# PHY 554
# Fundamentals of Accelerator Physics
# Lecture 22

# Hadron Beam Cooling


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# Outline

- Introduction
  - What is beam cooling?
  - Why cooling?
- Electron Cooling
  - Non-magnetized electron cooling
- Stochastic Cooling
  - Transverse stochastic cooling
- Coherent electron cooling

# Introduction

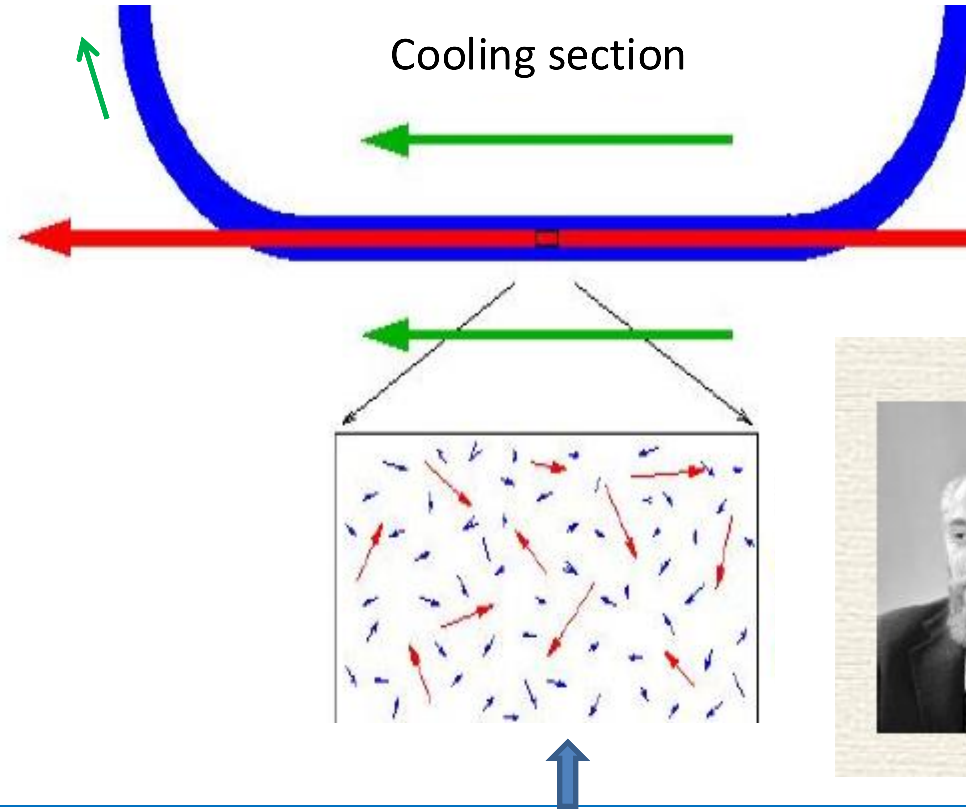
- Beam cooling is to reduce beam temperature, i.e. phase space volume, emittance and momentum spread.
- Beam cooling processes does not violate Liouville's theorem since it involves non-conservative forces, which typically depends on the velocities of the particles to be cooled.

$$F_{x,y,s} = -\alpha_{x,y,s} v_{x,y,s}$$

# Introduction

- Why is beam cooling needed?
  - Improve beam quality (higher density, smaller angular spread): higher luminosity in collider, brighter radiation in light source
  - Compensation of heating: IBS, beam-gas scattering, beam-beam interaction
  - Beam accumulation (antiprotons, rare isotopes): accumulate more beam from a weak particle source

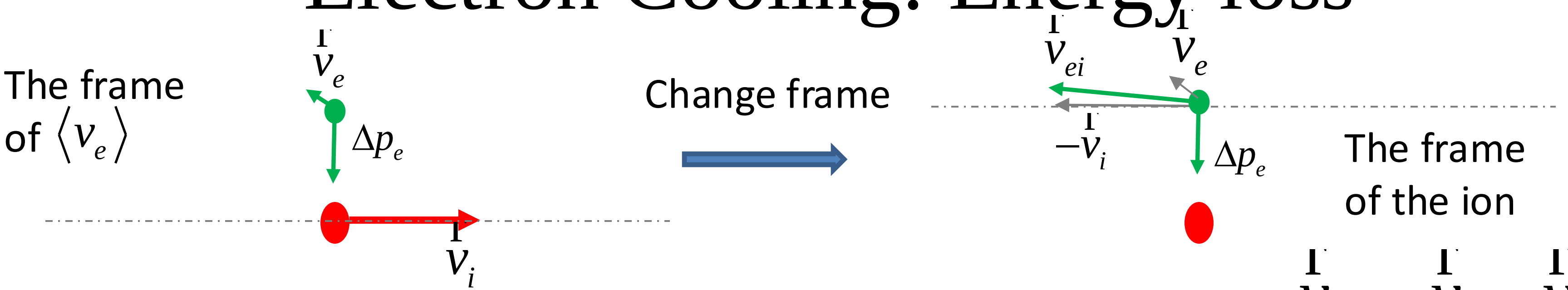
# Electron Cooling: introduction




**Electron cooling was invented by Gersh Budker in 1966**

In the beam frame, cold electrons interact with warm ions and take heat away from the ions.

# Electron Cooling: Energy loss




$$\vec{v}_{ei} = \vec{v}_e - \vec{v}_i$$

1. Derive $\Delta p_e$ in ion's frame

2. $\Delta p_e$ is also the momentum change in the frame of $\langle v_e \rangle$

3. Get the energy change of the electron in the frame of $\langle v_e \rangle$ from

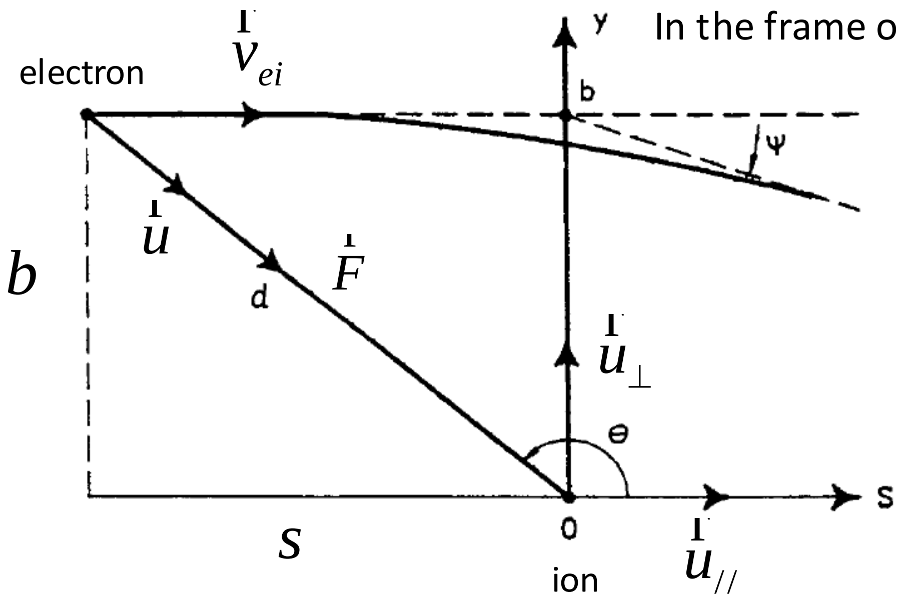
$$\Delta E_e = \frac{(p_{e0} + \Delta p_e)^2 - p_{e0}^2}{2m_e} \approx \frac{\Delta p_e^{\ 2}}{2m_e} \quad \text{for } p_{e0} << \Delta p_e \text{ (cold electrons)}$$

4. Get the energy loss of the ion in the frame of $\langle v_e \rangle$ from

$$\Delta E_{i,loss} = \sum_j \Delta E_{e,j} = \sum_j \frac{\Delta p_e^{\ 2}}{2m_e}$$

since the total energy is conserved.

# Electron Cooling: Energy loss




In the frame of the moving ion

The energy gained by the electron is equal to the energy loss by ion.

$$\vec{F} = \frac{Ze^2}{4\pi\varepsilon_0 d^2}\vec{u}$$

$$\vec{v}_{ei} = \vec{v}_e - \vec{v}_i$$

Assuming $\psi$ is small $\quad \Delta\vec{p} = \int_{-\infty}^{\infty} \vec{F}dt \approx \frac{1}{v_{ei}} \int_{-\infty}^{\infty} \vec{F}ds$

$$\Delta p_{//} \approx \frac{1}{v_{ei}} \int_{-\infty}^{\infty} F_{//}ds = -\frac{1}{v_{ei}} \int_{-\infty}^{\infty} F\cos\theta ds = -\frac{1}{v_{ei}} \int_{-\infty}^{\infty} \frac{Ze^2}{4\pi\varepsilon_0\left(s^2+b^2\right)} \frac{s}{\sqrt{s^2+b^2}} ds = 0$$

$$\Delta p_{\perp} \approx -\frac{1}{v_{ei}} \int_{-\infty}^{\infty} F\sin\theta ds = -\frac{1}{v_{ei}} \int_{-\infty}^{\infty} \frac{Ze^2 b}{4\pi\varepsilon_0\left(s^2+b^2\right)^{3/2}} ds = -\frac{1}{v_{ei}} \frac{2Ze^2 b}{4\pi\varepsilon_0 b^2} \int_{0}^{\infty} \frac{d}{ds}\left[\frac{s}{\left(s^2+b^2\right)^{1/2}}\right] ds = -\frac{1}{v_{ei}} \frac{2Ze^2}{4\pi\varepsilon_0 b}$$

# Electron Cooling: Energy loss

- Back to the frame of the electrons' average velocity, the electron's energy gain is

$$\Delta E(b) \approx \frac{\Delta p_{\perp}^2}{2m_e} = \frac{2Z^2 e^4}{m_e {v_{ei}}^2 (4\pi\varepsilon_0)^2 b^2}$$

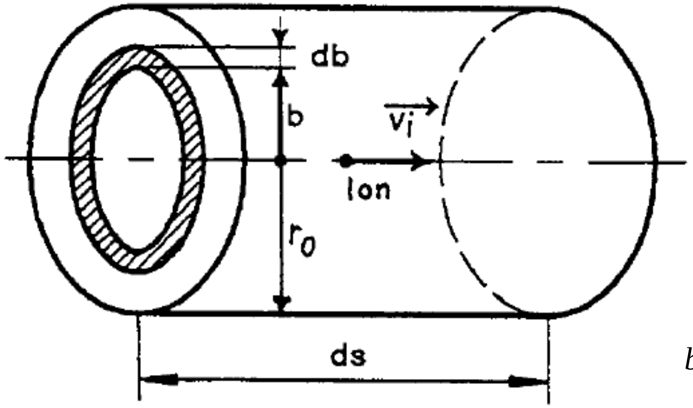
which has to be the energy loss by the ion.

$$\Delta E_{loss}(b) = \frac{2Z^2 e^4}{m_e {v_{ei}}^2 (4\pi\varepsilon_0)^2 b^2}$$

Energy loss by a moving ion due to its interaction with one electron sitting at impact parameter b.

# Electron Cooling: friction force

Energy loss by a moving ion due to its passing through a slice of electrons with velocity $\vec{v}_e$. (assuming electron spatial density is a constant)




$$b_{\min} = \frac{Ze^2}{4\pi\varepsilon_0 m_e v_i^2}$$

$b_{\max}$ ~ Debye length

$$L_c \equiv \ln\left(\frac{b_{\max}}{b_{\min}}\right) \approx 10$$

$$\Delta E_{slice} = 2\pi \Delta s n_e \int_{b_{\min}}^{b_{\max}} b \Delta E_{loss}(b)\, db$$

Coulomb Logarithm: $L_c$

Ion energy loss rate:
$$\frac{dE}{ds} = 2\pi n_e \int_{b_{\min}}^{b_{\max}} b \Delta E_{loss}(b)\, db = \frac{4\pi n_e Z^2 e^4}{m_e v_{ei}^2 (4\pi\varepsilon_0)^2} \ln\left(\frac{b_{\max}}{b_{\min}}\right)$$

Since the energy loss slows down the ion in the frame of the electron's initial velocity, it is equivalent to a friction force in the direction of $\vec{v}_e - \vec{v}_i$ and can be defined as

$$\vec{F} = \frac{dE}{ds} \frac{\vec{v}_e - \vec{v}_i}{|\vec{v}_e - \vec{v}_i|} = \frac{4\pi n_e Z^2 e^4 L_c}{m_e (4\pi\varepsilon_0)^2} \frac{\vec{v}_e - \vec{v}_i}{|\vec{v}_e - \vec{v}_i|^3}$$

# Electron Cooling: friction force

- The friction force due to electrons with velocity distribution $f(\vec{v}_e)$ is

$$\vec{F} = \frac{4\pi n_e Z^2 e^4 L_c}{m_e (4\pi\varepsilon_0)^2} \int_{-\infty}^{\infty} \frac{\vec{v}_e - \vec{v}_i}{\left|\vec{v}_e - \vec{v}_i\right|^3} f(\vec{v}_e)\, d^3 v_e$$

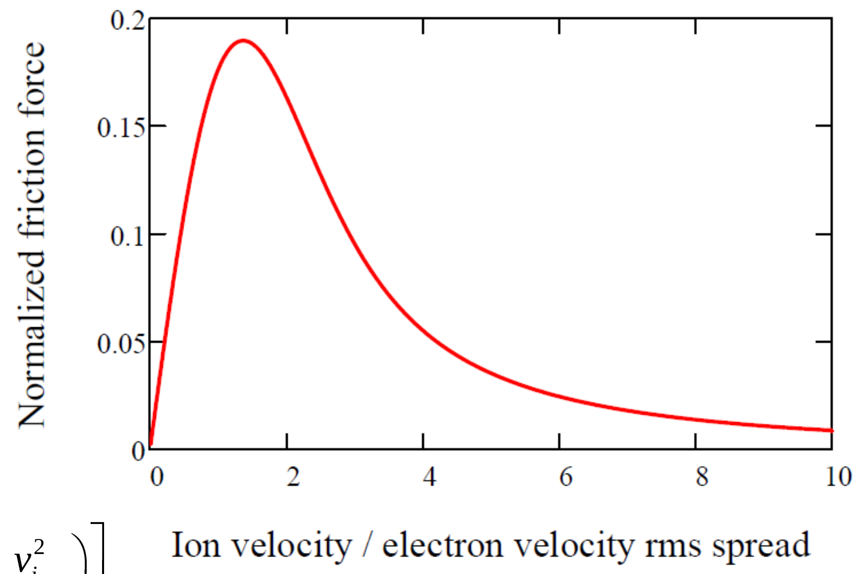
Similar to Coulomb force but in velocity space.

For isotropic Gaussian electron velocity distribution,

$$f(\vec{v}_e) = \exp\left(-\frac{v_e^2}{2\sigma_{ve}^2}\right)$$

this integral can be carried out

$$\vec{F} = -\left(\frac{\vec{v}_i}{v_i}\right)\frac{n_e Z^2 e^4 L_c}{2 m_e \pi^{3/2} \varepsilon_0^2 \sigma_{ve}^2} \cdot \frac{\sigma_{ve}^2}{v_i^2}\left[\frac{\sqrt{\pi}}{2} erf\left(\sqrt{\frac{v_i^2}{2\sigma_{ve}^2}}\right) - \sqrt{\frac{v_i^2}{2\sigma_{ve}^2}} \exp\left(-\frac{v_i^2}{2\sigma_{ve}^2}\right)\right]$$



# Electron Cooling: friction force

For anisotropic Gaussian distribution of electrons' velocity, numerical integration is required. However, the asymptotic solution has been derived.

$$f(v_e)=\frac{e^{-\left[\frac{v_{e\parallel}^2}{2\Delta_{e\parallel}^2}+\frac{v_{e\perp}^2}{2\Delta_{e\perp}^2}\right]}}{(2\pi)^{3/2}\,\Delta_{e\perp}^2\,\Delta_{e\parallel}}$$

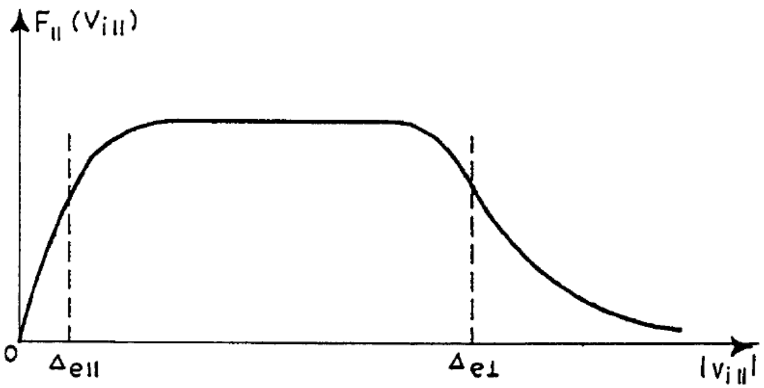



Fig. 5.6 Shape of the "non-magnetised" longitudinal cooling force

*In the longitudinal direction* ($v_{i\perp}=0$) (Fig. 5.6)

$$F_\parallel(v_{i\parallel})=-\frac{4\pi Z^2\cdot e^4}{m_e}n_e\cdot L_c\begin{cases}\frac{1}{v_{i\parallel}^2} & ; \quad |v_{i\parallel}|>>\Delta_{e\perp}\\ \frac{1}{\Delta_{e\perp}^2} & ; \quad \Delta_{e\perp}>>|v_{i\parallel}|>>\Delta_{e\parallel}\\ \frac{v_{i\parallel}}{(2\pi)^{3/2}\,\Delta_{e\perp}^2\,\Delta_{e\parallel}} & ; \quad |v_{i\parallel}|<<\Delta_{e\parallel}\end{cases}$$

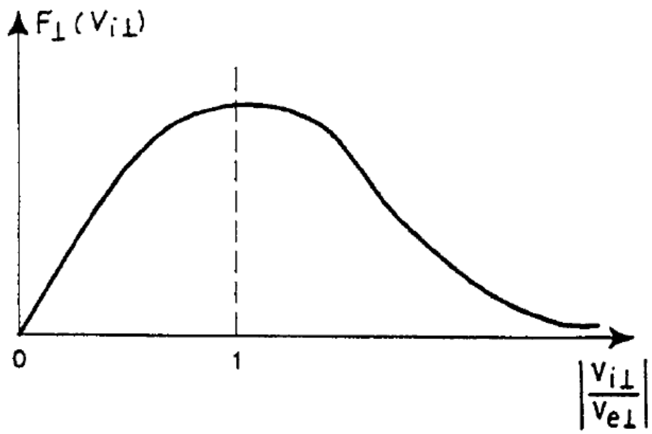



Fig. 5.7 Shape of the "non-magnetised" transverse cooling force

*In the transverse direction* ($v_{i\parallel}=0$) (Fig. 5.7)

$$F(v_{i\perp})=-\frac{4\pi Z^2.\,e^4}{m_e}n_e\cdot L_c\begin{cases}\frac{1}{v_{i\perp}^2} & ; \quad |v_{i\perp}|>>\Delta_{e\perp}\\ \frac{\sqrt{\pi}}{8}\frac{v_{i\perp}}{\Delta_{e\perp}^3} & ; \quad |v_{i\perp}|<<\Delta_{e\perp}\end{cases}$$

We observe that:

- the forces are not independent of the ion relative velocities
- for large ion velocities the forces scale as $1/(v_i^{\,2})$, suggesting that a beam with a relatively large emittance will have a large cooling time
- for small velocities the forces are proportional to $v_i$.

# Electron Cooling: cooling rate

$$\frac{1}{\tau_{v_i}} \equiv -\frac{1}{v_i}\frac{dv_i}{dt} = -\frac{1}{p_i}\frac{dE_i}{ds} = -\frac{F(v_i)}{p_i}$$

If $\tau_{v_i}$ is independent of $t$ → $v_i(t) = v_i(0)\exp\left(-\frac{t}{\tau_{v_i}}\right)$

In lab frame: $\tau_{v_i,lab} = \gamma_0 \tau_{v_i} = -\gamma_0 \frac{m_i v_i}{F(v_i)}$

$$\vec{F} = \frac{4\pi n_e Z^2 e^4 L_c}{m_e (4\pi\varepsilon_0)^2} \int_{-\infty}^{\infty} \frac{\vec{v}_e - \vec{v}_i}{|\vec{v}_e - \vec{v}_i|^3} f(\vec{v}_e) d^3 v_e$$

$$f(\vec{v}_e) = \delta(\vec{v}_e)$$

For cold beam: $\vec{F} = -\frac{4\pi Z^2 e^4 n_e L_c}{m_e (4\pi\varepsilon_0)^2} \frac{\vec{v}_i}{|v_i|^3}$

$$F(v_i) = -\frac{4\pi Z^2 e^4 n_e L_c}{m_e (4\pi\varepsilon_0)^2} \frac{1}{|v_i|^2}$$

$$\tau_{v_i,lab} = -\gamma_0 \frac{m_i v_i}{F(v_i)} = \gamma_0^2 \frac{m_i m_e (4\pi\varepsilon_0)^2 |v_i|^3}{4\pi Z^2 e^4 n_{e,lab} L_c} = \gamma_0^2 \frac{m_i m_e (4\pi\varepsilon_0)^2 |v_i|^3 ecS}{4\pi Z^2 e^4 L_c} \frac{1}{S n_{e,lab} ec}$$

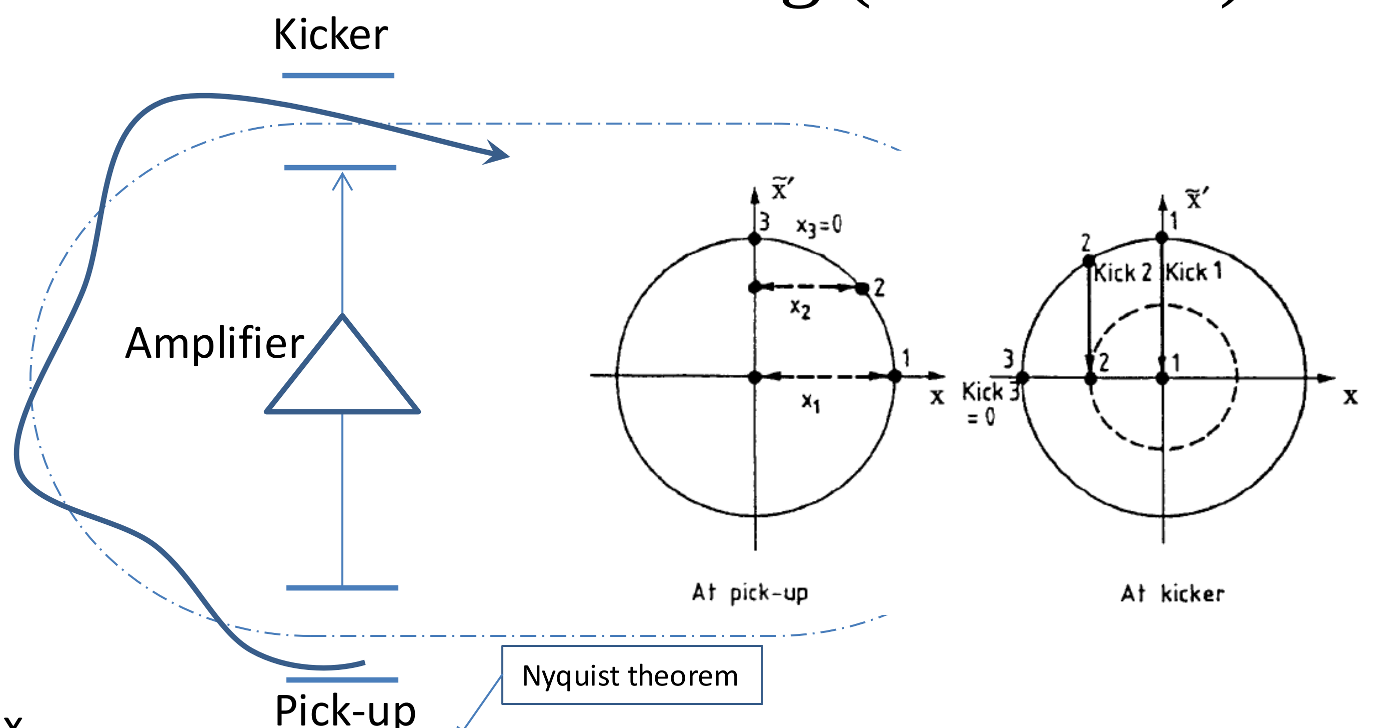
$$= \gamma_0^5 \frac{m_i m_e (4\pi\varepsilon_0)^2 c^3 ecS}{4\pi Z^2 e^4 L_c} \left( \frac{m_i^3 |v_i|^3}{m_i^3 c^3 \gamma_0^3} \right) \frac{1}{I_b}$$

$$= \gamma_0^5 \frac{m_i m_e (4\pi\varepsilon_0)^2 c^4 Se}{4\pi Z^2 e^4 L_c} \theta_i^3 \frac{1}{I_b} = \gamma_0^5 \frac{A}{Z^2} \frac{Se}{4\pi r_e r_p L_c} \theta_i^3 \frac{1}{I_b}$$

Cooling time increases dramatically with energy. Therefore, electron cooling is not efficient to cool high energy beam.

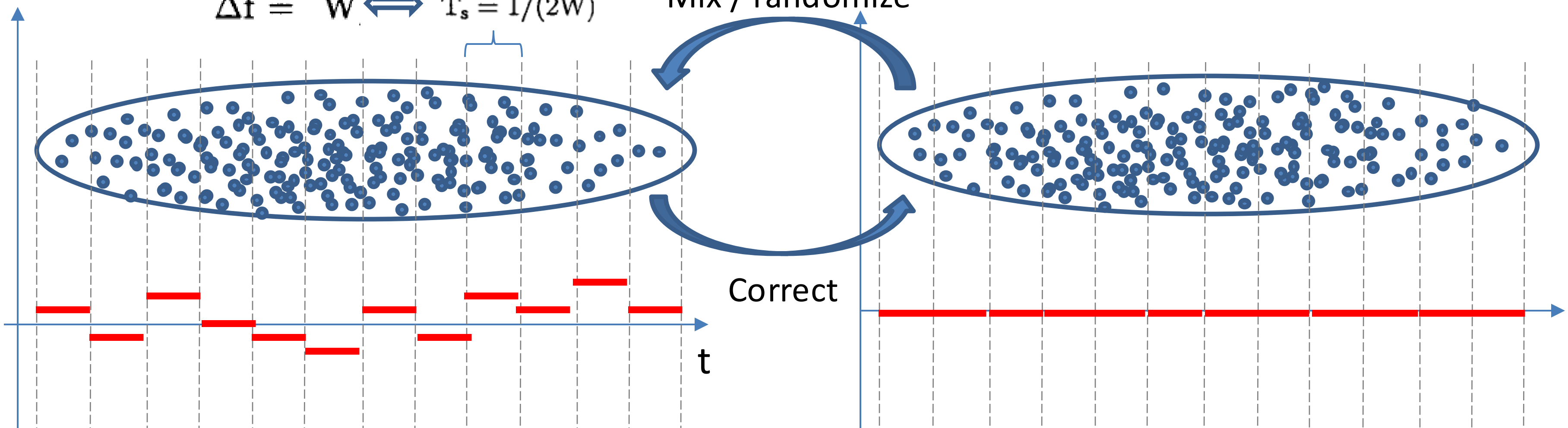
$S$: area of beam cross section

# Stochastic Cooling (transverse): Introduction




Stochastic cooling was invented by Simon van der Meer in 1972.

$$\Delta f = W \Leftrightarrow T_s = 1/(2W)$$



# Stochastic Cooling: cooling rate

For the $i^{th}$ particle, its transverse offset after one correction is

$$x_{c,i} = x_i - g\langle x\rangle_s \qquad \langle x\rangle_s = \frac{1}{N_s}\sum_{j=1}^{N_s} x_j$$

$$x_{c,i}^2 - x_i^{\,2} = -2g x_i\langle x\rangle_s + g^2\langle x\rangle_s^2$$

$$\left\langle x_c^2\right\rangle_s - \left\langle x^2\right\rangle_s = \left(-2g + g^2\right)\langle x\rangle_s^2$$

← One turn kick is random in nature.

For very large number of turns, we need to find out the average correction per turn, i.e. the expectation value

$$E\left(\left\langle x_c^2\right\rangle_s\right) - E\left(\left\langle x^2\right\rangle_s\right) = \left(-2g + g^2\right)E\left(\langle x\rangle_s^2\right)$$

# Stochastic Cooling: cooling rate

- For random samples drawn from a distribution, the expectation value of the variance of the sample is the same as the variance of the distribution

$$E\left(\left\langle x^2 \right\rangle_s\right) = \left\langle x^2 \right\rangle$$

- Central limit theory

Let $\{X_1, X_2 \ldots X_n\}$ be a random sample of size n drawn from distributions of expected values given by μ and variances given by $\sigma^2$. For large enough n, the distribution of the sample average

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is close to the normal distribution with mean μ and variance $\sigma^2/n$.

Thus the expectation value of the sample average is

$$E\left(\left\langle x \right\rangle_s^2\right) = \frac{\left\langle x^2 \right\rangle}{N_s}$$

# Stochastic Cooling: cooling rate

$$E\left(\left\langle x_c^2\right\rangle_s\right)-E\left(\left\langle x^2\right\rangle_s\right)=\left(-2g+g^2\right)E\left(\left\langle x\right\rangle_s^2\right)$$

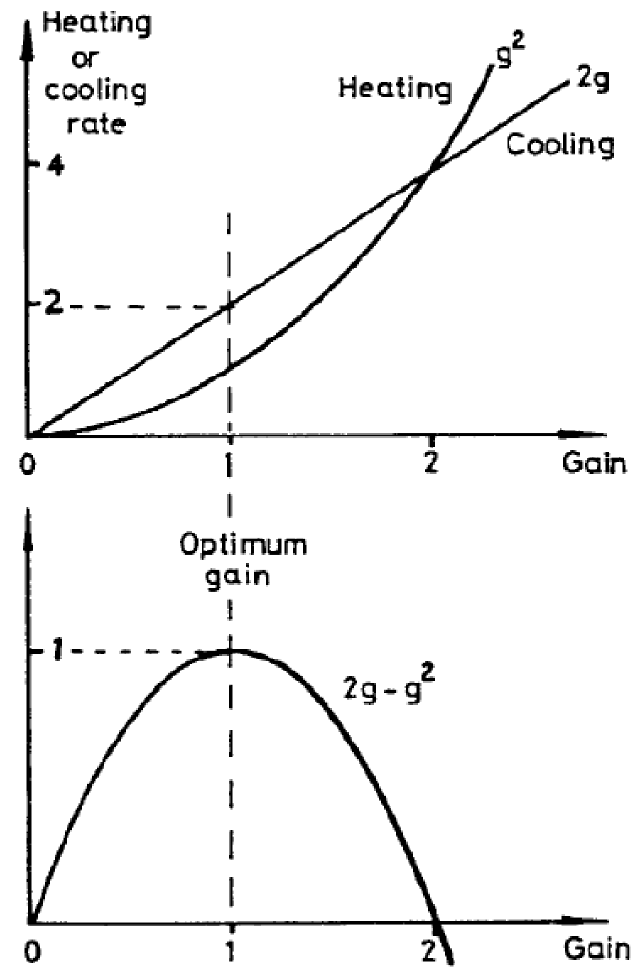
$$\Delta\left\langle x^2\right\rangle \equiv \left\langle x_c^2\right\rangle-\left\langle x^2\right\rangle$$

$$\Delta\left\langle x^2\right\rangle=\left(-2g+g^2\right)\frac{\left\langle x^2\right\rangle}{N_s}$$

Cooling due to self induced correction (Coherent kick)

Heating due to neighbors (Incoherent kick)

$$\Delta\left\langle x^2\right\rangle=\left[\left(g-1\right)^2-1\right]\frac{\left\langle x^2\right\rangle}{N_s}$$




In the ideal case (no noise, 1 turn good mixing and no bad mixing), the optimal gain is 1 and the optimal relative cooling rate is

$$\frac{1}{\tau_{x^2}}=\frac{1}{N_sT_{rev}}=\frac{\tau_b}{N_bT_s}\frac{1}{T_{rev}}=\frac{2W}{N_b}\frac{\tau_b}{T_{rev}}$$

Full bunch length

Revolution period

$N_b$: total number of particles in a bunch

Assuming $T_s \le \tau_b$

# Stochastic Cooling: cooling rate

- Cooling rate of $x_{rms}$

$$\Delta\left\langle x^2\right\rangle=\left[(g-1)^2-1\right]\frac{\left\langle x^2\right\rangle}{N_s}\Rightarrow\ \frac{dx_{rms}^2}{dN}=\left[(g-1)^2-1\right]\frac{x_{rms}^2}{N_s}\Rightarrow\ \frac{dx_{rms}}{dN}=\frac{1}{2}\left[(g-1)^2-1\right]\frac{x_{rms}}{N_s}$$

$$\boxed{\frac{1}{\tau_{x_{rms}}}=\frac{1}{2N_sT_{rev}}=\frac{\tau_b}{2N_bT_s}\frac{1}{T_{rev}}=\frac{W}{N_b}\frac{\tau_b}{T_{rev}}}$$

- For coasting beam, $\tau_b=T_{rev}$ and $N_b=N_{ring}$

$$\boxed{\frac{1}{\tau_{x_{rms}}}=\frac{W}{N_{ring}}}$$

# Stochastic Cooling: cooling rate

- In the presence of noise, bad mixing and non-ideal good mixing, the cooling rate as well as the optimal gain will be affected.

$$\frac{1}{\tau} = \frac{W}{N}\left[2g(1-\widetilde{M}^{-2}) - g^2(M+U)\right].$$

$$g_0 = \frac{1-\widetilde{M}^{-2}}{M+U},$$

$$\frac{1}{\tau_0} = \frac{W}{N}\left(\frac{(1-\widetilde{M}^{-2})^2}{M+U}\right).$$

The optimal cooling requires small $U$, small $M$ and large $\widetilde{M}$.

$U = E\left(x_n^2\right)/E\left(\langle x\rangle_s^2\right)$ is the ratio of the expected noise to the expected signal power, or noise to signal ratio.

$M$ is the number of turns required for good mixing / re-randomization. (Typically, it is for a particle of typical momentum error to move by one sample length with respect to the nominal particles. )

$\widetilde{M} = \left(l_{ring}/l_{PK}\right)M$ is the number of turns required for complete bad mixing and $l_{PK}$ is the distance from the pickup to the kicker

# Not fully convinced? Test it on your pc.

1. Generate an array of random numbers of dimension N;

$$x_1, x_2, x_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots x_N$$

2. Calculate and record the variance of the array;

3. Group the array into $N_{Slice}$ sub-group with each group having $M = N/N_{slice}$ random numbers and calculate the average of each sub-group (i.e. errors to be corrected);

$$\underbrace{(x_1, x_2, \ldots x_M)}_{\langle x \rangle_1 = \sum_{i=1}^{M} x_i}, \underbrace{(x_{M+1} \ldots, x_{2M})}_{\langle x \rangle_2}, \ldots\ldots\ldots\ldots, \underbrace{(x_{(N_{slice}-1)\cdot M+1} \ldots, x_{N_{slice}\cdot M})}_{\langle x \rangle_{N_{slice}}}$$

4. Subtract each element of the array by its sub-group average (apply correction);

$$\left(x_1 - \langle x \rangle_1, \ldots x_M - \langle x \rangle_1\right), \ldots\ldots, \left(x_{(N_{slice}-1)\cdot M+1} - \langle x \rangle_{N_{slice}} \ldots, x_{N_{slice}\cdot M} - \langle x \rangle_{N_{slice}}\right)$$

$$\Updownarrow$$

Array after correction $\rightarrow x_{c,1}, x_{c,2}, x_{c,3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots x_{c,N}$

5. Randomize the order of the elements in the corrected array $x_c$ generated in step 4 to get a new series

$$y_1, y_2, y_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots y_N$$

6. Repeat 2~5 with the new array $y$

# Numerical Testing Stochastic Cooling

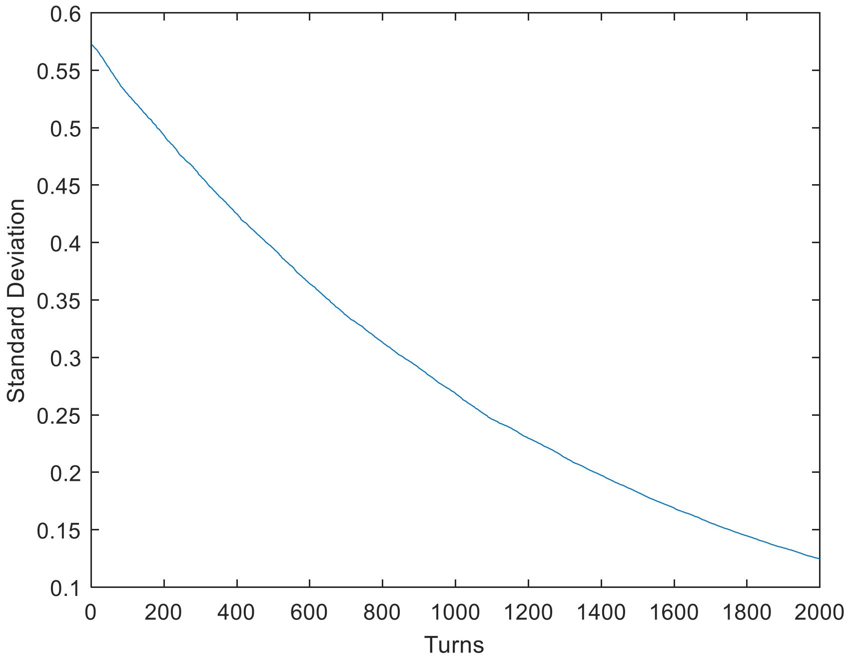
$$N = 6000$$

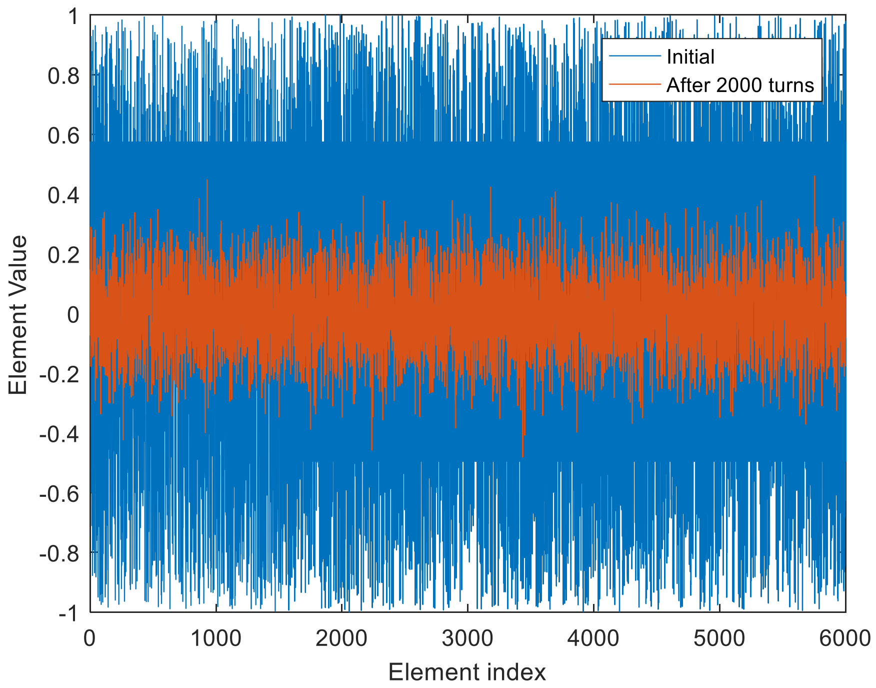
$$N_{slice} = 10$$

$$N_{turn} = 2000$$

As an example, a matlab script 'SC_test.m' to do the test is uploaded to the course webpage and you can play with it or write one of your own.

- Test how $N_{slice}$ , gain, noise affect the cooling rate

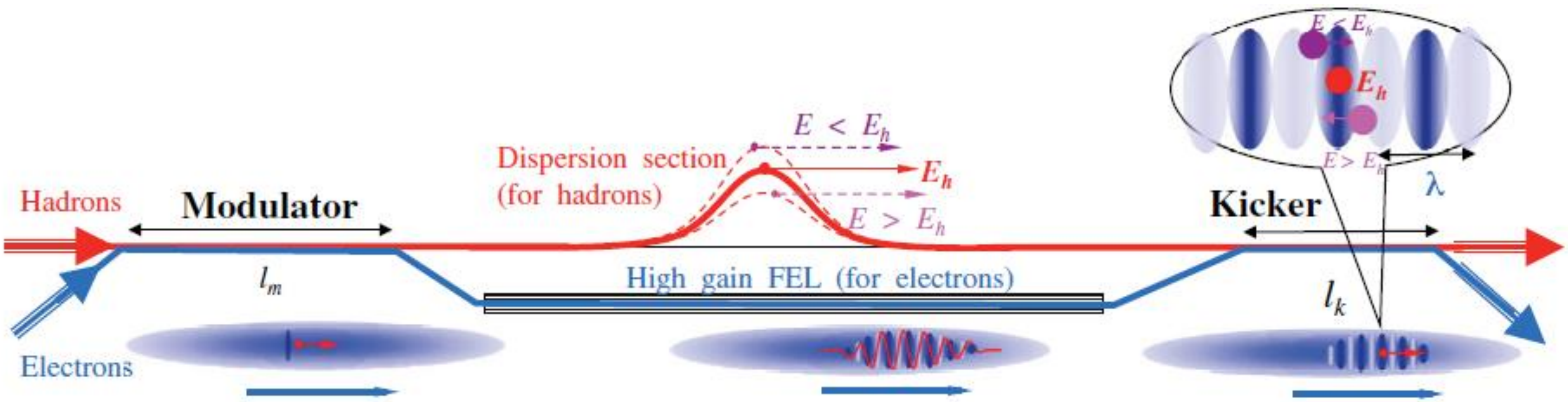


# Coherent electron Cooling

Coherent electron cooling is to use electron beam as the cooling media and to cool ions stochastically.

- The general idea was proposed by Y. Derbenev in 1980.
- A scheme based on using FEL as an amplifier was proposed by V. N. Litvinenko in 2007.
- A scheme based on Chicane based microbunching amplifier was proposed by D. Ratner in 2013.
- A scheme based on Plasma-cascade amplifier was proposed by V.N. Litvinenko in 2018.
- Unproven yet: a proof-of-principle experiment is undergoing in BNL to test the CeC concept.




Each ion imprint a density bump in the electron, (~pick-up session of stochastic cooling)

Each delta-like density bump generates a wave-packets with width of the FEL coherent length, i.e. the sample length (~amplifier in SC)

Each ion gets a kick determined by itself and by its neighbours in the sample. (~kicker in SC)

# CeC Cooling Rate

$$\langle \delta^2 \rangle' = -2\xi \langle \delta^2 \rangle + D;$$

One of the major advantage of CeC is that the bandwidth are orders of magnitude wider than the traditional Stochastic cooling, which make it possible to cool high intensity ion beam.

$$\xi = -g \langle \delta_i \mathrm{Im}[K(\Delta\zeta_i) e^{ik\Delta\zeta_i}] \rangle / \langle \delta^2 \rangle;$$

Delay of ion with respect to the on-momentum ion

Incoherent diffusive kick, i.e. heating due to neighbours → $D = g^2 N_{\mathrm{eff}}/2$

FEL coherent length

The number of particles in one sample length / coherent length → $N_{\mathrm{eff}} \cong N_h \frac{\Lambda_k}{\sqrt{4\pi}\sigma_{z,h}} + \frac{N_e}{X^2} \frac{\Lambda_k}{\sqrt{4\pi}\sigma_{z,e}},$

Efficiency of modulation. X~Z for effective modulation.

Maximal cooling rate → $\xi_{\max} \propto N_{\mathrm{eff}}^{-1}$

# What we learned today

- Beam cooling is to reduce beam phase space volume.
- We derived cooling rate for un-magnitized electron cooling and showed that electron cooling rate decreases dramatically for high energy hadron beam.
- We derived cooling rate for stochastic cooling and show that the optimal cooling rate is proportional to the bandwidth of the cooling system and inversely proportional to the number of particles to be cooled.
- We also learned the concept of coherent electron cooling and showed the expressions for calculating its cooling rate.