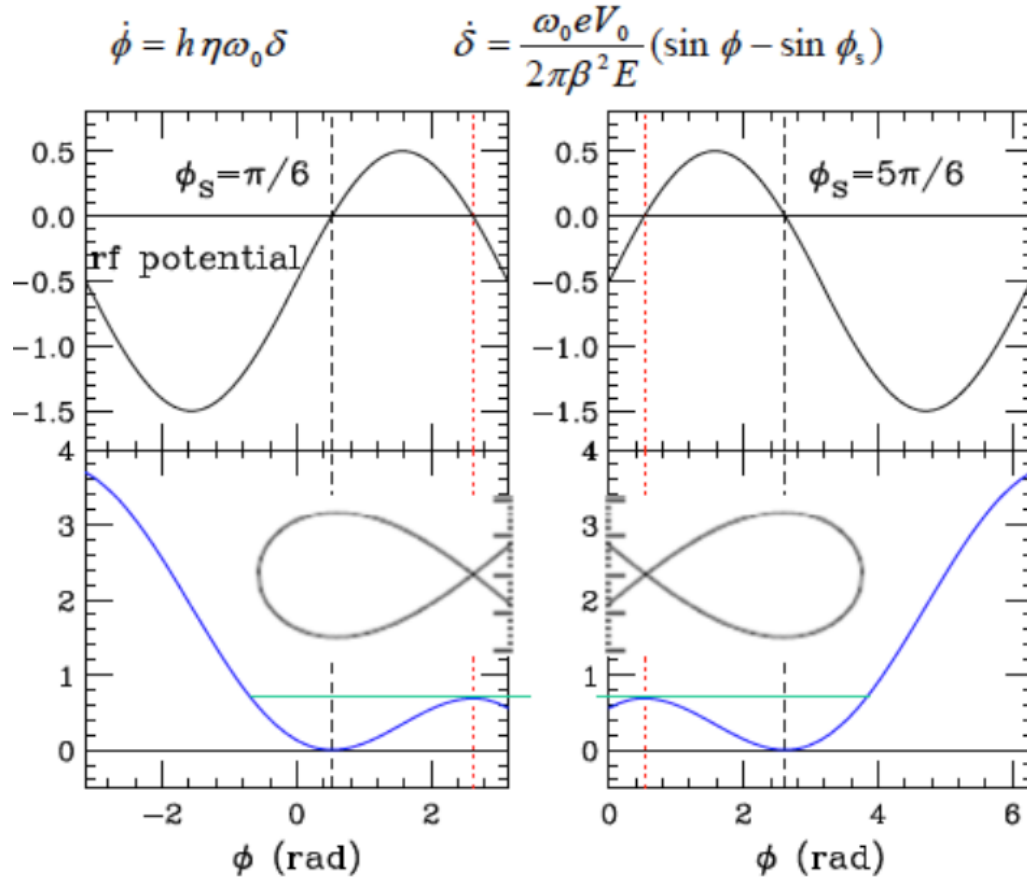
$$\left(\frac{\delta}{\hat{\delta}}\right)^2 + \left(\frac{\varphi}{\hat{\phi}}\right)^2 = 1, \quad \frac{\hat{\delta}}{\hat{\phi}} = \left(\frac{eV|\cos \phi_s|}{2\pi\beta^2 E h|\eta|}\right)^{1/2} = \frac{Q_s}{h|\eta|},$$

rms phase space of a Gaussian bunch:

$$\tilde{A}_{rms} = \pi\sigma_\delta\sigma_\phi.$$

$$\tilde{A}_{rms} = \pi\sigma_t\sigma_E$$

Synchrotron motion



Effective potential well
 Separatrix
 Stable fixed point
 Unstable fixed point
 Turning point

$$\begin{cases} \delta_{n+1} = \delta_n + \frac{eV}{\beta^2 E} (\sin\phi_n - \sin\phi_s), \\ \phi_{n+1} = \phi_n + 2\pi h\eta(\delta_{n+1})\delta_{n+1}. \end{cases}$$

$$\begin{aligned} \Delta E_{n+1} &= \Delta E_n + eV(\sin\phi_n - \sin\phi_s), \\ \phi_{n+1} &= \phi_n + \frac{2\pi h\eta}{\beta^2 E} \Delta E_{n+1}, \end{aligned}$$

