Homework 16

Problem 1. 10 points, 2D distribution function and RMS beam sizes

For the case of fully coupled transverse oscillations with eigen vectors

$$
Y_1 = \begin{vmatrix} w_{1x}e^{i\varphi_{1x}} \\ \left(u_{1x}+i\dfrac{q}{w_{1x}}\right)e^{i\varphi_{1x}} \\ w_{1y}e^{i\varphi_{1y}} \\ \left(u_{1y}+i\dfrac{1-q}{w_{1y}}\right)e^{i\varphi_{1y}} \end{vmatrix}; \quad
Y_2 = \begin{vmatrix} w_{2x}e^{i\varphi_{2x}} \\ \left(u_{2x}+i\dfrac{1-q}{w_{2x}}\right)e^{i\varphi_{2x}} \\ w_{2y}e^{i\varphi_{2y}} \\ \left(u_{2y}+i\dfrac{q}{w_{2y}}\right)e^{i\varphi_{2y}} \end{vmatrix}
$$

and known values of eigen emittances $\varepsilon_{1,2} \equiv I_{1,2} = \dfrac{\langle a_{1,2}^2 \rangle}{2}$ of stationary Gaussian distribution (solution of Fokker-Plank equation)

(a)   **6 points**; Write explicit expression for the distribution function in terms of x, Px, y and Py.

(b)   **4 points**; Write expression of the RMS beam sizes

$$
\sigma_x = \sqrt{<x^2>};\sigma_y = \sqrt{<y^2>}
$$

using beam emittances and necessary components of eigen vectors.

Solution: (a) The normalized distribution function in phase-action variable is

$$
f(I,\varphi) = \frac{1}{(2\pi)^2 \varepsilon_1 \varepsilon_2}\exp\left(-\frac{I_1}{\varepsilon_1}-\frac{I_2}{\varepsilon_2}\right);
$$

$$
\int_0^\infty dI_1 \int_0^\infty dI_2 \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \cdot f(I,\varphi) = 1.
$$

Now we need to find explicit expression for $I_{1,2}$ using

$$
Y_1 = \begin{bmatrix} w_{1x}e^{i\varphi_{1x}} \\ \left(u_{1x}+iv_{1x}\right)e^{i\varphi_{1x}} \\ w_{1y}e^{i\varphi_{1y}} \\ \left(u_{1y}+iv_{1y}\right)e^{i\varphi_{1y}} \end{bmatrix}; \quad
Y_2 = \begin{bmatrix} w_{2x}e^{i\varphi_{2x}} \\ \left(u_{2x}+iv_{2x}\right)e^{i\varphi_{2x}} \\ w_{2y}e^{i\varphi_{2y}} \\ \left(u_{2y}+iv_{2y}\right)e^{i\varphi_{2y}} \end{bmatrix};
$$

$$
v_{1x} = \frac{q}{w_{1x}}; v_{1y} = \frac{1-q}{w_{1y}}; v_{2x} = \frac{1-q}{w_{2x}}; v_{2y} = \frac{q}{w_{2y}}.
$$

were c.c. stand for complex conjugate components:

$$X = \text{Re}\left(a_1 Y_1 e^{i\psi_1} + a_2 Y_2 e^{i\psi_2}\right) = \frac{1}{2}\left(a_1 Y_1 e^{i\psi_1} + a_2 Y_2 e^{i\psi_2} + c.c.\right); Y_j^{*T} SY_k = 2i\delta_{jk}; Y_j^{*T} SY_k^* = 0;$$

$$Y_k^{*T} SX = \frac{1}{2} a_k Y_k^{*T} SY_k; Y_k^{*T} SY_k = 2i \rightarrow a_i = -iY_i^{*T} SX; I_k = \frac{|a_k|^2}{2} = \frac{\left|Y_k^{*T} SX\right|^2}{2} \equiv \frac{\left|X^T SY_k\right|^2}{2};$$

$$X^T S = \left[x, P_x, y, P_y\right]\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \left[-P_x, x, -P_y, y\right];$$

$$v_{kx} = \frac{q_{kx}}{w_{kx}}; v_{ky} = \frac{q_{kx}}{w_{1y}}; q_{1x} = q_{2y} = q; \; q_{2x} = q_{1y} = 1 - q;$$

$$X^T SY_k = e^{i\varphi_{kx}}\left(x\left(u_{kx} + iv_{kx}\right) - w_{kx} P_x\right) + e^{i\varphi_{ky}}\left(y\left(u_{1y} + iv_{1y}\right) - w_{1y} P_y\right);$$

$$\left|X^T SY_k\right|^2 = f_{kx} + f_{ky} + f_{kxy}$$

$$f_{kx} = \left|x\left(u_{kx} + iv_{kx}\right) - w_{kx} P_x\right|^2; f_{ky=}\left|y\left(u_{ky} + iv_{ky}\right) - w_{ky} P_y\right|^2$$

$$f_{kxy} = 2\,\text{Re}\,e^{i\left(\varphi_{kx} - \varphi_{ky}\right)}\left(x\left(u_{kx} + iv_{kx}\right) - w_{kx} P_x\right)\left(y\left(u_{1y} - iv_{1y}\right) - w_{1y} P_y\right)$$

with exlicit expression being (as expected) long and ugly:

$$f_{kx} = \left|x\left(u_{kx} + iv_{kx}\right) - w_{kx} P_x\right|^2 = x^2 v_{kx}^2 + \left(u_{kx} x - w_{kx} P_x\right)^2 = x^2\left(v_{kx}^2 + u_{kx}^2\right) - 2u_{kx} w_{kx} xP_x + w_{kx}^2 P_x^2$$

$$f_{ky} = \left|y\left(u_{ky} + iv_{ky}\right) - w_{ky} P_y\right|^2 = y^2 v_{ky}^2 + \left(u_{ky} x - w_{ky} P_y\right)^2 = y^2\left(v_{ky}^2 + u_{ky}^2\right) - 2u_{ky} w_{ky} yP_y + w_{ky}^2 P_y^2$$

$$f_{kxy} = 2\,\text{Re}\,e^{i\left(\varphi_{kx} - \varphi_{ky}\right)}\left(A + iB\right)$$

$$A = \left\{\left(xu_{kx} - w_{kx} P_x\right)\left(yu_{1y} - w_{1y} P_y\right) + xyv_{kx} v_{1y}\right\}$$

$$B = \left\{xv_{kx}\left(yu_{1y} - w_{1y} P_y\right) - yv_{1y}\left(xu_{kx} - w_{kx} P_x\right)\right\}$$

$$f_{kxy} = 2\cos\left(\varphi_{kx} - \varphi_{ky}\right)\left\{\left(xu_{kx} - w_{kx} P_x\right)\left(yu_{1y} - w_{1y} P_y\right) + xyv_{kx} v_{1y}\right\}$$

$$-2\sin\left(\varphi_{kx} - \varphi_{ky}\right)\left\{xv_{kx}\left(yu_{1y} - w_{1y} P_y\right) - yv_{1y}\left(xu_{kx} - w_{kx} P_x\right)\right\}$$

with 4D distribution function:

$$f(X) = \frac{1}{(2\pi)^2 \varepsilon_1 \varepsilon_2} e^{-\frac{f_{1x} + f_{1y} + f_{1xy}}{2\varepsilon_1}} e^{-\frac{f_{2x} + f_{2y} + f_{2xy}}{2\varepsilon_2}};$$

$$\int\int\int\int f(X)\, dx\, dP_x\, dy\, dP_y = \int dX^4 = 1.$$

One can try to introduce α and β components similar to 1D motion

$$f_{kx} = \frac{x^2\left(q_{kx}^2 + \alpha_{kx}^2\right)}{\beta_{kx}} + 2\alpha_{kx}P_x + \beta_{kx}P_x^2; \quad \beta_{kx} = w_{kx}^2; \alpha_{kx} = -w_{kx}u_{kx};$$

$$f_{ky} = \frac{y^2\left(q_{ky}^2 + \alpha_{ky}^2\right)}{\beta_{ky}} + 2\alpha_{ky}P_y + \beta_{ky}P_y^2; \quad \beta_{ky} = w_{ky}^2; \alpha_{ky} = -w_{ky}u_{ky};$$

but it does not make expressions better, especially for f_{xy}.

Another way of writing it is to use real and imaginary parts of the eigen vectors

$$Y_k = R_k + iQ_k;$$

$$\left|a_k\right|^2 = \left|X^T SR_k + iX^T SQ_k\right| = \left(X^T SR_k\right)^2 + \left(X^T SQ_k\right)^2;$$

$$\left|a_k\right|^2 = \left(xr_{k2} - P_x r_{k1} + yr_{k4} - P_y r_{k3}\right)^2 + \left(xq_{k2} - P_x q_{k1} + yq_{k4} - P_y q_{k3}\right)^2;$$

$$f(X) = f_1(X)f_2(X);$$

$$f_k(X) = \frac{1}{2\pi\varepsilon_k}\exp\left(-\frac{\left(xr_{k2} - P_x r_{k1} + yr_{k4} - P_y r_{k3}\right)^2 + \left(xq_{k2} - P_x q_{k1} + yq_{k4} - P_y q_{k3}\right)^2}{2\varepsilon_x}\right).$$

(b) This is much easier task:

$$x = a_1 w_{1x}\cos\varphi_{1x} + a_2 w_{2x}\cos\varphi_{2x};$$

$$y = a_1 w_{1y}\cos\varphi_{1y} + a_2 w_{2y}\cos\varphi_{2y};$$

$$\langle x^2 \rangle = a_1^2 w_{1x}^2 \langle \cos^2\varphi_{1x}\rangle + 2a_1 w_{1x} a_2 w_{2x}\langle \cos\varphi_{1x}\cos\varphi_{2x}\rangle + a_2^2 w_{2x}^2\langle \cos^2\varphi_{2x}\rangle;$$

$$\langle y^2 \rangle = a_1^2 w_{1y}^2 \langle \cos^2\varphi_{1y}\rangle + 2a_1 w_{1y} a_2 w_{2y}\langle \cos\varphi_{1y}\cos\varphi_{2y}\rangle + a_2^2 w_{2y}^2\langle \cos^2\varphi_{2y}\rangle;$$

Since phases of oscillation are random are not correlated between to modes, we have

$$\langle \cos^2\varphi_{kx}\rangle = \langle \cos^2\varphi_{ky}\rangle = \frac{1}{2}; \langle \cos\varphi_{kx}\cos\varphi_{kx}\rangle = 0;$$

$$\langle x^2\rangle = \frac{a_1^2 w_{1x}^2 + a_2^2 w_{2x}^2}{2} = \varepsilon_1\beta_{1x} + \varepsilon_2\beta_{2x}; \sigma_x = \sqrt{\varepsilon_1\beta_{1x} + \varepsilon_2\beta_{2x}};$$

$$\langle y^2\rangle = \frac{a_1^2 w_{1y}^2 + a_2^2 w_{2y}^2}{2} = \varepsilon_1\beta_{1y} + \varepsilon_2\beta_{2y}; \sigma_y = \sqrt{\varepsilon_1\beta_{1y} + \varepsilon_2\beta_{2y}}.$$

Problem 2. 10 points, 3D distribution function and RMS beam sizes

(a) 5 points: For the case of fully coupled transverse oscillations with eigen vectors

$$Y_k(s) = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + i\dfrac{q_{kx}}{\mathrm{w}_{kx}} \right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + i\dfrac{q_{ky}}{\mathrm{w}_{ky}} \right) e^{i\chi_{ky}} \\ \mathrm{w}_{k\tau} e^{i\chi_{k\tau}} \\ \left(\mathrm{v}_{k\tau} + i\dfrac{q_{k\tau}}{\mathrm{w}_{k\tau}} \right) e^{i\chi_{k\tau}} \end{bmatrix} ; k = 1,2,3$$

and known values of eigen emittances $\varepsilon_k \equiv I_k = \dfrac{\langle a_k^2 \rangle}{2}; k = 1,2,3$ of stationary Gaussian distribution, write expression of the RMS beam sizes

$$\sigma_x = \sqrt{<x^2>}; \sigma_y = \sqrt{<y^2>}; \sigma_\tau = \sqrt{<\tau^2>}$$

using the beam emittances and necessary components of eigen vectors.

(b) 5 points: For the case of slow synchrotron oscillations and approximate expressions for the eigen vectors:

$$Y_k = \begin{bmatrix} Y_{k\beta} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + \dfrac{iq_k}{\mathrm{w}_{kx}} \right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + \dfrac{i(1-q_k)}{\mathrm{w}_{ky}} \right) e^{i\chi_{ky}} \\ \mathrm{y}_{k\tau} = \eta^T S Y_{k\beta} \\ 0 \end{bmatrix} ; k = 1,2; Y_\delta = \begin{bmatrix} \eta \\ \chi_\tau \\ 1 \end{bmatrix} = \begin{bmatrix} \eta_x \\ \eta_{px} \\ \eta_y \\ \eta_{py} \\ \chi_\tau \\ 1 \end{bmatrix};$$

and known values of eigen emittances $\varepsilon_k \equiv I_k = \dfrac{\langle a_k^2 \rangle}{2}; k = 1,2$ and RMS values of the

relative energy spread $\sigma_\delta = \sqrt{\langle \delta^2 \rangle}$ write expressions for transverse beam sizes:

$$\sigma_x = \sqrt{<x^2>}; \sigma_y = \sqrt{<y^2>}$$

Solution:

(a) Treatment is identical to the RMS beam sizes in the previous problem:

$$x = a_1 w_{1x} \cos\varphi_{1x} + a_2 w_{2x} \cos\varphi_{2x} + a_3 w_{3x} \cos\varphi_{3x};$$
$$y = a_1 w_{1y} \cos\varphi_{1y} + a_2 w_{2y} \cos\varphi_{2y} + a_3 w_{3y} \cos\varphi_{3y};$$
$$\tau = a_1 w_{1\tau} \cos\varphi_{1\tau} + a_2 w_{2\tau} \cos\varphi_{2\tau} + a_3 w_{3y} \cos\varphi_{3\tau};$$

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\frac{a_1^2 w_{1x}^2 + a_2^2 w_{2x}^2 + a_3^2 w_{3x}^2}{2}} = \sqrt{\varepsilon_1 w_{1x}^2 + \varepsilon_2 w_{2x}^2 + \varepsilon_3 w_{3x}^2};$$

$$\sigma_y = \sqrt{\langle y^2 \rangle} = \sqrt{\frac{a_1^2 w_{1y}^2 + a_2^2 w_{2y}^2 + a_3^2 w_{3y}^2}{2}} = \sqrt{\varepsilon_1 w_{1y}^2 + \varepsilon_2 w_{2y}^2 + \varepsilon_3 w_{3y}^2};$$

$$\sigma_\tau = \sqrt{\langle \tau^2 \rangle} = \sqrt{\frac{a_1^2 w_{1\tau}^2 + a_2^2 w_{2\tau}^2 + a_3^2 w_{3\tau}^2}{2}} = \sqrt{\varepsilon_1 w_{1\tau}^2 + \varepsilon_2 w_{2\tau}^2 + \varepsilon_3 w_{3\tau}^2};$$

(b) In this specific case we have

$$Y_k = \begin{bmatrix} Y_{k\beta} \\ y_{k\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} w_{kx} e^{i\chi_{kx}} \\ \left(v_{kx} + \dfrac{iq_k}{w_{kx}}\right) e^{i\chi_{kx}} \\ w_{ky} e^{i\chi_{ky}} \\ \left(v_{ky} + \dfrac{i(1-q_k)}{w_{ky}}\right) e^{i\chi_{ky}} \\ y_{k\tau} = \eta^T S Y_{k\beta} \\ 0 \end{bmatrix}; k = 1,2; \quad Y_\delta = \begin{bmatrix} \eta \\ \chi_\tau \\ 1 \end{bmatrix} = \begin{bmatrix} \eta_x \\ \eta_{px} \\ \eta_y \\ \eta_{py} \\ \chi_\tau \\ 1 \end{bmatrix}$$

$$\sigma_x^2 = \varepsilon_1 w_{1x}^2 + \varepsilon_2 w_{2x} + \eta_x^2 \sigma_\delta^2$$
$$\sigma_y^2 = \varepsilon_1 w_{1y}^2 + \varepsilon_2 w_{2y} + \eta_y^2 \sigma_\delta^2$$
$$\sigma_\tau^2 = \varepsilon_1 \left(\eta^T S Y_{1\beta}\right)^2 + \varepsilon_2 \left(\eta^T S Y_{2\beta}\right)^2 + +\chi_\tau^2 \sigma_\delta^2;$$