PHY 564
Advanced Accelerator Physics
Lecture 7
How to build a magnet

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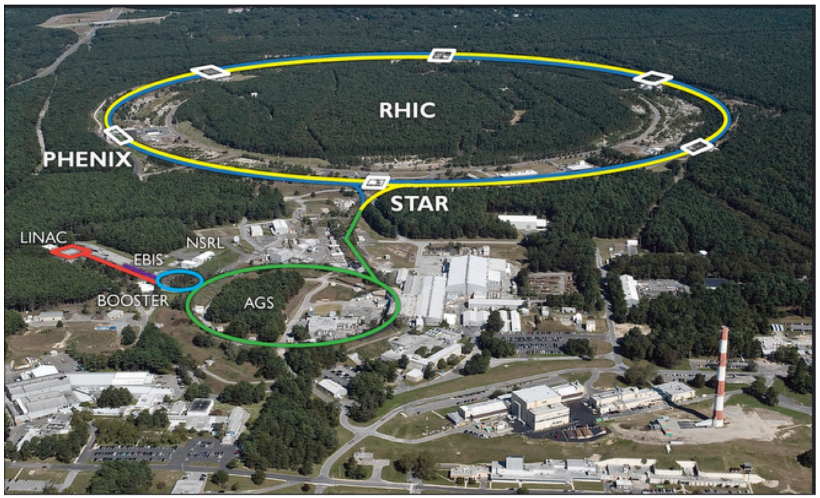
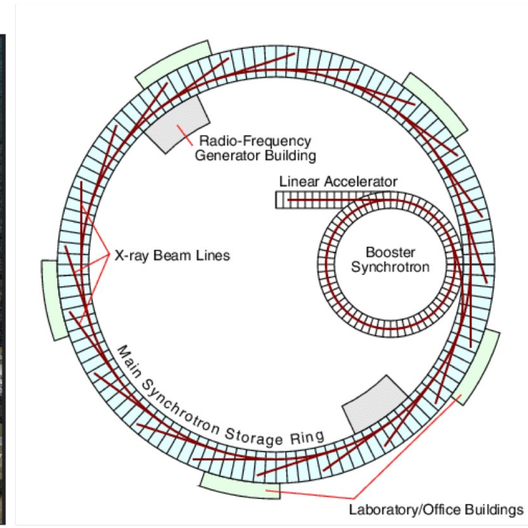
Each modern accelerator has a large number of elements serving various purposes: guns (or sources, including targets producing positrons, muons, antiprotons…) generating charged particles, which are both accelerated and transported for their intended use. Accelerated particles either dumped into a target or circulated in synchrotron or storage rings. Modern accelerator complexes build for high energy colliders or generating synchrotron radiation are comprised of a multiple dedicated accelerators and transport channels connecting the later.



(a)



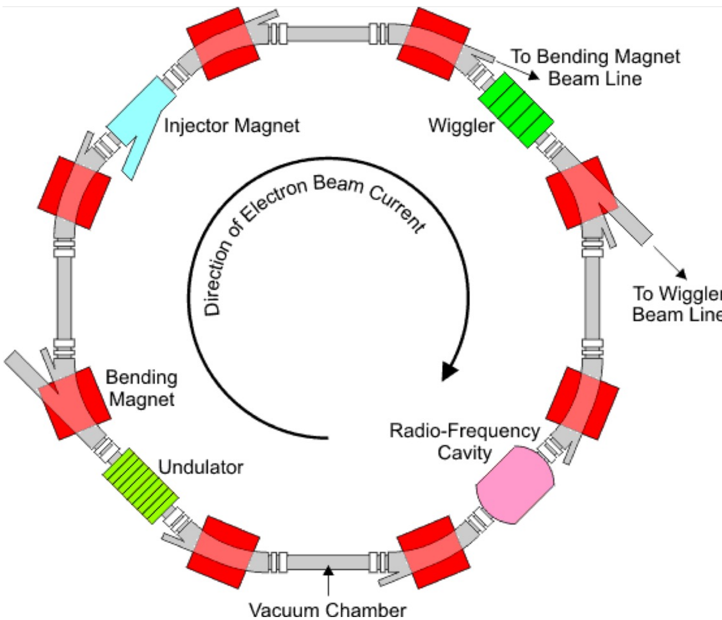
(b)

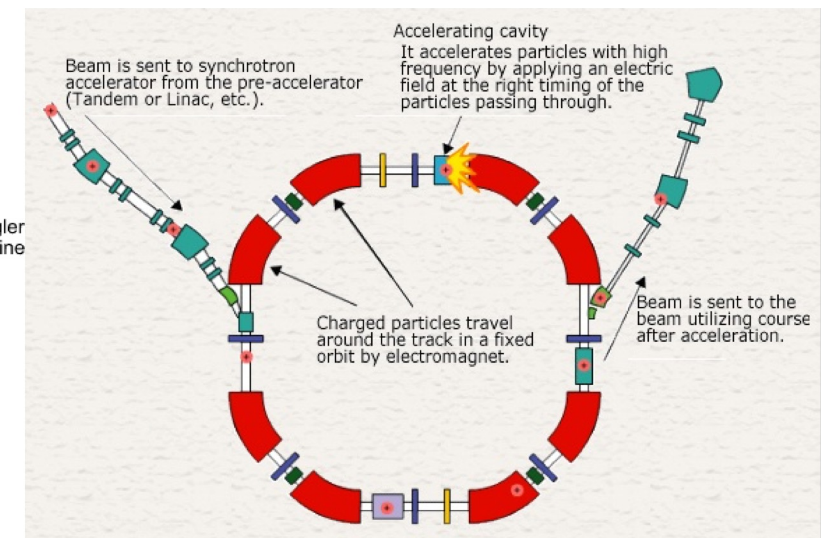
(a) RHIC (Relativistic Heavy Ion Collider) has sources of polarized protons and various ions. A linear accelerator, a booster and AGS (Alternating gradient Synchrotron) and two super-conducting (blue and yellow) RHIC rings– all serving for staged accelerating protons to 250 GeV and heavy ions to 100 GeV/u. Accelerated beams are circulating and colliding in RHIC for many hours. The chain of accelerator/rings are connected to each other by transport channels.

(b) A typical synchrotron radiation source has a booster synchrotron for acceleration to the final energy (typically 3-8 GeV). Beam at the top energy is injected into the storage ring where it circulates for many hours.

We will discuss the sources of particles later in this course, while dedicating one of next classes for discussion of linear accelerator and RF accelerating cavities for storage rings and boosters. In general, time dependent elements of accelerators may require computer simulation when analytical solutions are not available.

In this class we will focus on majority of elements, which are used in accelerators. These elements either have time independent EM field (DC) or it varies very slow when compared with the time required for particles to pass through the elements. These elements are used for bending and focusing beams of particles in accelerators and transport channels or as elements of detectors. They include dipoles, quadrupoles, skew-quadrupoles, wigglers, undulators, solenoids, etc.





Lecture 7. How to build a magnet

First, let discuss why a vast majority of accelerator elements for high-energy accelerators are magnetic and not electric devices? The most important is the relative strength of the devices. For a ultra-relativistic particle, moving with velocity close to the speed of light, a 1 Gs (0.1 mT) of magnetic field is equivalent to 300 V/cm of electric field:

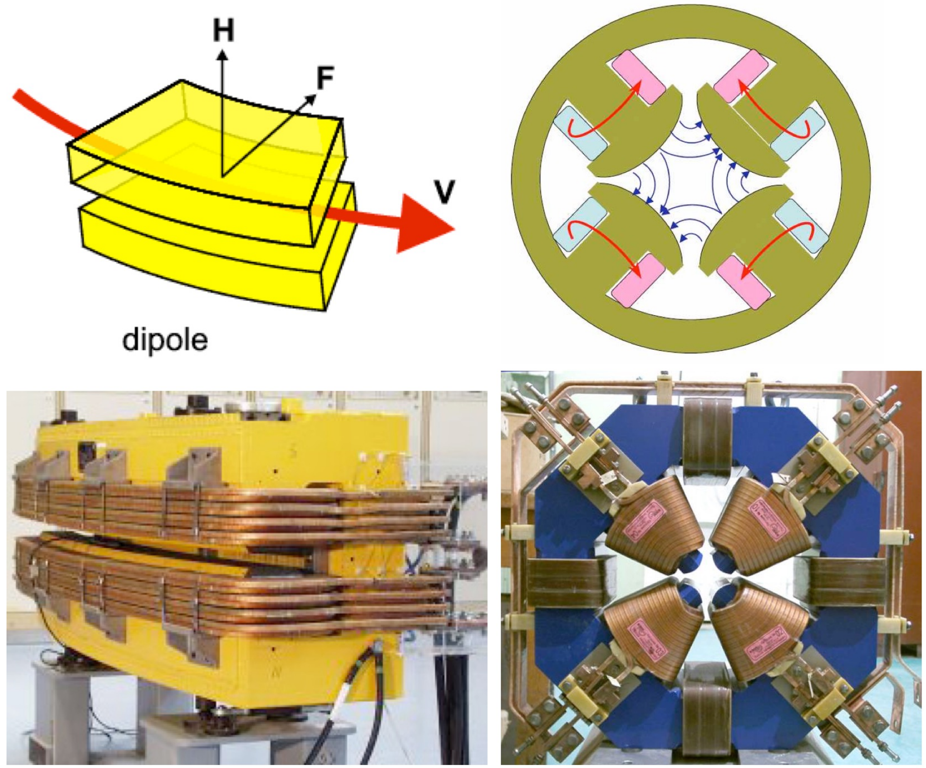
$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \left[\frac{\vec{v}}{c} \times \vec{B}\right]\right) \tag{L8-1}$$

With typical aperture of few centimeters, 1 Gs electric field will require voltage of few kVs. But detailed reasons for this are threefold:

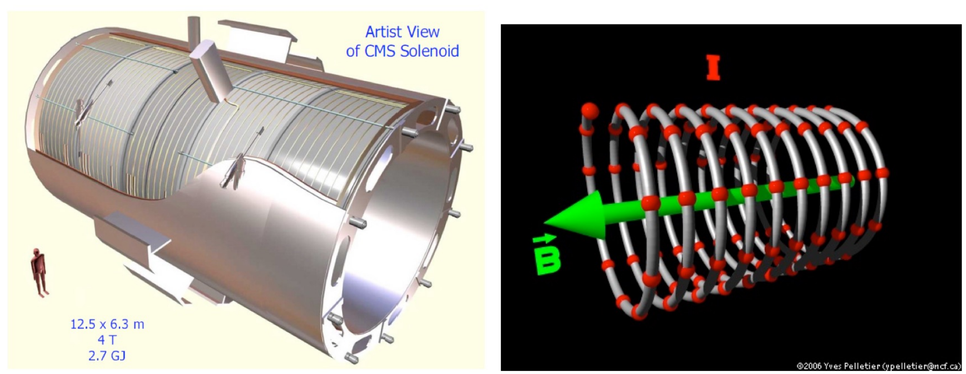
(a) for particles moving with speed close to the speed of light magnetic element are much stronger and much more effective. A typical magnetic field of 20 kGs (2 T) for room temperature steel-dominated magnets corresponds to 6 MV/cm electric field, which would very hard (most likely impossible because of arcing) to achieve with typical gap of few cm. Needles to say that superconducting magnets are edging towards 100 kGs (10T) field, and 30 MV/cm E-field is beyond reach.

(b) Electric fields do not penetrate through metal vacuum chamber, which is used for high vacuum systems. It means that there should be penetrations and internal electrostatic structures – both limiting the available voltages/E-fields and creating beam-unfriendly environment: we will discuss wakefields and collective effects later in the course.

(c) Electrostatic system are much more complicated and much more hazardous for humans – you can be killed by just 100 V, not mentioning 10 MV. In contrast, magnetic field does not arc – the only danger is that it attracts magnetic materials.



Dipole and quadrupole magnets.



Superconducting solenoid for CMS detector at LHC.

Let's further clarify the task for today: magnets with static (or almost static) fields in vacuum. The later condition is typical for most of the accelerators, where beams are propagating inside a vacuum chamber with a good vacuum to prevent particle's scattering on gas and particle loss. Since there is no material in the vacuum, there is no currents and charges. Then general Maxwell equations

$$\vec{E} = -grad\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t};\ div\vec{E} = 4\pi\rho;\ curl\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t};$$
$$\vec{B} = curl\vec{A};\qquad div\vec{B} = 0;\ curl\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}. \qquad \text{(L8-2)}$$

significantly simplify

$$\vec{E} = -grad\varphi;\ div\vec{E} = 0;\ curl\vec{E} = 0;$$
$$\vec{B} = curl\vec{A};\quad div\vec{B} = 0;\ curl\vec{B} = 0, \qquad \text{(L8-3)}$$

assuming the time dependence of the fields is either absent or so slow that fields modification are negligibly small. It means that DC fields in vacuum are determined by boundary conditions.

What is also striking that equations for magnetic field are identical to those for electric field. It means that we can introduce effective magnetic potential inside vacuum

$$\vec{E} = -grad\varphi_E;\quad div\vec{E} = 0;\ curl\vec{E} = 0; \Rightarrow \Delta\varphi_E = 0;$$
$$\vec{B} = -grad\varphi_M;\quad div\vec{B} = 0;\ curl\vec{B} = 0; \Rightarrow \Delta\varphi_M = 0, \qquad \text{(L8-4)}$$

which allows us to use potential theory for describing DC fields in the vacuum. For magnetic field the later caused by $curl\vec{B} = 0$ - hence, beware that as soon you encounter current (needed to excite the field), $\vec{B} \neq -grad\varphi_M$.

Let's start from a simple case with Cartesian geometry, e.g. a transport beam-line without bending.

$$\varphi_M = \varphi_M(x,y,s);\ s \equiv z;$$
$$\Delta\varphi_M = \frac{\partial^2 \varphi_M}{\partial x^2} + \frac{\partial^2 \varphi_M}{\partial y^2} + \frac{\partial^2 \varphi_M}{\partial z^2} = 0. \qquad \text{(L8-5)}$$

further simplified by assumption that magnet has transverse fields, is very long compared with its aperture, and its magnetic field does not depend on s, except the magnet edges (to be considered later):

$$\varphi_M = \varphi_M(x,y);\ \Delta\varphi_M = \frac{\partial^2 \varphi_M}{\partial x^2} + \frac{\partial^2 \varphi_M}{\partial y^2} = 0. \qquad \text{(L8-6)}$$

Equation (L8-6) is identical to definition of analytical functions in x-y plane. We can also use cylindrical coordinate system with natural expansion in Fourier series:

$$x = \rho\cos\varphi; y = \rho\sin\varphi;\ \varphi_M = \varphi_M(\rho,\varphi) = \sum_{n=0}^{\infty} \varphi_n e^{in\varphi};$$
$$\Delta\varphi_M = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi_M}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\varphi_M}{\partial\varphi^2} = 0 \Rightarrow \rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi_n}{\partial\rho}\right) = n^2\varphi_n; \qquad \text{(L8-7)}$$

The solution are well known

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi_n}{\partial\rho}\right) = n^2\varphi_n \rightarrow \varphi_n = a_n\rho^n + \frac{b_n}{\rho^n} \qquad \text{(L8-8)}$$

with b_n =0 for finite fields at ρ=0. Thus, the field expansion – known as multipoles -

$$\varphi_M(\rho,\varphi) = \mathrm{Re}\sum_{n=0}^{\infty} a_n\rho^n e^{in\varphi} \equiv \mathrm{Re}\sum_{n=0}^{\infty} a_n(x+iy)^n \qquad \text{(L8-9)}$$

where we used trivial

$$\rho^n e^{in\varphi} = \left(\rho e^{i\varphi}\right)^n = \left(\rho\cos\varphi + i\rho\sin\varphi\right)^n = (x+iy)^n \qquad \text{(L8-10)}$$

Multipoles are named by counting number of extrema (minima and maxima) of the potential:

n=1, two poles, one negative, one positive: dipole;

n=2, four poles: quadrupole;

n=3, six poles: sextupole;

n=4, eight poles: octupole;

n=5; ten poles: dodecapole……

Magnetic fields generated by such magnetic potential are indeed transverse and can be easily found by differentiating (L8-9).

$$\vec{B} = \vec{\nabla}\varphi_M(\rho,\varphi) = -\mathrm{Re}\sum_{n=0}^{\infty} n a_n (\hat{x} + i\hat{y})(x+iy)^{n-1};$$
$$B_x = -\mathrm{Re}\sum_{n=0}^{\infty} n a_n (x+iy)^{n-1};\ B_y = \mathrm{Im}\sum_{n=0}^{\infty} n a_n (x+iy)^{n-1}. \qquad \text{(L8-11)}$$

Dipole (n=1) generates constant magnetic field, while higher multipoles have zero field on the magnet axis (x=y=0):

$$\vec{B}_1 = \hat{y}\,\mathrm{Im}\,a_1 - \hat{x}\cdot\mathrm{Re}\,a_1$$
$$\vec{B}_2 = 2\,\mathrm{Im}\,a_2(\hat{y}\cdot x + \hat{x}\cdot y) + 2\,\mathrm{Re}\,a_2(\hat{y}\cdot y - \hat{x}\cdot x) \qquad \text{(L8-12)}$$
$$\vec{B}_3 = 3\,\mathrm{Im}\,a_3\left(\hat{y}\cdot x^2 - 2\hat{x}\cdot xy\right) + 2\,\mathrm{Re}\,a_3\left(\hat{y}\cdot y^2 - \hat{x}\cdot x^2\right)$$

.....

We also can describe these transverse fields using a single component A_z (A_2) of vector potential:

$$A_2(\rho,\varphi) = \operatorname{Im}\sum_{n=0}^{\infty} a_n \rho^n e^{in\varphi} \equiv \operatorname{Im}\sum_{n=0}^{\infty} a_n (x+iy)^n$$

$$B_y = \frac{\partial A_s}{\partial x} = \operatorname{Im}\sum_{n=0}^{\infty} n a_n (x+iy)^{n-1}; \qquad \text{(L8-13)}$$

$$B_x = -\frac{\partial A_s}{\partial y} = -\operatorname{Re}\sum_{n=0}^{\infty} n a_n (x+iy)^{n-1}.$$

As any complicated design, building good magnets is an art form. It requires both good understanding of magnetostatics, electrical and mechanical engineering and sophisticated dedicated 3D programs, such as OPERA3D or similar. Here we are discussing basic principles of the building magnets – so you can develop an intuition about magnets and understand what is reasonable and what is not. Important features of magnetic filed are encapsulated in its integral properties:

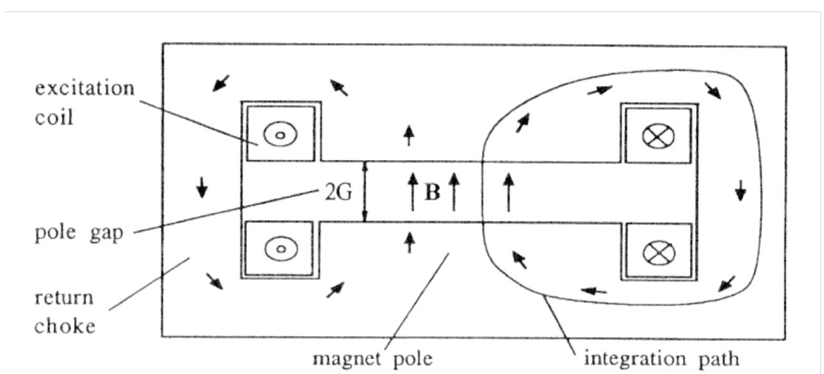
$$\begin{gathered} div\vec{B} = 0 \rightarrow \oint \vec{B} \cdot d\vec{a} = 0; \quad \vec{B} = \mu(\vec{B}) \cdot \vec{H}; \\ curl\vec{H} = \frac{4\pi}{c}\vec{j} \rightarrow \oint \vec{H} \cdot d\vec{l} = \frac{4\pi}{c}\oint \vec{j} \cdot d\vec{a} \equiv \frac{4\pi}{c} I \end{gathered} \tag{L8-14}$$

The first is stating that flux of closed surface is zero – e.g. the flux of magnetic field is returning to its origin. It means that flux pass must be properly defined and design. Frequently it is very desirable to have zero (or at least very low) fields outside the magnet. This is one of the reasons to have a magnetic yoke/core build of magnetic (ferromagnetic) steel with high permeability μ: it makes steel with high μ nearly ideal conductor of the magnetic flux.

Second, $\vec{B} = \mu(\vec{B}) \cdot \vec{H}$ indicates that magnetic permeability can be function of the field. The most important fact that good magnetic steel has very high value of permeability $\mu > 1{,}000$ in the operation range $\leq$ 10 kGs (1 T) and it saturates at about 20 kGs (2 T). Some dedicated materials can have $\mu \sim 10^4$-10^6, but they usually saturate at well below 20 kGs.

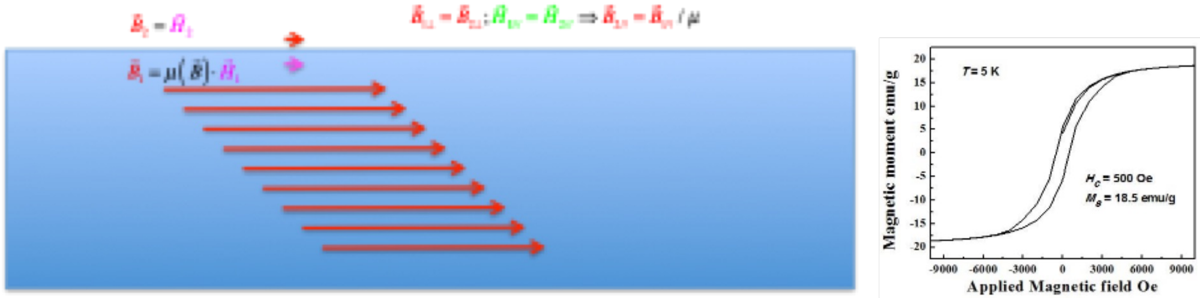
Third, the integral of the magnetic induction over a closed pass (loop) is proportional to the current ***I*** flown through this loop. Sine values of magnetic field's B and H are equal in the vacuum, while H is μ-fold smaller in the steel, one can generate much higher magnetic field with the same amount of current:

$$\oint \frac{\vec{B}}{\mu(\vec{B})} \cdot d\vec{l} = \frac{4\pi}{c} \oint \vec{j} \cdot d\vec{a} \equiv \frac{4\pi}{c} I$$
$$\int_{vacuum} \vec{B} \cdot d\vec{l} + \int_{steel} \frac{\vec{B}}{\mu(\vec{B})} \cdot d\vec{l} = \frac{4\pi}{c} I \qquad \text{(L8-15)}$$



Magnetic susceptibility and permeability data for selected materials

Medium	Susceptibility, volumetric, SI, χ_m	Permeability, μ (H/m)	Relative permeability, μ/μ_0
Metglas 2714A (annealed)		1.26×10^{0}	1 000 000[7]
Iron (99.95% pure Fe annealed in H)		2.5×10^{-1}	200 000[8]
NANOPERM®		1.0×10^{-1}	80 000[9]
Mu-metal		2.5×10^{-2}	20 000[10]
Mu-metal		6.3×10^{-2}	50 000[11]
Cobalt-iron (high permeability strip material)		2.3×10^{-2}	18 000[12]
Permalloy	8000	1.0×10^{-2}	8000[10]
Iron (99.8% pure)		6.3×10^{-3}	5000[8]
Electrical steel		5.0×10^{-3}	4000[10]
Ferritic stainless steel (annealed)		1.26×10^{-3} – 2.26×10^{-3}	1000 – 1800[13]
Martensitic stainless steel (annealed)		9.42×10^{-4} – 1.19×10^{-3}	750 – 950[13]
Ferrite (manganese zinc)		$>8.0 \times 10^{-4}$	640 (or more)
Ferrite (nickel zinc)		2.0×10^{-5} – 8.0×10^{-4}	16 – 640
Carbon steel		1.26×10^{-4}	100[10]



(a) (b)

Fig. L8-1. (a) Illustration of the steel yoke operation. (b) A hysteresis curve for a ferromagnetic.

Forth, for a qualitative analysis one can use an approximation of $\mu \to \infty$. The consequence of these approximations are rather dramatic:

(a) Component of the magnetic field parallel to the surface of steel disappears

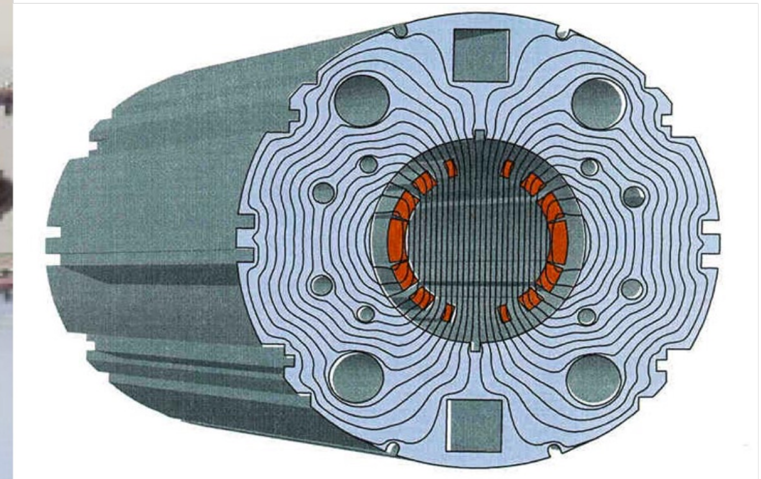
$$\vec{B}_{1\perp} = \vec{B}_{2\perp}; \vec{H}_{1_{//}} = \vec{H}_{2_{//}} \Rightarrow \vec{B}_{2//} = \vec{B}_{1//}/\mu; \qquad \text{(L8-16)}$$
$$\vec{B}_{2//}\Big|_{\mu\to\infty} \to 0.$$

e.g. the magnetic field is perpendicular to the steel. It makes the steel surface to become an equipotential: $\varphi_M\Big|_{surface} = const;$

(b) We can estimate required current in the magnet coil to excite necessary magnetic field by integrating field in the magnet gap:

$$\int_{magnet\ gap} \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I; \quad \int_{magnet\ gap} \vec{B}[Gs]\cdot d\vec{l}\,[cm] = 0.4\pi\cdot I[A]; \qquad \text{(L8-17)}$$
$$\int_{magnet\ gap} \vec{B}[T]\cdot d\vec{l}\,[m] = 0.4\pi\cdot I[MA].$$

with two last equations giving this ratio in practical units. Note that if one uses a multi-turn conductor coil (a typical fit), then the total current is the product of the current in the conductor and the number of turns – engineers call them *ampere-turns*. Now, empowered by these facts, let's consider a typical magnet.



Room temperature magnets with magnetic steel cores. The most popular and wide-spread magnets are using soft magnetic steel for returning magnetic flux.

Dipole magnet. Fig. L8.2 shows a simplified cross-section of a O-type dipole magnet with vertical magnetic field **B**. Two coils carry the current: into the page on the right and out of the page on the left.

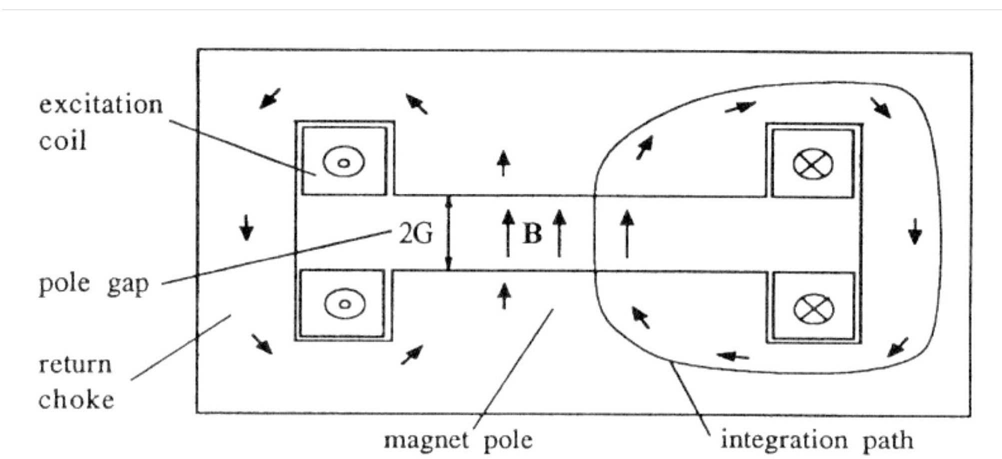


Fig. L8-2. A simplified cross-section of dipole magnet (©: H. Wideman, Particle Acc. Physics)

To have an ideal vertical dipole field we have to consider the poles to be equipotential of

$$\vec{B}_1 = B_y \hat{y}; \;\; \varphi_{M1} = B_y y: \quad \varphi_{M1} = const \rightarrow y = const \tag{L8-18}$$

e.g. two infinite parallel poles. It is worth noticing here that uniform vertical dipole field solution for plane curved coordinate system, where y component remains to be Cartesian.

Having infinitely wide poles is not possible in practice and the poles have final width, with Fig. L8-2 showing a typical design. In addition, designer should be sure that the returning yoke of the magnet is not saturated, since it is carrying back all magnetic flux through the pole and somewhat more from its sides. First question to ask is “what magnetic field in such dipole is?”. With gap between poles g=2G, the average value of the dipole field

$$\bar{B}[Gs] = \frac{\int_{gap} B\,dy}{g} = \frac{0.4\pi \cdot I[A]}{g[cm]}; \tag{L8-19}$$

For example, if we want 15 kGs magnetic field and magnet aperture of 5 cm, we will need to have approximately 60 kA turns in two coils. With current in the conductor of 500 A (typical), we will need 60 turns in each of the coils.

It is natural to expect that far from the pole-edges the magnetic field is uniform and $\vec{B} \cong \hat{y}\bar{B}$. Since

$$div\vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0; curl\vec{B} = -\hat{z}\left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right) = 0$$

any variation of the vertical magnetic field component required corresponding variation of the horizontal component. But in our approximation,

$$B_x\big|_{poles} = 0 \Rightarrow B_x(x, y = \pm G) = 0$$

e.g. we can expend it into Fourier series:

$$B_x(x,y) = \sum_{n=1}^{\infty} B_{xn}(x)\cos\left(2n-1\right)ky; k = \frac{\pi}{g}; \qquad \text{(L8-20)}$$

from where using the field symmetry about x=0 (magnet center) and $\Delta B_x = 0$ we can find

$$B_{xn}(x) = b_{xn}\left(e^{(2n-1)kx} - e^{-(2n-1)kx}\right) = 2b_{xn}\sinh(2n-1)kx \qquad \text{(L8-21)}$$

that perturbations of magnetic field from the pole tips (located at x=±a) are exponentially decay at least as fast as $\exp\left(-\frac{\pi}{g}\left\|x\right| - a\right|\right)$. Thus, by stepping from the edge by two gaps, field errors are reduced by more than factor of 500. In fact, there are number of trick that magnet designers use to make the field even more uniform (called shims) and typically the accuracy of $\Delta B/B \sim 10^{-3}$ can be achieved at the distance of a one gap. Hence, the gap plays important role in the magnet design: with a proper choice of the design nearly uniform field as soon as we step one or two gaps from any edge the magnet's pole inside.

Transport matrix of dipole is very easy to calculate for the case of sector magnet with hard edges, e.g. when

$$B_y = B_o \forall 0 < s < L; \quad B_y = 0, elswhere$$

$$\tilde{h}_n = \frac{\pi_x^2 + \pi_y^2}{2} + f\frac{x^2}{2}; \quad f = K^2 = \left(\frac{eB_o}{pc}\right)^2$$

$$\frac{dx}{ds} = \pi_x; \frac{d\pi_x}{ds} = -fx; \tag{L8-22}$$

$$\frac{dy}{ds} = \pi_y; \quad \frac{d\pi_y}{ds} = 0;$$

which has decoupled x and y motion, with y motion being just a drift:

$$M_x = \begin{pmatrix} \cos Ks & \dfrac{\sin Ks}{K} \\ -K\sin Ks & \cos Ks \end{pmatrix}; M_y = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}; \tag{l8-23}$$

This dipole design corresponds to the center in Fig. 8.3, when the beam enters and exit magnet perpendicular to the edge of the magnet. But it is possible to have variation of bending magnet with either negative or positive exit angle(s).

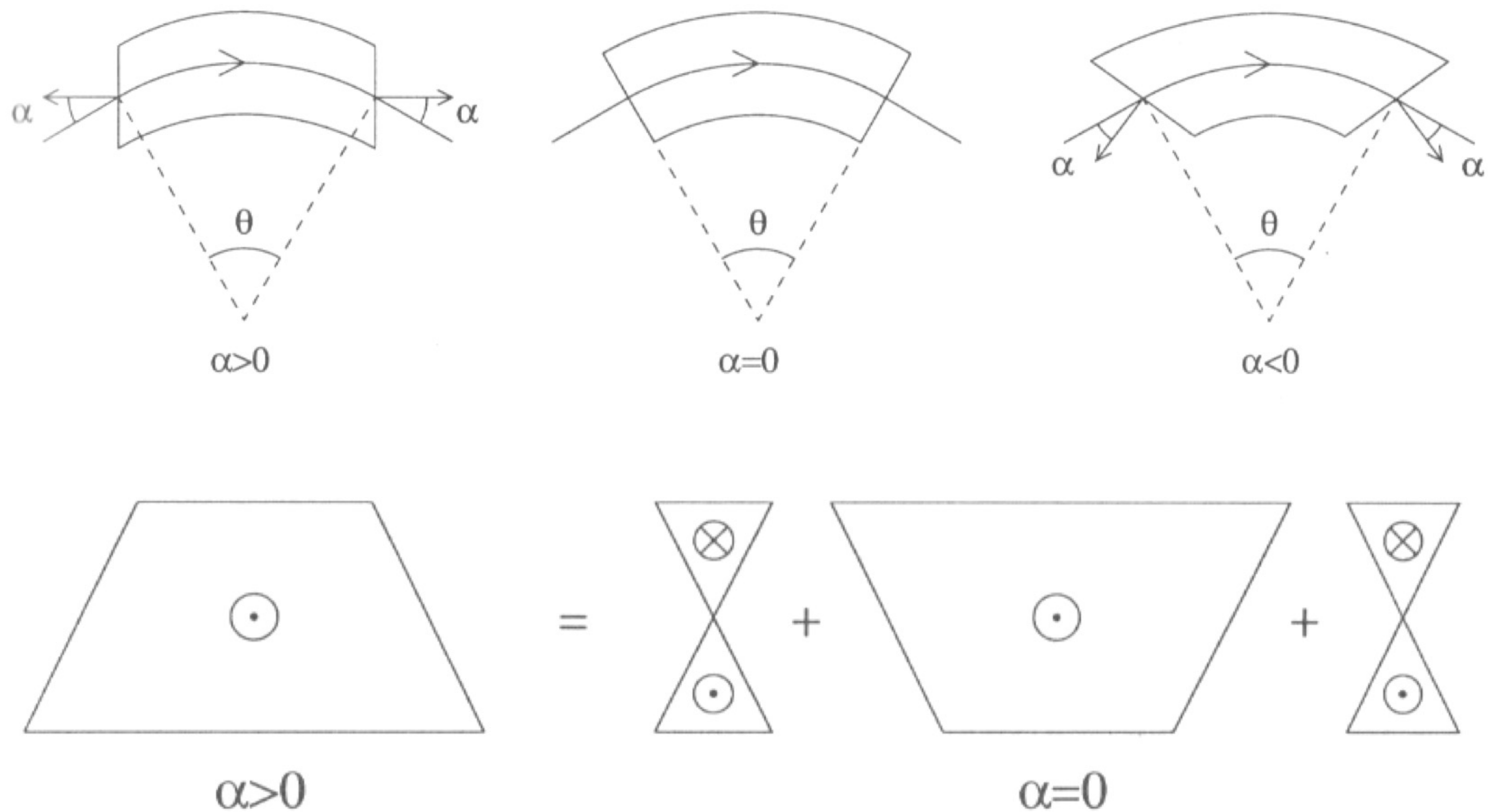


Fig. 8.3. Three typical configurations of bending magnets. (©: W. McKay)

One of the simplest way of considering such edges of the dipole magnet is present it as a combination of the sector magnet plus two short "quadrupoles".

We will return to this case as soon as we have considered quadrupoles:

$$\vec{B}_Q = G\left(\hat{y}\cdot x + \hat{x}\cdot y\right)$$

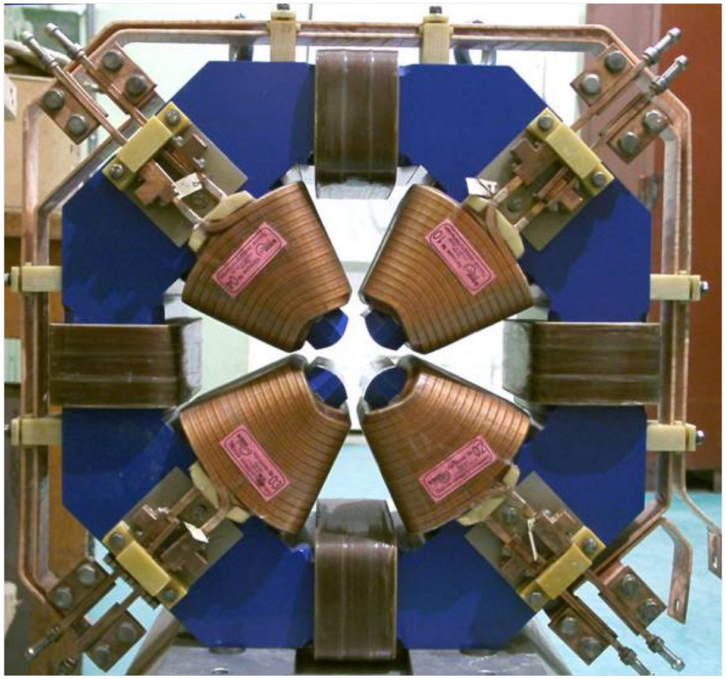
$$\varphi_Q = -Gxy; \quad A_{sQ} = \frac{G}{2}\left(x^2 - y^2\right) \tag{L8-24}$$

with Hamiltonian

$$\tilde{h}_n = \frac{\pi_x^2 + \pi_y^2}{2} + f\left(\frac{x^2}{2} - \frac{y^2}{2}\right); \quad f = -\frac{eG}{pc}$$

$$\frac{dx}{ds} = \pi_x; \frac{d\pi_x}{ds} = -fx; \tag{l8-22}$$

$$\frac{dy}{ds} = \pi_y; \quad \frac{d\pi_y}{ds} = fy;$$



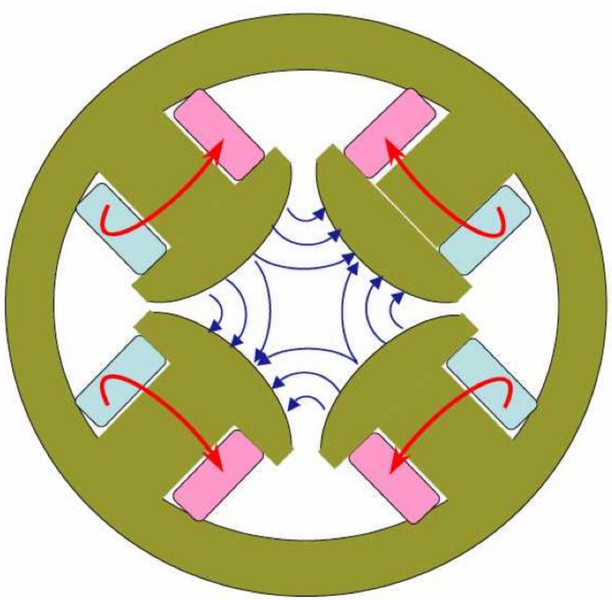
The equipotential surfaces for quadrupole are hyperbolas

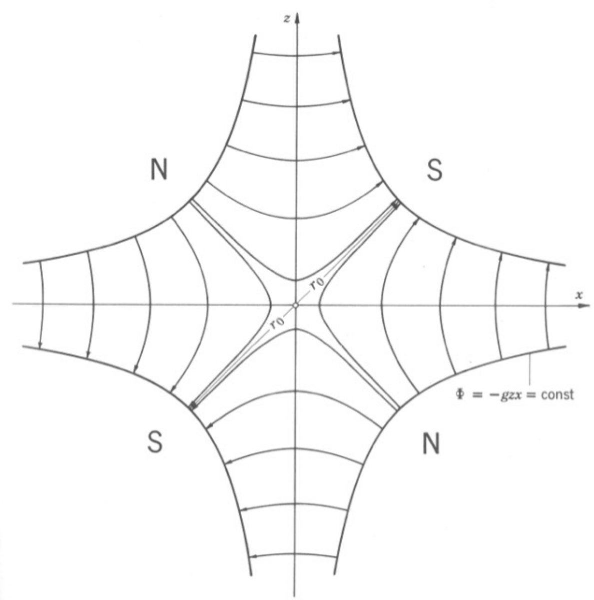
$$xy = const = \pm\frac{r_o^2}{2} \qquad \text{(L8-23)}$$

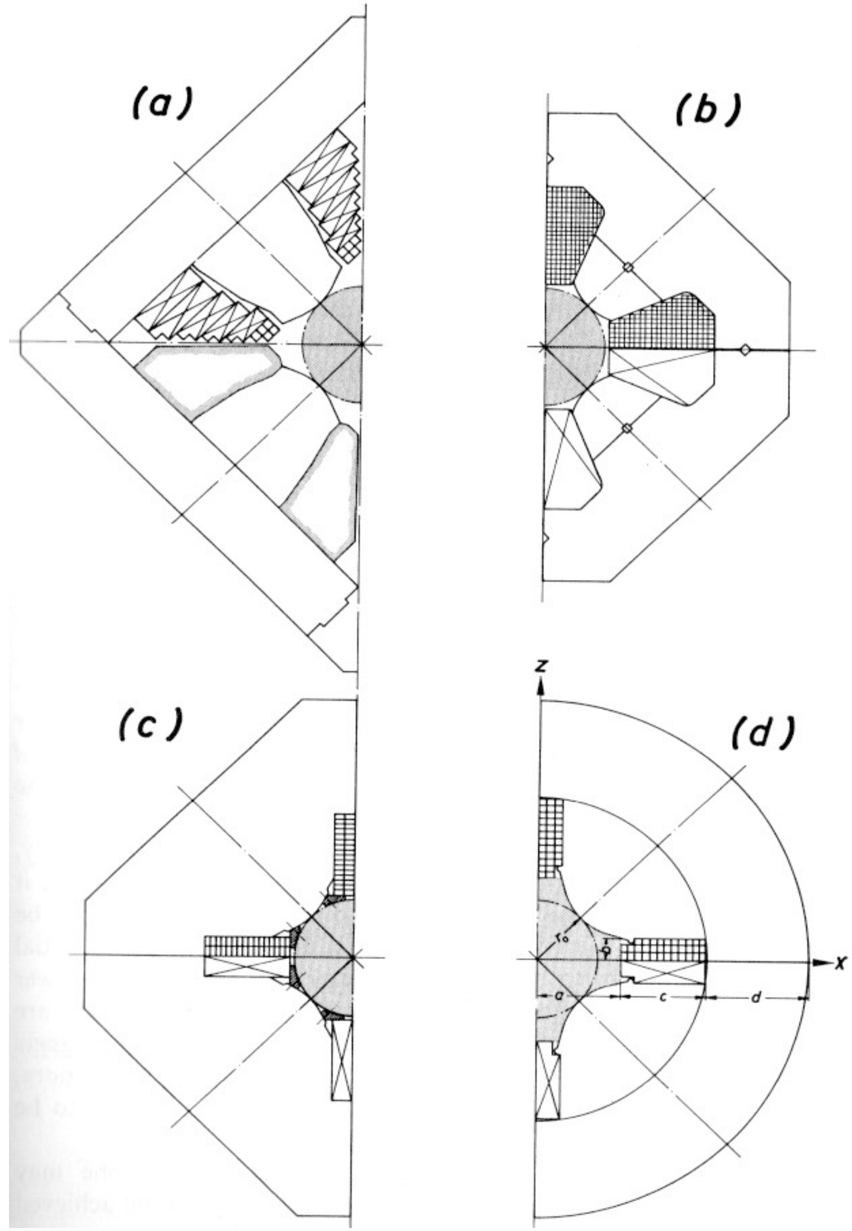
where $\boldsymbol{r_o}$ is the radius of inner circle fit between four quadrupole poles. Again, the poles can not be extended to infinity and typically cut (with some shimming) at distance of few radii. The requirement for the coil current can be again calculated by integrating from zero to the pole:

$$\int_0^{r_o} B_r dr = \int_0^{r_o} Grdr = G\frac{r_o^2}{2} = 0.4\pi \cdot I \Rightarrow G[Gs/cm] = \frac{0.8\pi \cdot I[A]}{r_o^2[cm]} \qquad \text{(L8-24)}$$

Getting 3 kGs/cm gradient for quadrupole with aperture of 4 cm ($\boldsymbol{r_o}$ =2cm) one will need four coils with approximately 4.8 kA turns each.






(a)
(b)
(c)
(d)
z
x
r_0
a
c
d

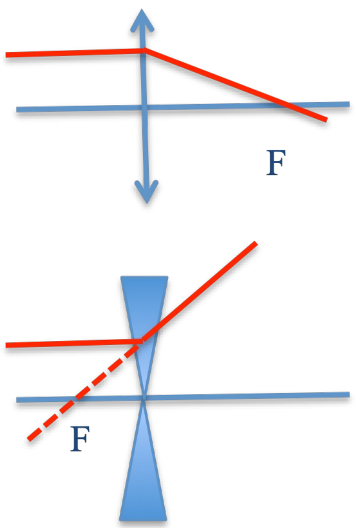
The matrix of hard edge quadrupole was derived in last class:

$$D_x = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix}; D_y = \begin{bmatrix} 0 & 1 \\ K_1 & 0 \end{bmatrix}; K_1 = -\frac{e}{p_o c}\frac{\partial B_y}{\partial x} = -\frac{eG}{p_o c}$$

$$\begin{pmatrix} x \\ \pi_1 \end{pmatrix} \equiv \begin{pmatrix} x \\ x' \end{pmatrix}; \begin{pmatrix} y \\ \pi_3 \end{pmatrix} \equiv \begin{pmatrix} y \\ y' \end{pmatrix};$$

$$\left\{ \begin{array}{l} K_1 > 0; \ \ M_x = M_F; M_y = M_D \\ K_1 < 0; \ \ M_x = M_D; M_y = M_F \end{array} \right\}; \ \omega = \sqrt{|K|}; \varphi = \omega s; \tag{7-23}$$

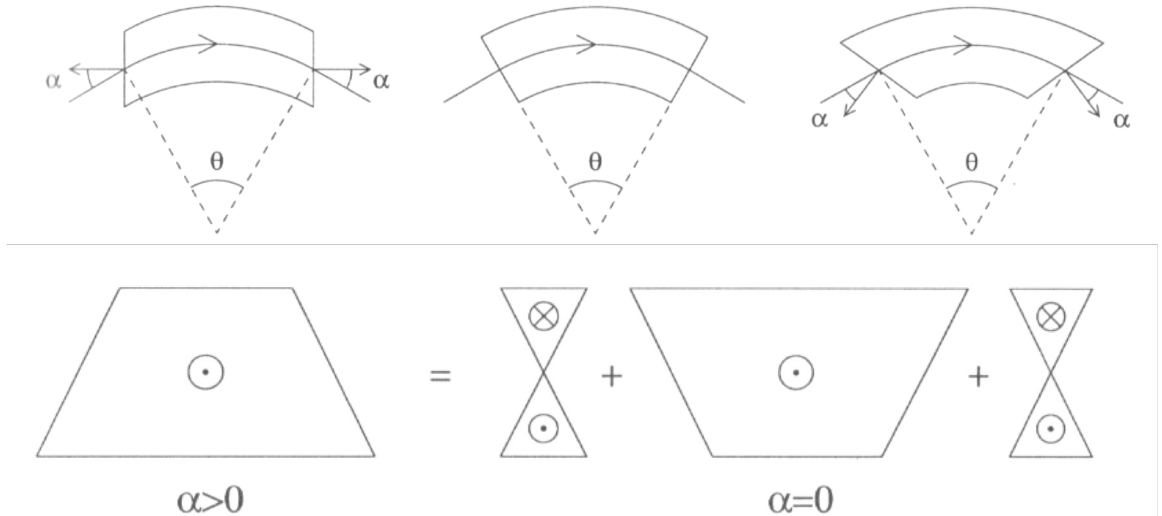
$$M_F = \begin{bmatrix} \cos\varphi & \dfrac{\sin\varphi}{\omega} \\ -\omega \sin\varphi & \cos\varphi \end{bmatrix}; M_D = \begin{bmatrix} \cosh\varphi & \dfrac{\sinh\varphi}{\omega} \\ \omega \sinh\varphi & \cosh\varphi \end{bmatrix};$$



which means that quadrupole focuses in one direction and defocuses in the other. We will see in one future classes that combination of focusing and defocusing quadrupoles (quads) can focus in both direction, so call strong focusing.

For a short quadrupole $\varphi << 1; \quad q = G \cdot l = fixed$ we can rewrite the matrix as

$$M_x = \begin{bmatrix} 1 & 0 \\ -\dfrac{eG}{p_o c} l & 1 \end{bmatrix}; M_y = \begin{bmatrix} 1 & 0 \\ +\dfrac{eG}{p_o c} l & 1 \end{bmatrix} \tag{7-24}$$



Now we ready to address the edge effect in the dipole: the integrated angular kick in x direction is equal to

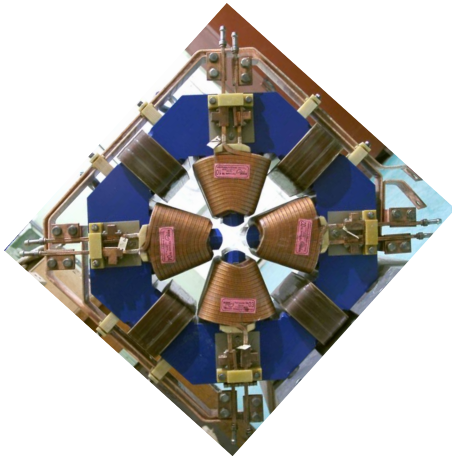
$$\Delta x' \equiv \Delta \pi_x = K \cdot x \cdot \tan\alpha; \tag{7-25}$$

Hence

$$M_{xedge} = \begin{bmatrix} 1 & 0 \\ K \cdot \tan\alpha & 1 \end{bmatrix}; M_{yedge} = \begin{bmatrix} 1 & 0 \\ -K \cdot \tan\alpha & 1 \end{bmatrix};$$

$$M_{x\ \alpha-dipole} = M_{xedge} \begin{pmatrix} \cos Kl & \dfrac{\sin Kl}{K} \\ -K\sin Kl & \cos Kl \end{pmatrix} M_{xedge};\ \ M_{y\ \alpha-dipole} = M_{xedge} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} M_{xedge}.$$

(7-26)

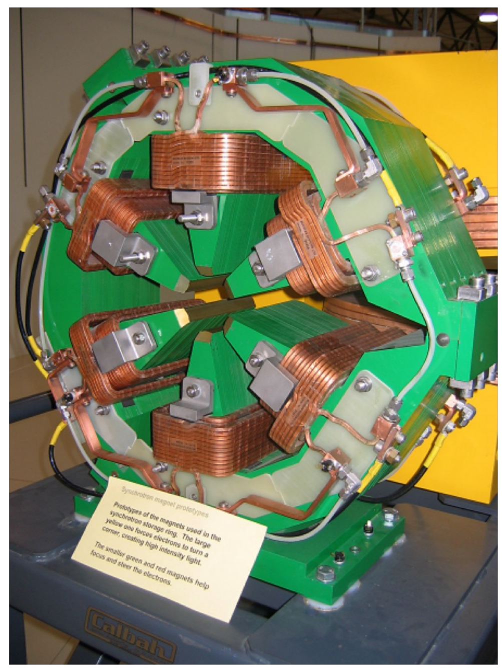


Skew-quadrupoles is nothing else as a quadrupole turned 45-degrees. It naturally couples x and y motion. It means that its matrix can be calculated by rotating 45-degrees, going through quadrupole (4x4 matrix of quad), rotating back (-45 degrees):

$$M_{SQ} = \frac{1}{2}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}\begin{bmatrix} M_{xQ} & 0 \\ 0 & M_{yQ} \end{bmatrix}\begin{bmatrix} I & -I \\ I & I \end{bmatrix} = \frac{1}{2}\begin{bmatrix} M_{xQ}+M_{yQ} & M_{yQ}-M_{xQ} \\ M_{yQ}-M_{xQ} & M_{xQ}+M_{yQ} \end{bmatrix} \quad (7\text{-}26)$$

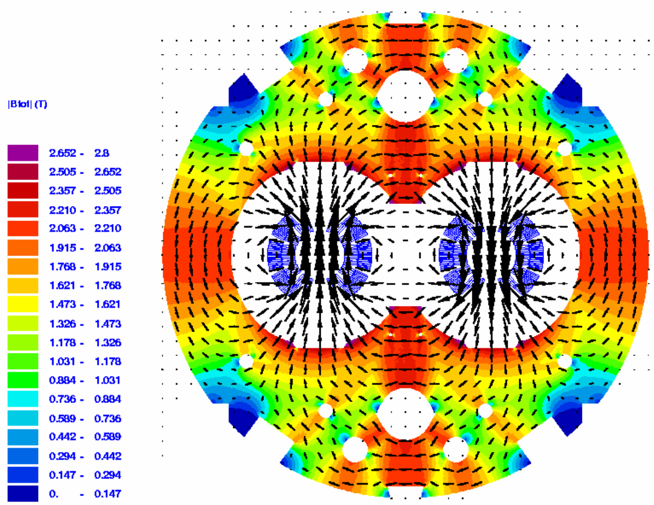
Higher order multipoles. Building higher order multipole magnets, like sextupoles, follows the same logic. Use corresponding number of poles and follow the equipotential.

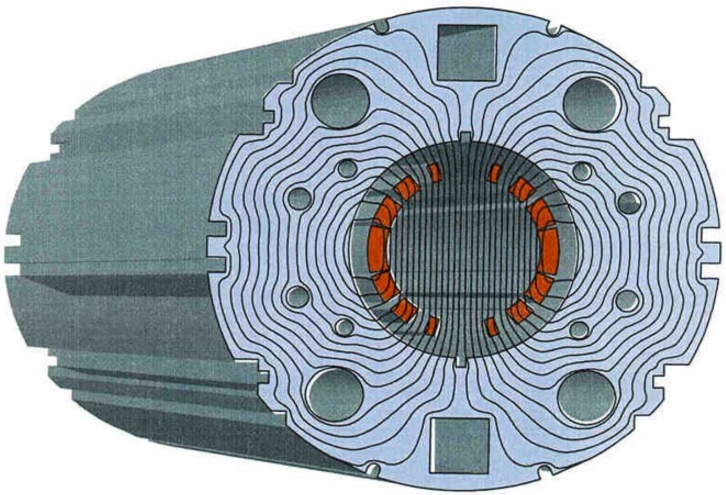
Sextupole



Octupole

Superconducting and air-coil magnets: $I_n = I \cdot \cos(n\varphi)$





How sin/cos coils work?

- By the nature of cylindrical system, it is natural that angular harmonics of current will excite the same harmonic in magnetic field

$$\vec{j} = \delta(\rho - R)\mathrm{Re}\left(\frac{I}{4R}e^{in\theta}\right) \to \vec{B} = \mathrm{Re}\left(\vec{B}(\rho)e^{in\theta}\right)$$

- Outside the coil the field satisfied equations in vacuum and we already know solutions

$$\varphi_n = \mathrm{Re}\left(a_n\rho^n + \frac{b_n}{\rho^n}\right)e^{in\theta} \to \vec{H} = \vec{\nabla}\varphi_n = n\cdot\mathrm{Re}\left\{\hat{\rho}\left(a_n\rho^{n-1} - \frac{b_n}{\rho^{n+1}}\right) + i\hat{\theta}\left(a_n\rho^{n-1} + \frac{b_n}{\rho^{n+1}}\right)\right\}e^{in\theta}$$

- Which mist be finite inside and outside the coil

$$\vec{H}_{in} = \mathrm{Re}\,A_n(\hat{\rho} + i\hat{\varphi})\rho^{n-1}e^{in\theta},\ \vec{H}_{out} = \mathrm{Re}\,B_n(\hat{\rho} - i\hat{\varphi})\rho^{-n-1}e^{in\theta}$$

- And using continuity equation for radial component and integral equation for the angular component we can connect to the current in the coil.

$$\hat{\rho}\cdot\vec{H}_{in}(R) = \hat{\rho}\cdot\vec{H}_{in}(R) \Rightarrow \mathrm{Re}\,A_n R^{n-1}e^{in\theta} = \mathrm{Re}\,B_n R^{-n-1}e^{in\theta} \to B_n = A_n R^{2n} = H_o R^{n+1}$$

$$\vec{H}_{in} = \mathrm{Re}\,H_o(\hat{\rho} + i\hat{\varphi})\left(\frac{\rho}{R}\right)^{n-1}e^{in\theta},\ \vec{B}_{out} = \mathrm{Re}\,H_o(\hat{\rho} - i\hat{\varphi})\left(\frac{R}{\rho}\right)^{n+1}e^{in\theta}$$

$$\oint \vec{B}\,d\vec{l} = \frac{4\pi}{c}\oint \vec{j}\cdot d\vec{A};\ d\vec{A} = \hat{z}\rho d\theta d\rho = \hat{z}Rd\theta d\rho \to \frac{4\pi}{c}\oint \vec{j}\cdot d\vec{A} = \frac{\pi}{c}\Delta\theta\cdot\mathrm{Re}\,I_o e^{in\theta}$$

$$\left(\hat{\theta}\vec{B}_{in}(R-\varepsilon) + \hat{\theta}\cdot\vec{B}_{in}(R+\varepsilon)\right)R\Delta\theta = \frac{\pi}{c}\Delta\theta\cdot\mathrm{Re}\,I_o e^{in\theta}$$

How sin/cos coils work?

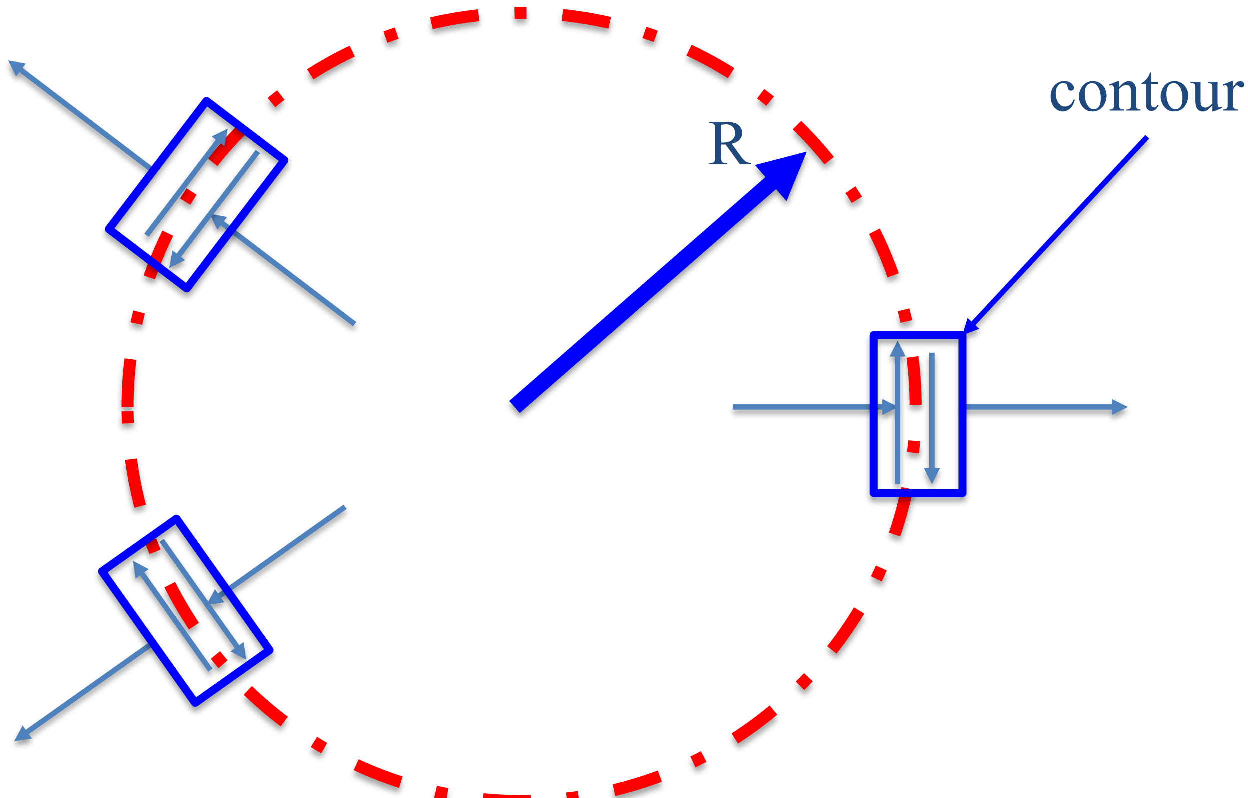
- Infinitesimally small contour around the piece of the coil

$$\hat{\theta}\cdot\left(\vec{B}_{in}\left(R-\varepsilon\right)-\vec{B}_{in}\left(R+\varepsilon\right)\right)R\Delta\theta=\frac{\pi}{c}\Delta\theta\cdot\mathrm{Re}\,I_o e^{in\theta}$$

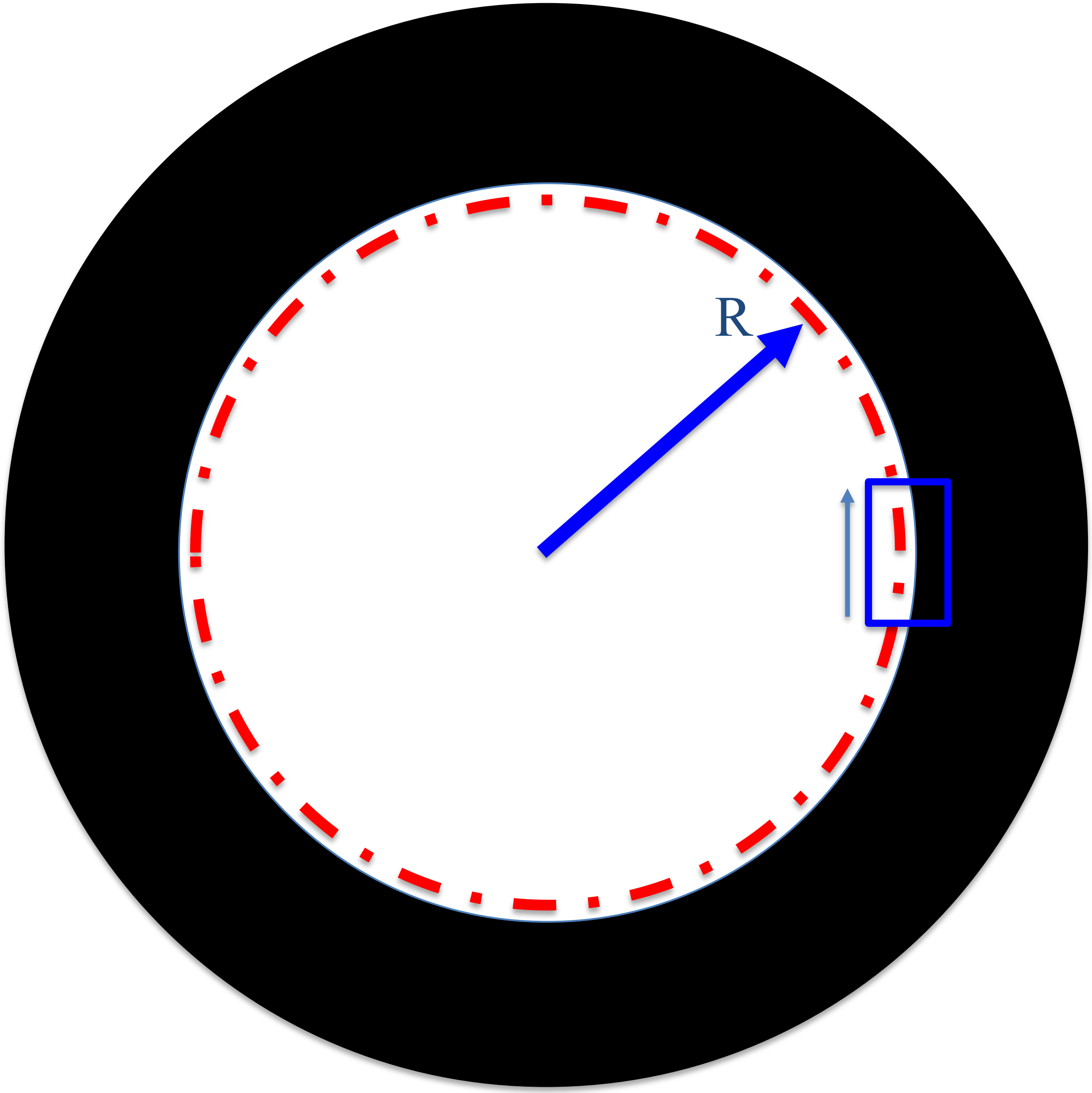
$$\hat{\theta}\cdot\left(\vec{B}_{in}\left(R_-\right)-\vec{B}_{in}\left(R_+\right)\right)=\frac{\pi}{c}\mathrm{Re}\,I_o e^{in\theta}$$

....

- You have to solve this in your homework…



Sin/Cos coils inside the magnetic yoke



s-dependence: Note that we can add only a linear z-dependence to the multipole's coefficients

$$\Delta\sum_{n=0}^{\infty} a_n(z)(x+iy)^n = \sum_{n=0}^{\infty} \frac{\partial^2 a_n(z)}{\partial^2 z}(x+iy)^n = 0 \rightarrow \frac{\partial^2 a_n(z)}{\partial^2 z} \equiv 0 \rightarrow a_n(z) = \alpha_n + \beta_n z; \quad \text{(L8-27)}$$

which does not allow to make a magnet with zero field outsize it without introducing terms which do not have form of multipoles. This is the nasty feature of magnet's edge fields; which accelerator designers are studying for years. The fields are always nonlinear, and we will consider their inclusion when we switch to nonlinear beam dynamics.