Homework 8.

Problem 1. 3x5 points. Beam envelope in straight section.
For a one-dimensional motion consider beam propagating in a straight section starting as s_o and having length L. Let's eigen vector (beam envelope) at s_o is given by:

$$Y(s_o)=\begin{bmatrix} w_o \\ w'_o+\dfrac{i}{w_o} \end{bmatrix};\beta_o \equiv w_o^2;\ \ \alpha_o=-\frac{\beta'}{2}\equiv -w_o w'_o;$$

(1)

(a) Propagate the eigen vector along the straight section. Show that β-function can be expressed as

$$\beta(s)=\beta^* + \frac{(s-s^*)^2}{\beta^*};$$

where β^*, s^* can be found from initial conditions (1). Hint, use derivative of β-function to find s^*. β^* is frequently used in colliders to describe the beam envelope in detectors.
(b) Calculate the (betatron) phase advance acquired in the straight section. Express the phase advance as function of β^*, s^*. Write expression for x(s) and x'(s). Show that x'=const.
(c) What is the maximum possible phase advance in a straight section (e.g. when s_o ,L are unlimited)?

Solution: (a) Propagating the eigen vector through a drift is just multiplying it by the drifts transport matrix:

$$\tilde{Y}(s)=\begin{bmatrix} 1 & \Delta s \\ 0 & 1 \end{bmatrix} Y(s_o)=\begin{bmatrix} w_o+\Delta s\left(w'_o+\dfrac{i}{w_o}\right) \\ w'_o+\dfrac{i}{w_o} \end{bmatrix}=\begin{bmatrix} w(s) \\ w'(s)+\dfrac{i}{w(s)} \end{bmatrix} e^{i\Delta\psi};\Delta s=s-s_o; \qquad (2)$$

$$\beta(s)=w^2(s)=\left|w_o+\Delta s\left(w'_o+\frac{i}{w_o}\right)\right|^2=\left(w_o+\Delta s w'_o\right)^2+\frac{\Delta s^2}{w_o^2}=\beta_o-2\alpha_o\Delta s+\frac{\Delta s^2}{\beta_o}\left(1+\alpha_o^2\right)$$

It is clearly a positively defined parabola and we just should find where it has a minimum:

$$\beta'\left(s^*\right)=2\left(w_o+\Delta s^* w'_o\right)w'_o+2\frac{\Delta s^*}{w_o^2}=0\rightarrow \Delta s^*=-w_o^2\frac{w_o w'_o}{1+\left(w_o w'_o\right)^2}=\frac{\alpha_o\beta_o}{1+\alpha_o^2}$$

$$\beta^*=\beta\left(s^*\right)=\frac{\beta_o}{1+\alpha_o^2};w^*=\sqrt{\frac{\beta_o}{1+\alpha_o^2}};w'^*=0;$$

Now we need just to apply (2) again with $s_o=s^*$:

$$\beta(s)=w^2(s)=\left|w^*+\frac{i\left(s-s^*\right)}{w^*}\right|^2=\beta^*+\frac{\left(s-s^*\right)^2}{\beta^*}\#.$$

(b) Using (2) again we have:

$$\mathrm{w}(\mathrm{s})e^{i\psi(s)} = \mathrm{w}_o + (s - s_o)\mathrm{s}\left(\mathrm{w}_o' + \frac{i}{\mathrm{w}_o}\right) = \mathrm{w}^* + i\frac{s - s^*}{\mathrm{w}^*} = \mathrm{w}^*\left(1 + i\frac{s - s^*}{\beta^*}\right);$$

$$\psi(s) = \tan^{-1}\left(\frac{s - s^*}{\beta^*}\right) \to \psi(s_2) - \psi(s_1) = \tan^{-1}\left(\frac{s_2 - s^*}{\beta^*}\right) - \tan^{-1}\left(\frac{s_1 - s^*}{\beta^*}\right).$$

Trajectory:

$$x(z) = a\sqrt{\beta(z)}\cos(\psi(z) + \varphi); \beta(z) = \beta^* + \frac{z^2}{\beta^*}; \tan\psi(z) = \frac{z}{\beta^*};$$

$$x'(z) = a\left(\frac{\beta'(z)}{2\sqrt{\beta(z)}}\cos(\psi(z) + \varphi) - \frac{1}{\sqrt{\beta(z)}}\sin(\psi(z) + \varphi)\right)$$

We should note that:

$$\frac{\beta(z)}{\beta^*} = 1 + \frac{z^2}{\beta^{*2}} = 1 + \tan^2\psi = \frac{1}{\cos^2\psi}; \tan\psi(s) = \frac{z}{\beta^*};$$

$$x(z) = a\sqrt{\beta(z)}(\cos\psi\cos\varphi - \sin\psi\sin\varphi) = \frac{a\sqrt{\beta^*}}{\cos\psi}(\cos\psi\cos\varphi - \sin\psi\sin\varphi)$$

$$x(z) = a\sqrt{\beta^*}(\cos\varphi - \tan\psi\sin\varphi) = a\sqrt{\beta^*}\left(\cos\varphi - \frac{z}{\beta^*}\sin\varphi\right)$$

e.g. the trajectory is a straight line with constant

$$x'(z) = -\frac{a}{\sqrt{\beta^*}}\sin\varphi$$

(c) Assuming an very long drift

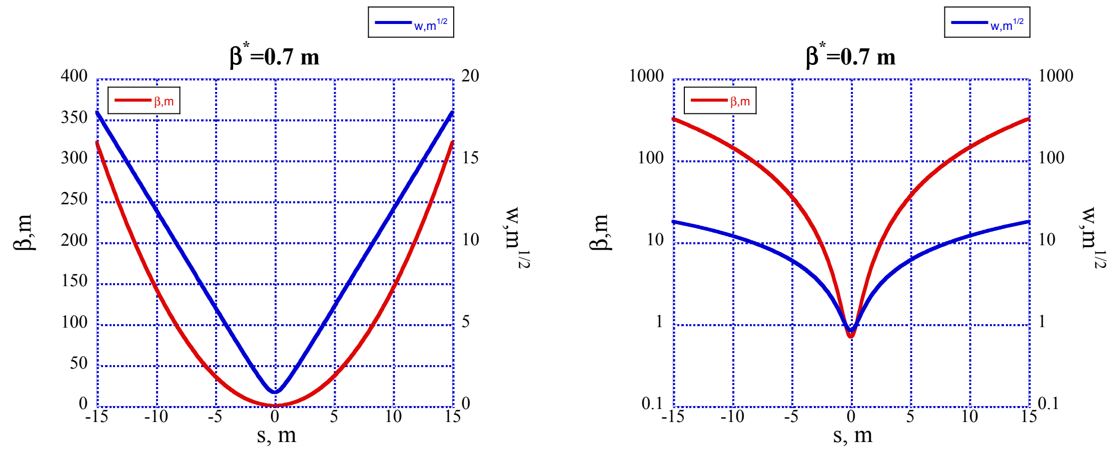
$$s_1 \to -\infty; s_2 \to +\infty$$

$$\psi(s_2) - \psi(s_1) \to \tan^{-1}\left(\frac{\to +\infty}{\beta^*}\right) - \tan^{-1}\left(\frac{\to -\infty}{\beta^*}\right) = \pi$$

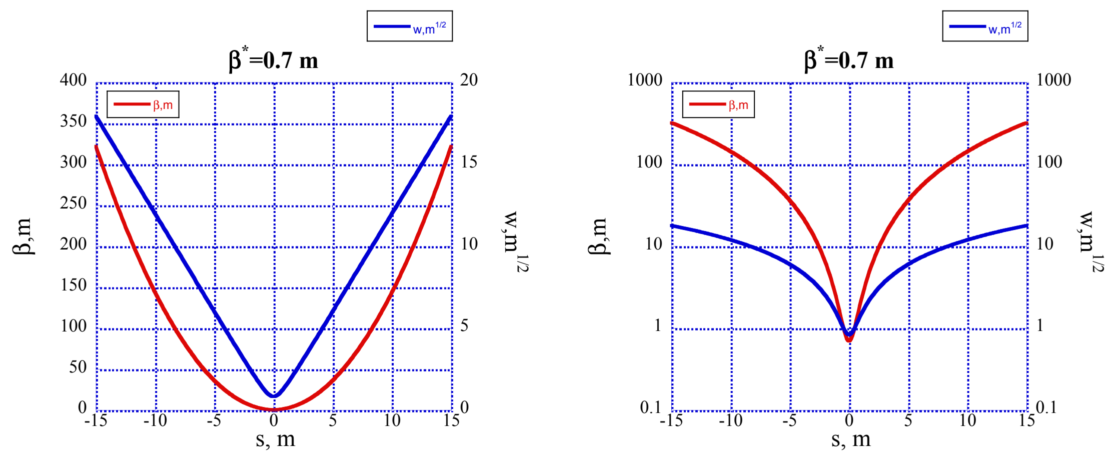
Naturally, you can get exactly the same result by integrating the phase advance using

$$\frac{d\psi}{ds} = \frac{1}{\beta(s)} \to \psi(s_2) - \psi(s_1) = \int_{s_1}^{s_2} \frac{ds}{\beta^* + \frac{(\mathrm{s}\text{-}\mathrm{s}^*)^2}{\beta^*}} =$$

$$\tan^{-1}\left(\frac{s_2 - s^*}{\beta^*}\right) - \tan^{-1}\left(\frac{s_1 - s^*}{\beta^*}\right)$$



Plot of beta-function and beam envelope in 30-m long straight section with β*=0.7 m – typical for RHIC interection region.



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