Some 3D effects in MBEC

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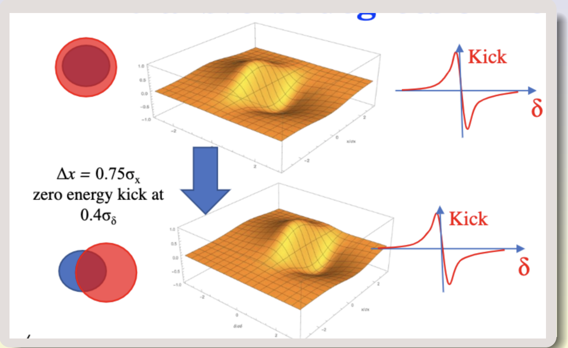
Outline

MBEC for EIC is currently based on theory and simulations of a quasi-1D model in which point charges (ions and electrons) are replaced by disks (beam slices) with a Gaussian transverse surface charge distribution. We want to better understand what 3D adds to/changes in this model.

- 3D energy kick depends on particle offset (work done with W. Bergan)
- 3D plasma oscillations
  - Cold plasma, uniform density profile
  - Large transverse Debye radius, Gaussian profile
- Summary

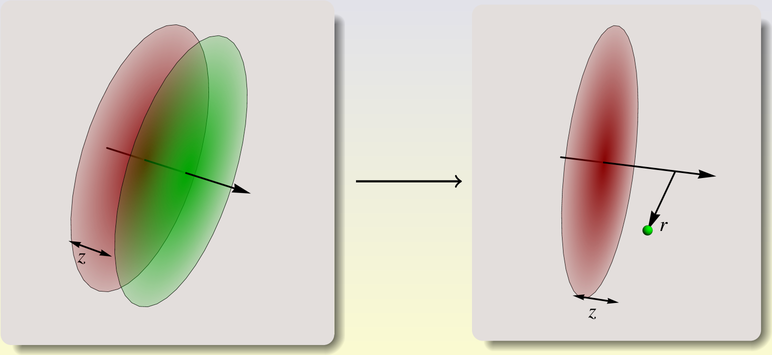
Introduction

At the previous meetings V. Litvinenko (10/15/202) and M. Blaskiewicz (11/08/2021) discussed the idea that if the energy kick depends on r, in combination with the horizontal dispersion in the kicker, it will lead to transverse cooling. To achieve this, it was proposed to horizontally shift the centers of the hadron and electron beams in the kicker.

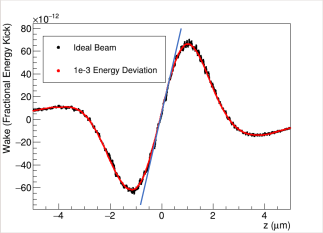


Introduction

Even without this shift (the beams are centered) the energy kick depends on x because the electron density (and the longitudinal field) is smaller near the edge than at the center of the beam. Does this effect contribute to/modify the transverse cooling (we ignored it in the current quasi-1D slice model)? We explore this issue both theoretically and computationally (work done with W. Bergan, presented on 11/18/2021).



Quasi-1D model (disk-disk), transverse cooling



Let us assume that the energy kick is proportional to the longitudinal shift of the hadron Δz when it travels from the modulator to the kicker ($\eta = \Delta\mathcal{E}/\mathcal{E}$)

$$\Delta\eta(\Delta z) = w'(0)\Delta z$$

Assuming $\alpha_1 = \alpha_2 = 0$ the transverse cooling time t_c is (see Ref.[1])

$$t_c^{-1} = w'(0)\left[\frac{D_2 D_1}{\sqrt{\beta_1\beta_2}}\sin\mu - D_2 D_1'\sqrt{\frac{\beta_1}{\beta_2}}\cos\mu + D_2' D_1\sqrt{\frac{\beta_2}{\beta_1}}\cos\mu + D_1' D_2'\sqrt{\beta_1\beta_2}\sin\mu\right]$$

Here D_2 and D_2' are in the kicker and D_1 and D_1' are in the modulator, μ is the phase advance between K and M.

[1] P. Baxevanis and G. Stupakov. PRAB, **22**, p.081003, (2019).

Disk-point model, energy kick depends on x

We now assume

$$w'(0,x) = Ae^{-x^2/2\sigma_{e,\perp}^2}$$

In reality w radially extends somewhat beyond $\sigma_{e,\perp}$.

Calculations can be done analytically for $D_1 = D_1' = 0$. Here is the result for the case when also $D_2' = \alpha_2 = 0$ in the modulator (remember, there is no cooling in this case if w' does not depend on x).

$$t_c^{-1} = \frac{1}{T} R_{56} A D_2^2 \frac{4\sigma_{e,\perp}^2 \sigma_{h,\eta}^3}{\beta^2 (\epsilon_{h,x})^2 \sigma_0} \frac{1}{(1+b)\sqrt{b^2-1}}$$

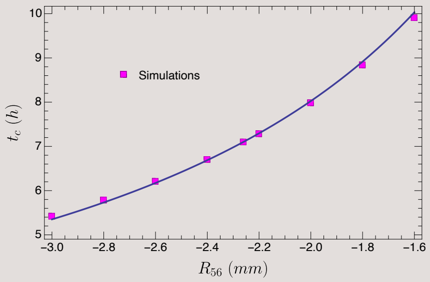
where

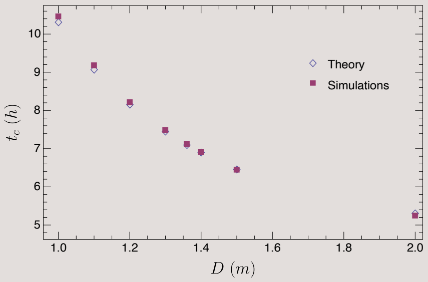
$$b = 1 + \frac{2\sigma_{e,\perp}^2 \sigma_{h,\eta}^2}{\beta \sigma_0^2 \epsilon_{h,x}}, \quad \sigma_0^{-2} = \sigma_{h,\eta}^{-2} + \frac{D^2}{\sigma_{e,\perp}^2}$$

We now have cooling with R_{56} and D_2 in the kicker only!

Simulations

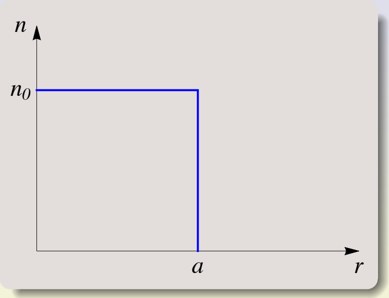
We've chosen $A = 10^{-6}\ \mathrm{m}^{-1}$ in the model of a coasting electron and hadron beams.





Plasma oscillations in the drifts

In reality, the dependence of the wake on the transverse coordinates of the ion in the kicker, $w(x, y, z)$, will be determined by the radial profile of plasma eigenmodes in the electron beam. These eigenmodes can be relatively easy found for a cylindrical beam of *zero temperature*.



We work *in the beam frame* and locally consider the electron beam as a plasma cylinder ($-\infty < z < \infty$) with constant density n_0.

Plasma waves in the cylinder

This problem was studied in Ref.[2] We seek potential perturbations (subscript $_1$) in the beam $\boldsymbol{E}_1 = -\nabla\phi_1$. Assume the time dependence $\propto e^{-i\omega t}$

$$\nabla \cdot \boldsymbol{D}_1 = \nabla \cdot \epsilon \boldsymbol{E}_1 = 0$$

where $\boldsymbol{D}_1$ is the electric induction and

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$$

is the dielectric function for the cold beam with $\omega_p = \sqrt{4\pi n_0 e^2/m}$ the plasma frequency (in the lab frame $\omega_p^{(\mathrm{lab})} \to \gamma^{-3/2}\omega_p^{(\mathrm{beam})}$). Assume the dependence $\propto e^{in\theta + ik_z z}$ in the cylindrical coordinate system. Inside and outside of the beam we have $\Delta\phi_1 = 0$ that is

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \phi_1}{\partial r} - \frac{n^2}{r^2}\phi_1 - k_z^2 \phi_1 = 0$$

The solution

$$\phi_1 = A I_n(k_z r), \quad 0 < r < a, \qquad \phi_1 = B K_n(k_z r), \quad a < r$$

[2] A. W. Trivelpiece and R. W. Gould. Space charge waves in cylindrical plasma columns. Journal of Applied Physics, **30**, 1784 (1959).

The eigen frequencies

The boundary conditions at $r = a$ are: the continuity of ϕ_1 and the transverse component D_r, They give the following dispersion relation for the plasma eigenmodes

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} = \frac{K_n'(k_z a) I_n(k_z a)}{I_n'(k_z a) K_n(k_z a)}$$

from which we find the frequency of plasma oscillations

$$\omega_n = \omega_p \left(1 - \frac{K_n'(k_z a) I_n(k_z a)}{I_n'(k_z a) K_n(k_z a)} \right)^{-1/2}$$

Charge is localized on the surface

There is no density perturbations inside the plasma—only on the surface of the cylinder.

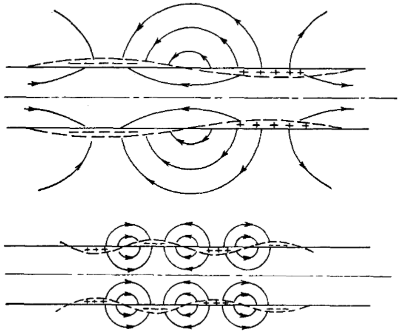
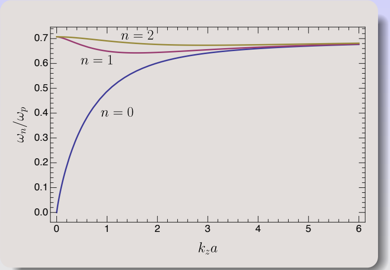


FIG. 4. Electric field configuration at the time of maximum field for a circularly symmetric surface wave.

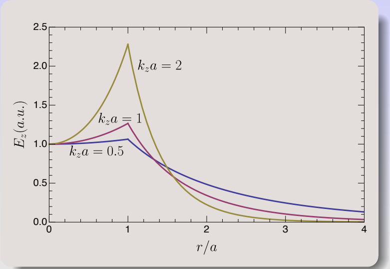
There is however a velocity perturbation, $\boldsymbol{v}_1$, inside the plasma. If $\phi_1 = \phi_1^{(0)} I_n(k_z r) e^{in\theta + ik_z z}\cos(\omega_n t)$ for $r < a$ then for the z-component of the velocity we have

$$v_{1z} = \frac{iek_z}{m\omega_n}\phi_1^{(0)} I_n(k_z r) e^{in\theta + ik_z z}\sin(\omega_n t)$$

The eigenmodes

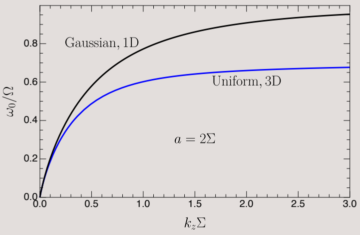


The plot of the eigenmode frequencies as a function of $k_z a$. Asymptotically, $\omega_n \to \omega_p/\sqrt{2}$ at $k_z \to \infty$.

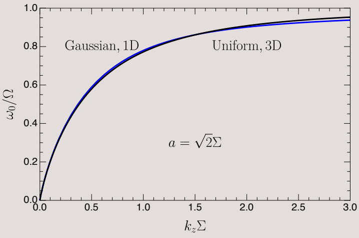


Distribution of the electric field in the $n = 0$ mode for different values of the longitudinal wavenumber k_z.

Comparison with quasi-1D disk model



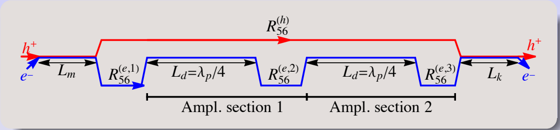
Comparison of the plasma cylinder (uniform distribution) $n = 0$ frequency with 1D Gaussian beam (1D model) with rms size Σ. The rms size of the beams is equal when $a = 2\Sigma$.



Comparison of frequencies for $a = \sqrt{2}\Sigma$. Here $\Omega^2 = \nu e^2/m\Sigma^2$ with ν the number of particle per unit length.

In 3D, an ion in the modulator will excite many modes in the electron beam with different n and k_z. Our immediate interest is in the axisymmetric modes, $n = 0$. Other modes with $n \geq 1$ will contribute to the noise in the beam. Hopefully they are not as strongly amplified as the $n = 0$ modes.

Effect of chicanes in the beam frame



What happens when an eigenmode passes through a chicane in the beam frame? This can be figured out through a sequence of Lorentz transformations: transform to the lab frame before the chicane; then transform the variable with the transport matrix R through the chicane; transform back to the beam frame.

In the lab frame the chicane is treated as a longitudinal shift of particles $\Delta z = R_{56}\Delta\mathcal{E}/\mathcal{E}$. In the Lorentz transformations

$$\frac{\Delta\mathcal{E}}{\mathcal{E}} = \frac{v_z}{c}$$

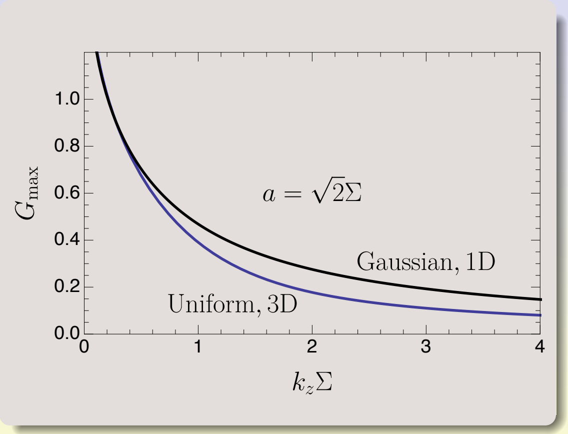
and $\Delta z^{(\mathrm{beam})} = \gamma\Delta z^{(\mathrm{lab})}$. Hence

$$\Delta z^{(\mathrm{beam})} = \gamma R_{56}\frac{v_z}{c}$$

In the beam frame after passing through a chicane particles are shifted by $\gamma R_{56} v_z/c$.

Maximum amplification of $n = 0$ modes by a chicane

A mode with initial density perturbation $n_1^{(i)}$ makes a quarter period oscillation, $t = \pi/\omega_0$, and then is sent through a chicane with R_{56}. At the exit the density perturbation is amplified, $n_1^{(f)} = G n_1^{(i)}$.



$G_{\max}$ is normalized by $(c/\sigma_{v_z})\sqrt{I_e/I_A}$.

Debye radius in the electron beam in the amplifier

We calculate the Debye radius of electrons in the beam frame. The beam has different temperatures in the longitudinal, $T_\parallel$, and transverse, $T_\perp$, directions,

$$T_\parallel = mc^2 \left(\frac{\sigma_E}{E_0}\right)^2, \qquad T_\perp = mc^2\gamma\frac{\epsilon_N}{\beta}$$

Use the nominal SHC parameters: $\gamma = 293$, $\sigma_E/E_0 = 10^{-4}$, $\epsilon = 2.8$ µm, $\beta = 1$ m ($\sigma_\perp \approx 100$ µm). This gives

$$T_\parallel = 5.1 \times 10^{-3}\ \mathrm{eV}, \qquad T_\perp = 420\ \mathrm{eV}$$

Now we calculate the Debye length, $r_D^2 = T/4\pi n e^2$, using $I_e = 10$ A

$$r_{D,\parallel} = 4.9\ \mu\mathrm{m}, \qquad r_{D,\perp} = 1.4\ \mathrm{mm}$$

We have $r_{D,\perp} \gg \sigma_\perp$ and $r_{D,\parallel} \ll \sigma_\perp$. This means that the cold model is not applicable in the transverse direction, but it is valid in the longitudinal direction.

Vlasov equation for a beam with large $r_{D,\perp}$

We need to used (linearized) Vlasov equation to study plasma oscillations (*in the beam frame*). Assume axisymmetry. The plasma is confined in the transverse direction by the focusing potential $V(x, y)$,

$$V(x, y) = \frac{1}{2}\omega_0^2(x^2 + y^2)$$

The Vlasov equation for the distribution function $F(\boldsymbol{r}, \boldsymbol{v}, t)$ is (the electron charge is $-e$)

$$\frac{\partial F}{\partial t} + \boldsymbol{v} \cdot \nabla F + \frac{1}{m}(-\nabla V - e\boldsymbol{E}) \cdot \frac{\partial F}{\partial \boldsymbol{v}} = 0$$

In equilibrium we have $F_0(r, v_\perp, v_z)$ (it does not depend on t and z).

Linerized Vlasov equation

The linearized Vlasov equation is formulated in the cylindrical coordinate system, r, θ, z. The Hamiltonian:

$$H = \frac{1}{2m}{p_r}^2 + \frac{1}{2mr^2}p_\theta^2 + \frac{1}{2}m\omega_0^2 r^2$$

where ω_0 is the betatron frequency in the beam frame. Equations of motion

$$\dot{p}_r = -m\omega_0^2 r, \qquad \dot{r} = \frac{p_r}{m}, \qquad \dot{p}_\theta = 0, \qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$

Assume axisymmetric plasma eigenmodes. The perturbation of the distribution function is $f_1(r, p_r, \cancel{\theta}, p_\theta, z, v_z, t)$. The linearized Vlasov equation:

$$\frac{\partial f_1}{\partial t} + v_z\frac{\partial f_1}{\partial z} + v_r\frac{\partial f_1}{\partial r} + \xcancel{\frac{p_\theta}{mr^2}\frac{\partial f_1}{\partial \theta}} - m\omega_0^2 r\frac{\partial f_1}{\partial p_r} - \frac{e}{m}E_{1,z}\frac{\partial F_0}{\partial v_z} = 0$$

We ignored terms with $E_{1,x}$ and $E_{1,y}$ (P. Baxevanis). Seek solutions $\propto e^{-i\omega t + ik_z z}$, use $E_{1,z} = -ik_z\phi_1$. This equation is averaged over the variables r, p_r and combined with the Poisson equation for ϕ_1.

Equation for eigenmodes

In the limit $T_\parallel = 0$

$$\Delta_\perp \phi_1 - k_z^2 \phi_1 = -k_z^2 \frac{\omega_p^2}{\omega^2} \hat{U} \phi_1$$

where

$$\hat{U}\phi_1 = \frac{2}{\pi^2} \int_0^\infty \frac{r' dr'}{\sigma_\perp^2} R(r, r') \phi_1(r')$$

and

$$R(r, r') = \int_0^\infty d\xi K_0 \left(\frac{|r^2 - r'^2|}{4\sigma_\perp^2} |1 - \xi^2| \right) \exp \left(-\frac{r^2 + r'^2}{4\sigma_\perp^2} \left(1 + \xi^2\right) \right)$$

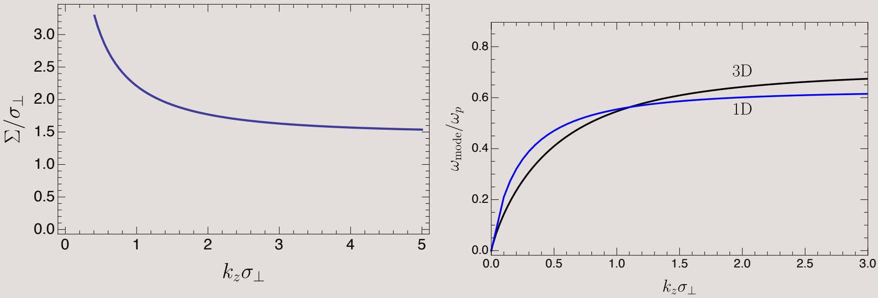
The kernel is symmetric, so the equation can be formulated as a variational problem:

$$\lambda = \min \frac{\int r dr [(\partial \phi_1 / \partial r)^2 + k_z^2 \phi_1^2]}{k_z^2 \int r dr \, \phi_1 \hat{U} \phi_1}$$

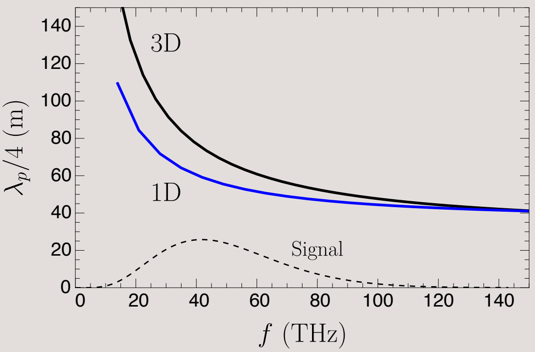
where $\lambda = \omega_p^2 / \omega^2$. Different eigenfunctions are orthogonal.

The lowest axisymmetric mode

We find the lowest axisymmetric mode using the trial function $\phi_1 = e^{-r^2/2\Sigma^2}$ where Σ is the variational parameter.



Lab frame



3D
1D
Signal
140
120
100
80
60
40
20
0
0
20
40
60
80
100
120
140
$\lambda_p/4$ (m)
f (THz)

Summary

- The dependence of the longitudinal kick on the transverse coordinate (x) leads to a new mechanism of the transverse cooling (which works even when D, D' in the modulator is zero). More work is needed to incorporate it into the MBEC for EIC optimization routine.
- There is an easy model for plasma oscillations in the drift: a rectangular density profile with zero temperature. The eigenmodes can be found and the MBEC amplification is calculated analytically.
- A more realistic model with $r_{D,\perp} \gg \sigma_\perp$ gives an integro-differential equation for eigenmodes of plasma oscillations. They allow a variational formulation and, in principle, can be used to find the gain in MBEC in 3D.