PHY 564
Advanced Accelerator Physics
Lecture 7

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Review of Linear Algebra

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Matrix: definition and properties

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \ldots & a_{1m} \\
  a_{21} & a_{22} & \ldots & a_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \ldots & a_{nm}
\end{pmatrix}
\]

Addition: \( A + B = C \iff a_{ij} + b_{ij} = c_{ij} \)

Multiplied by a constant: \( kA = B \iff ka_{ij} = b_{ij} \)

Equality: \( A = B \iff a_{ij} = b_{ij} \)

Multiplication (inner product): \( AB = C \iff \sum_k a_{ik}b_{kj} = c_{ij} \)

\[
(AB)C = A(BC), \quad A(B + C) = AB + AC
\]

In general \( AB \neq BA \)

Multiplication demands that \( A \) has the same number of columns as \( B \) has rows.
Matrix: special cases I

• Diagonal matrix:  \( a_{ij} = 0 \) for \( i \neq j \)

\[
A = \begin{pmatrix}
    a_{11} & 0 & \ldots & 0 \\
    0 & a_{22} & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & a_{nn}
\end{pmatrix}
\]

If A and B are both diagonal matrix, they are commutative:

\[ AB = BA \]

• Identity matrix:

\[
I = \begin{pmatrix}
    1 & 0 & \ldots & 0 \\
    0 & 1 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 1
\end{pmatrix}
\]

\[
I_{ij} = \delta_{ij} = \begin{cases} 
1 & \text{for } i = j \\
0 & \text{for } i \neq j
\end{cases}
\]
Matrix: special cases II

- Block diagonal matrix: $A$ and $A_i$ are square matrix.

$$A = \begin{bmatrix}
  A_1 & O & \cdots & O \\
  O & A_2 & \cdots & O \\
  \vdots & \vdots & \ddots & \vdots \\
  O & O & \cdots & A_k
\end{bmatrix}, \quad A = \begin{bmatrix}
  1 & 3 & 2 & 0 & 0 & 0 \\
  7 & 0 & 2 & 0 & 0 & 0 \\
  1 & 1 & 2 & 0 & 0 & 0 \\
  0 & 0 & 0 & 6 & 0 & 0 \\
  0 & 0 & 0 & 0 & 2 & 1 \\
  0 & 0 & 0 & 0 & 3 & 3
\end{bmatrix}$$

- Triangular matrix:
  
  Upper diagonal matrix: elements below diagonal are all zero

$$U = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  0 & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & a_{nn}
\end{pmatrix}$$

  Lower diagonal matrix: elements below diagonal are all zero

$$L = \begin{pmatrix}
  a_{11} & 0 & \cdots & 0 \\
  a_{21} & a_{22} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}$$
Matrix V: transpose matrix

- A matrix, B, is called the **transpose matrix** of a matrix A if

  \[ A_{ij} = B_{ji} \]

  The transpose matrix is often denoted as \( A^T \), i.e.

  \[ A_{ij} = (A^T)_{ji} \]

- A square matrix A is called an **orthogonal matrix** if

  \[ A^T = A^{-1} \]

- A square matrix A is called **symmetric matrix** if

  \[ A^T = A \]

  and **anti-symmetric** if

  \[ A^T = -A \]
Matrix: trace

• In any square matrix, the sum of the diagonal elements is called the trace.

\[ Tr(A) = \sum_i a_{ii} \]

• A useful property: \( Tr(AB) = Tr(BA) \)

• In general, \( Tr(ABC) = Tr(BCA) \neq Tr(BAC) \)

• Trace is a linear operator:

\[ Tr(A + kB) = Tr(A) + k \cdot Tr(B) \]
Matrix: determinant of a matrix

• For a square matrix,

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \ldots \\
  a_{n1} & a_{n2} & \ldots & a_{nn}
\end{pmatrix}
\]

• The determinant

\[
D = \begin{vmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \ldots \\
  a_{n1} & a_{n2} & \ldots & a_{nn}
\end{vmatrix} = \det(A)
\]

is called the determinant of matrix A and is denoted by \(\det(A)\).
# Determinant I

The determinant of an $n \times n$ matrix $D$ is defined as:

$$D = \begin{vmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nn}
\end{vmatrix} = \sum_{i,j,k} \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} \ldots$$

where the Levi-Civita symbol $\varepsilon_{ijk\ldots}$ is defined as:

$$\varepsilon_{ijk\ldots} = \begin{cases} 
1 & \text{if (i, j, k...) is even permutation of (1, 2, 3...)} \\
-1 & \text{if (i, j, k...) is odd permutation of (1, 2, 3...)} \\
0 & \text{if any of the two indices is repeated}
\end{cases}$$

The determinant $D$ is a scalar value that can be computed from the entries of the matrix. It is used in various areas of mathematics, such as linear algebra, where it indicates whether the matrix is invertible or not. The Levi-Civita symbol is a tensor used in the calculation of the determinant, and it is antisymmetric, meaning that if any two indices are swapped, the symbol changes sign. The determinant of a matrix is zero if and only if the matrix is singular, meaning it does not have an inverse.
A determinant of n dimension can be expanded over a column (or a row) into a sum of n determinants of n-1 dimension:

\[
D = \begin{vmatrix}
  a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
  a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nj} & \cdots & a_{nn}
\end{vmatrix} = \sum_i C_{ij} a_{ij}
\]

\(C_{ij} = (-1)^{i+j} M_{ij}\) is called the \(ij\)th cofactor of \(D\).

A determinant of n dimension can be expanded over a column (or a row) into a sum of n determinants of n-1 dimension:

\[
D = \begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}
\]
Determinant III

• Multiplied by a constant

\[
\begin{vmatrix}
ka_{11} & a_{12} & a_{13} \\
ka_{21} & a_{22} & a_{23} \\
ka_{31} & a_{32} & a_{33}
\end{vmatrix} = k
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix}
\]

• The value of a determinant is unchanged if a multiple of one column (row) is added to another column (row)

\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = \begin{vmatrix}
a_{11} + ka_{12} & a_{12} & a_{13} \\
a_{21} + ka_{22} & a_{22} & a_{23} \\
a_{31} + ka_{32} & a_{32} & a_{33}
\end{vmatrix}
\]

• A determinant is equal to zero if any two columns (rows) are proportional

\[
\begin{vmatrix}
a_{12} & ka_{12} & a_{13} \\
a_{22} & ka_{22} & a_{23} \\
a_{32} & ka_{32} & a_{33}
\end{vmatrix} = \begin{vmatrix}
0 & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{vmatrix} = 0
\]
Linear equation system

• Existence of non-trivial solution of homogeneous equations

\[
\begin{align*}
  a_{11}x + a_{12}y + a_{13}z &= 0 \\
  a_{21}x + a_{22}y + a_{23}z &= 0 \\
  a_{31}x + a_{32}y + a_{33}z &= 0
\end{align*}
\]

\[
x \cdot D \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}x + a_{12}y + a_{13}z & a_{12} & a_{13} \\ a_{21}x + a_{22}y + a_{23}z & a_{22} & a_{23} \\ a_{31}x + a_{32}y + a_{33}z & a_{32} & a_{33} \end{vmatrix} = 0
\]

Similarly, \( y \cdot D = 0 \) and \( z \cdot D = 0 \)

• Thus a set of homogenous linear equations have non-trivial solutions only if the determinant of the coefficients, \( D \), vanishes.
Matrix: properties of the determinant of a matrix

- Some properties of the determinant of matrices
  - \( \det(A^T) = \det(A) \)
  - \( \det(kA) = k^n \det(A) \)
  - \( \det(AB) = \det(A) \det(B) \)

Proof of \( \det(AB) = \det(A) \det(B) \):

1. \[
\sum_{i,j,k} \varepsilon_{ijk} a_{\beta i} a_{\alpha j} a_{\gamma k} \ldots
\]
2. \[
= \sum_{j,i,k} \varepsilon_{jik} a_{\alpha j} a_{\beta i} a_{\gamma k} \ldots
\]
3. \[
= \sum_{i,j,k} \varepsilon_{ijk} a_{\alpha i} a_{\beta j} a_{\gamma k} \ldots
\]
4. \[
= - \sum_{i,j,k} \varepsilon_{ijk} a_{\alpha i} a_{\beta j} a_{\gamma k} \ldots
\]
5. \[
\sum_{i,j,k} \varepsilon_{ijk} a_{\alpha i} a_{\beta j} a_{\gamma k} \ldots
\]
6. \[
= \varepsilon_{\alpha \beta \gamma} \sum_{i,j,k} \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} \ldots
\]
7. \[
= \varepsilon_{\alpha \beta \gamma} |A|
\]
Matrix IV: inversion

• Inversion of a square matrix $A$ is to find a square matrix $B$ such that

$$AB = BA = I$$

$B$ is called the inverse matrix of $A$ and often denoted by $A^{-1}$, i.e.

$$AA^{-1} = A^{-1}A = I$$

• One way to find the inverse matrix is by

$$(A^{-1})_{ij} = \frac{C_{ji}}{|A|}, \text{ where } C_{ji} \text{ is the } ji^{th} \text{ cofactor of } A$$
Matrix VI: similarity transformation and diagonalization

• Two matrix, $A$ and $B$, are called similar if there exists a invertible matrix $P$ such that

$$B = P^{-1}AP,$$

and the transformation from $A$ to $B$ is called similarity transformation.

• Diagonolization of a matrix, $A$, is to find a similarity transformation matrix, $P$, such that

$$P^{-1}AP$$

is a diagonal matrix:
Matrix VII: diagonalization

\[ P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix} \]

- If we look at the j\textsuperscript{th} column of the second equation, it follows

\[ \sum_k a_{ik} p_{kj} = \sum_k p_{ik} \lambda_k \delta_{kj} = \lambda_j p_{ij} \quad (*) \]

Defining a nx1 matrix (i.e. a column vector) \( |P^j\rangle \) such that

\[ |P^j\rangle_i = p_{ij} \quad \text{(note: j is fixed)} \]

Equation (*) becomes: \( A|P^j\rangle = \lambda_j |P^j\rangle \)
Matrix VIII: eigenvalue and eigenvector

• For a matrix $A$, a vector matrix $X$ is called an **eigenvector** of $A$ if
\[
A \cdot X = \lambda X
\]
where $\lambda$ is called the **eigenvalue** associated with the eigenvector $X$.

• The eigenvalues are found by solving the following polynomial equation
\[
(A - \lambda I) \cdot X = 0 \implies \det(A - \lambda I) = 0
\]
Defective Matrix

• Not all square matrix can be diagonalized:

\[
A = \begin{pmatrix}
2 & -3 \\
3 & -4 \\
\end{pmatrix}; \quad \det (A - \lambda I) = \begin{vmatrix}
2 - \lambda & -3 \\
3 & -4 - \lambda \\
\end{vmatrix} = 0 \Rightarrow (\lambda + 1)^2 = 0 \Rightarrow \lambda = -1
\]

\[
\begin{pmatrix}
2 & -3 \\
3 & -4 \\
\end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases}
3x_1 - 3x_2 = 0 \\
3x_1 - 3x_2 = 0 \\
\end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix};
\]

We end up with only one eigenvector.

• A square matrix that does not have a complete set of eigenvectors is not diagonalizable and is called a defective matrix.

• If a matrix, A, is defective (and hence is not similar to a diagonal matrix), then what is the simplest matrix that A is similar to?
Jordan form matrix

• Definition: a **Jordan block** with value $\lambda$ is a square, upper triangular matrix whose entries are all $\lambda$ on the diagonal, all 1 on the entries immediately above the diagonal, and zero elsewhere:

$$J(\lambda) = \begin{bmatrix}
\lambda & 1 & 0 & \ldots & 0 & 0 \\
0 & \lambda & 1 & \ldots & 0 & 0 \\
0 & 0 & \lambda & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \lambda & 1 \\
0 & 0 & 0 & \ldots & 0 & \lambda \\
\end{bmatrix}$$

1D: $[\lambda] \\
2D: \begin{bmatrix}
\lambda & 1 \\
0 & \lambda \\
\end{bmatrix} \\
3D: \begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda \\
\end{bmatrix}$

• Definition: a Jordan form matrix is a block diagonal matrix whose blocks are all Jordan blocks

• Theorem: Let $A$ be a nxn matrix. Then there is a Jordan form matrix that is similar to $A$. 

$$\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 1 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{bmatrix}$$
Symplectic Matrix

\[ M^T SM = S \quad (***) \]

\[ S = \begin{pmatrix}
    S_{1D} & 0 & \ldots & 0 \\
    0 & S_{1D} & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & S_{1D}
\end{pmatrix}; \quad S_{1D} = \begin{pmatrix}
    0 & 1 \\
    -1 & 0
\end{pmatrix} \]

- A matrix satisfying condition (***), called a **symplectic matrix**.
  - **Inverse:** \( M^T SM = S \Rightarrow SM^T SM = S^2 = -I \Rightarrow (-SM^T S)M = I \Rightarrow M^{-1} = -SM^T S \)
  - If \( M \) and \( N \) are both symplectic, then their product, \( MN \), is also symplectic \( (MN)^T S(MN) = N^T M^T SMN = N^T SN = S \)
  - If \( M \) is symplectic, \( M^T \) is also symplectic \( (M^T SM)^{-1} = -S \Rightarrow M^{-1} S(M^T)^{-1} = S \Rightarrow S = MSM^T \Rightarrow (M^T)^T SM^T = S \)
Symplectic Matrix II

• If $\lambda$ is eigenvalue then $1/\lambda$ is also an eigenvalue and the multiplicity of $\lambda$ and $1/\lambda$ is the same.
  – It implies that the eigenvalues are coming in pairs $\{\lambda, 1/\lambda\}$.

• As a consequence of above property, the determinant of a symplectic matrix is 1.