# Free Electron Lasers

G. Wang

# Outline

- Introduction
  - What is free electron laser (FEL)
  - Applications and some FEL facilities
  - Basic setup
  - Different types of FEL
- How FEL works
  - Electrons' trajectory and resonant condition
  - Analysis of FEL process at small gain regime (Oscillator)
  - Analysis of FEL process at high gain regime (Amplifier)

# Introduction I: What is free electron lasers

- A free-electron laser (FEL), is a type of laser whose lasing medium consists of very-high-speed electrons moving freely through a magnetic structure, hence the term free electron.
- The free-electron laser was invented by John Madey in 1971 at Stanford University.
- Advantages:
  - ✓ Wide frequency range
  - ✓ Tunable frequency
  - ✓ May work without a mirror (SASE)
- Disadvantages: large, expensive

# Introduction II: Applications and FEL facilities




- Medical, Chemistry, Biology (small wavelength and short pulse are required for imaging proteins), materials, Military (~Mwatts)…
- FEL Facilities (~33):

FREE ELECTRON LASERS

| LOCATION | NAME | WAVELENGTHS | TYPE | STATUS |
|---|---|---|---|---|
| RIKEN (Japan) | SACLA FEL | 0.63 - 3 Å | Linac | operating user facility |
| SLAC-SSRL (USA) | LCLS FEL | 1.2 - 15 Å | Linac | operating user facility |
| DESY (Germany) | FLASH FEL | 4.1 - 45 nm | SC Linac | operating user facility |
| ELETTRA Trieste, Italy | FERMI | 4 - 100 nm | Linac | operating user facility |
| SDL(NSLS) Brookhaven (USA) | HGHG FEL | 193 nm | Linac | operating experiment |
| Duke Univ. NC (USA) | OK-4 | 193 - 400 nm | storage ring | operating user facility |
| iFEL (Japan) | 3<br>2<br>1<br>4<br>5 | 230 nm - 1.2 μm<br>1 - 6 μm<br>5 - 22 μm<br>20 - 60 μm<br>50 - 100 μm | linac | operating user facility |
| Univ. of Hawaii (USA) | MK-V | 1.7 - 9.1 μm | linac | operating experiment |
| Vanderbilt TN (USA) | MK-III | 2.1 - 9.8 μm | linac | no longer operating |
| Radboud University (Netherlands) | FLARE<br>FELIX1<br>FELIX2 | 327 - 420 μm<br>3.1 - 35 μm<br>25 - 250 μm | linac | operating user facility |
| Stanford CA (USA) | SCA-FEL<br>FIREFLY | 3-10 μm<br>15-65 μm | SC-linac | no longer operating |
| LURE - Orsay (France) | CLIO | 3 - 150 μm | linac | operating user facility |
| Jefferson Lab VA (USA) |  | 3.2 - 4.8 μm<br>363 - 438 nm | SC-linac | operating user facility |
| Science Univ. of Tokyo (Japan) | FEL-SUT | 5 - 16 μm | linac | operating user facility |
| FZ Rossendorf (Germany) |  | 4-22 μm<br>18-250 μm |  | operating user facility |
| UCSB CA (USA) | FIR-FEL<br>MM-FEL<br>30 μ-FEL | 63 - 340 μm<br>340 μm - 2.5 mm<br>30 - 63 μm | electrostatic | operating user facility |
| ENEA - Frascati (Italy) |  | 3.6 - 2.1mm | microtron | operating user facility |
| ETL - Tsukuba (Japan) | NIJI-IV | 228 nm | storage ring | operating experiment |
| IMS - Okazaki (Japan) | UVSOR | 239 nm | storage ring | operating experiment |
| Dortmund, Univ. (Germany) | Felicita 1 | 470 nm | storage ring | operating expriment |
| LANL NM (USA) | AFEL<br>RAFEL | 4 - 8 μm<br>16 μm | linac | operating experiment |
| Darmstadt Univ. (Germany) | IR-FEL | 6.6 - 7.8 μm | SC-linac | operating experiment |
| IHEP (China) | Beijing FEL | 5 - 25 μm | linac | operating experiment |
| CEA - Bruyeres (France) | ELSA | 18-24 μm | linac | operating experiment |
| ISIR - Osaka (Japan) |  | 21-126 μm | linac | operating experiment |
| JAERI (Japan) |  | 22 μm<br>6 mm | SC-linac<br>induction linac | operating experiment |
| Univ. of Tokyo (Japan) | UT-FEL | 43 μm | linac | operating experiment |
| ILE - Osaka (Japan) |  | 47 μm | linac | operating experiment |
| LASTI (Japan) | LEENA | 65 - 75 μm | linac | operating experiment |
| KAERI (Korea) |  | 80 - 170 μm<br>10 mm | microtron<br>electrostatic | operating experiment |
| Budker Inst. Novosibirsk, Russia |  | 110 - 240 μm | linac | operating experiment |
| Univ. of Twente (Netherlands) | TEU-FEL | 200-500 μm | linac | operating experiment |
| FOM (Netherlands) | Fusion FEM |  |  | no longer operating |
| Tel Aviv Univ. (Israel) |  | 3 mm | electrostatic | operating experiment |

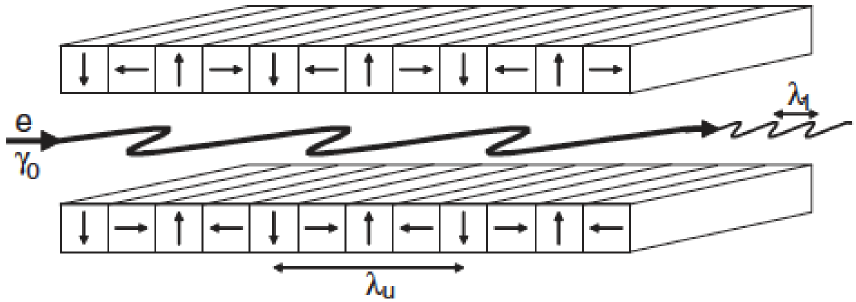
[1]So far only operating FEL oscillators with wavelength < 10 mm are included.
[2]"user facility" means a dedicated scientific research facility open to outside researchers.
[3]Order is first by type of facility and second roughly by wavelength.

Commissioned and in operation since 2017

# Introduction III: Basic Setup

Planar undulator

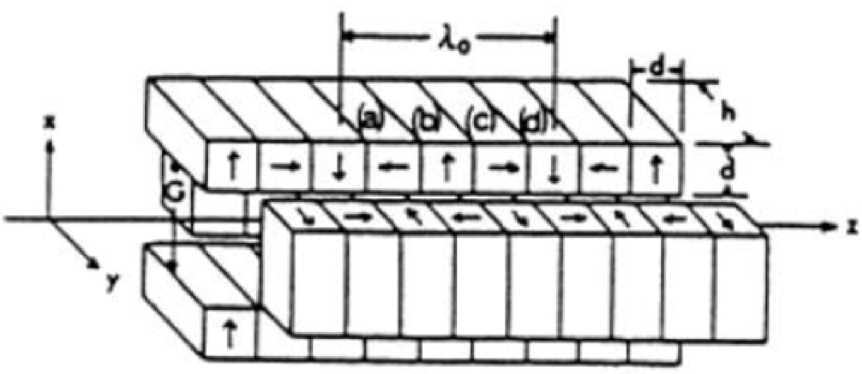
$$B_y(x,y,z) = B_0 \sin(k_u z)$$
for $x, y << \text{gap size}$

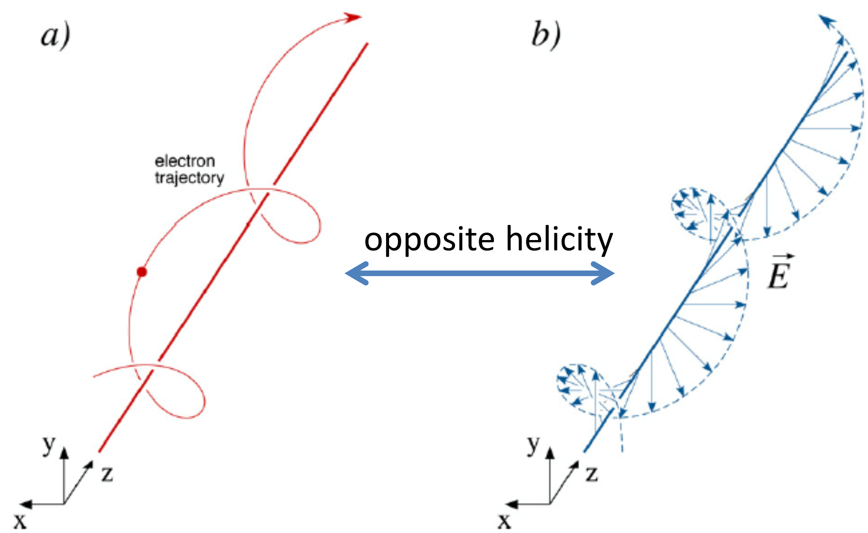



Helical wiggler for CeC PoP


Helical undulator

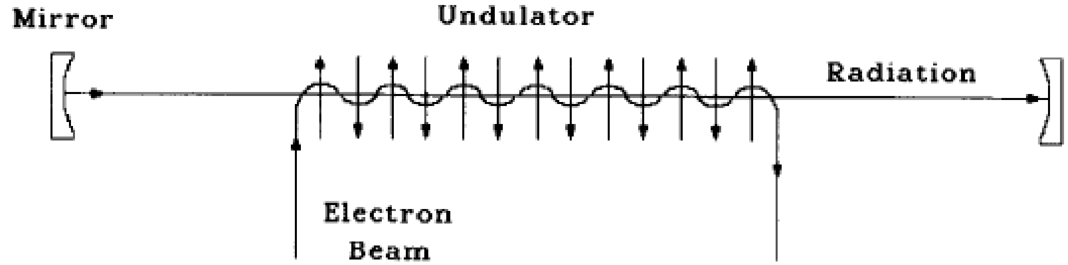
$$B_x(x,y,z) = B_0 \cos(k_u z)$$
$$B_y(x,y,z) = B_0 \sin(k_u z)$$
for $x, y << \text{gap size}$



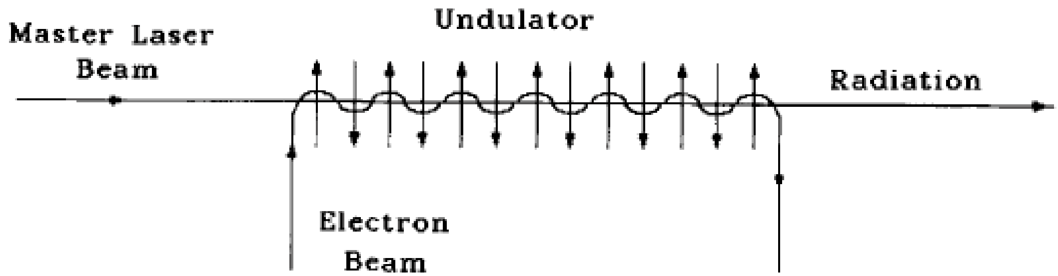



# Introduction IV: different types of FEL

FEL Oscillator
(Low gain regime)

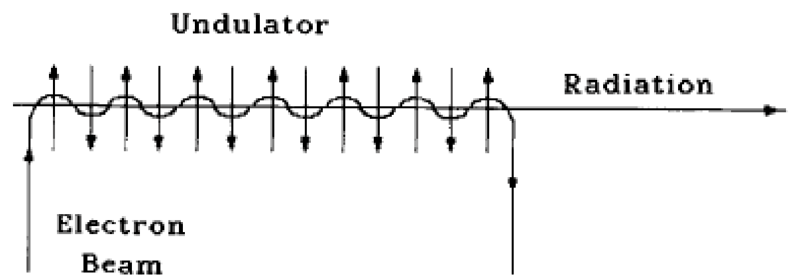



FEL Amplifier
(High gain regime)

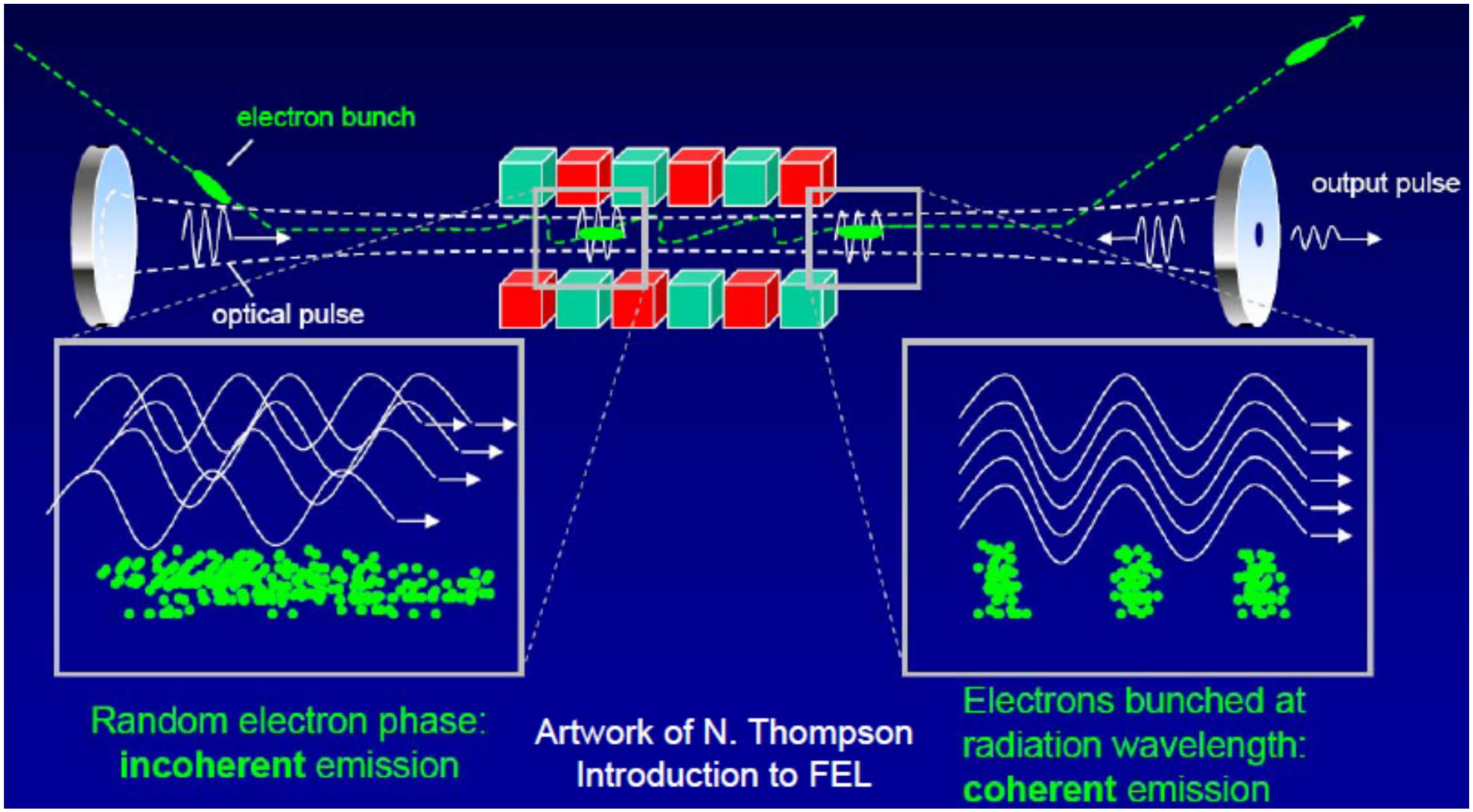



SASE FEL
(High gain regime)

Self-Amplified Spontaneous Emittion (SASE)



# FEL Oscillator (Low Gain)



# Unperturbed Electron motion in helical wiggler (in the absence of radiation field)

$$\vec{B}_w(x,y,z) = B_w\left[\cos(k_w z)\hat{x} - \sin(k_w z)\hat{y}\right]$$

$$\vec{F}(x,y,z) = -e\vec{v}\times\vec{B} = -ev_z\hat{z}\times\vec{B} = -ev_z B_w\left[\cos(k_w z)\hat{y} + \sin(k_w z)\hat{x}\right]$$

$$\frac{d(m\gamma v_x)}{dt} = m\gamma\frac{dv_x}{dt} = -ev_z B_w\sin(k_w z) \qquad \frac{d(m\gamma v_y)}{dt} = m\gamma\frac{dv_y}{dt} = -ev_z B_w\cos(k_w z)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad v = \sqrt{v_x^2 + v_y^2 + v_z^2} \qquad \tilde{v} \equiv v_x + iv_y$$

$$m\gamma\frac{d\tilde{v}}{dt} = -iev_z B_w\left(\cos(k_w z) - i\sin(k_w z)\right) = -iev_z B_w e^{-ik_w z}$$

$$m\gamma\frac{d\tilde{v}}{dt} = m\gamma\frac{dz}{dt}\frac{d\tilde{v}}{dz} = -iev_z B_w e^{-ik_w z} \Rightarrow m\gamma\frac{d\tilde{v}}{dz} = -ieB_w e^{-ik_w z}$$

Undulator parameter, also called $a_w$

$$K \equiv \frac{eB_w\lambda_w}{2\pi mc}$$

Electron rotation angle in undulator:

$$\theta_s = K/\gamma$$

$$\frac{\tilde{v}(z)}{c} = \frac{-ieB_w}{mc\gamma}\int e^{-ik_w z_1}dz_1 = \frac{eB_w}{mc\gamma k_w}e^{-ik_w z} = \frac{K}{\gamma}e^{-ik_w z}$$

*Assume the initial velocity of the electron make the integral constant vanishing.

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos(k_w z)\hat{x} - \sin(k_w z)\hat{y}\right] \quad v_z = const. \qquad \vec{x}(z) = \int_0^z \vec{v}(t_1)\,dt_1 + \vec{x}(z=0)$$

# Energy change of electrons due to radiation field

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos(k_w z)\,\hat{x} - \sin(k_w z)\,\hat{y}\right]$$

Consider a circularly polarized electromagnetic wave (plane wave is an assumption for 1D analysis, which is usually valid for near axis analysis) propogating along z direction

$$\vec{E}_\perp(z,t) = E\left[\cos(kz-\omega t)\,\hat{x} + \sin(kz-\omega t)\,\hat{y}\right] \qquad E_z = 0$$

$$= E\left[\cos\big(k(z-ct)\big)\,\hat{x} + \sin\big(k(z-ct)\big)\,\hat{y}\right] \qquad \omega = kc$$

Energy change of an electron is given by

$$\frac{d\mathcal{E}}{dt} = \vec{F}\cdot\vec{v} = -e\vec{v}_\perp\cdot\vec{E}_\perp$$

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s\frac{c}{v_z}\cos(\psi) \approx -eE\theta_s\cos(\psi)$$

Pondermotive phase:

$$\boxed{\psi = k_w z + k(z-ct)}$$

To the leading order, electrons move with constant velocity and hence $z = v_z(t-t_0)$

# Resonant Radiation Wavelength

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \cos\left[\left(k_w + k - k\frac{c}{v_z}\right)z + \psi_0\right]$$

Detuning parameter:

$$C \equiv k_w + k - \frac{kc}{v_z(\mathcal{E}_0)}$$

We define the resonant radiation wavelength such that

$$k_w + k_0 - k_0\frac{c}{v_z} = 0 \Rightarrow \lambda_0 = \lambda_w\left(\frac{c}{v_z} - 1\right) \approx \frac{\lambda_w}{2\gamma_z^2}$$

$$\gamma_z^{-2} \equiv 1 - v_z^2/c^2 = 1 - \left(v_z^2 + v_\perp^2\right)/c^2 + v_\perp^2/c^2 = \gamma^{-2} + \theta_s^2 = \gamma^{-2}\left(1 + K^2\right)$$

FEL resonant frequency:

$$\boxed{\lambda_0 \approx \frac{\lambda_w\left(1 + K^2\right)}{2\gamma^2}}$$

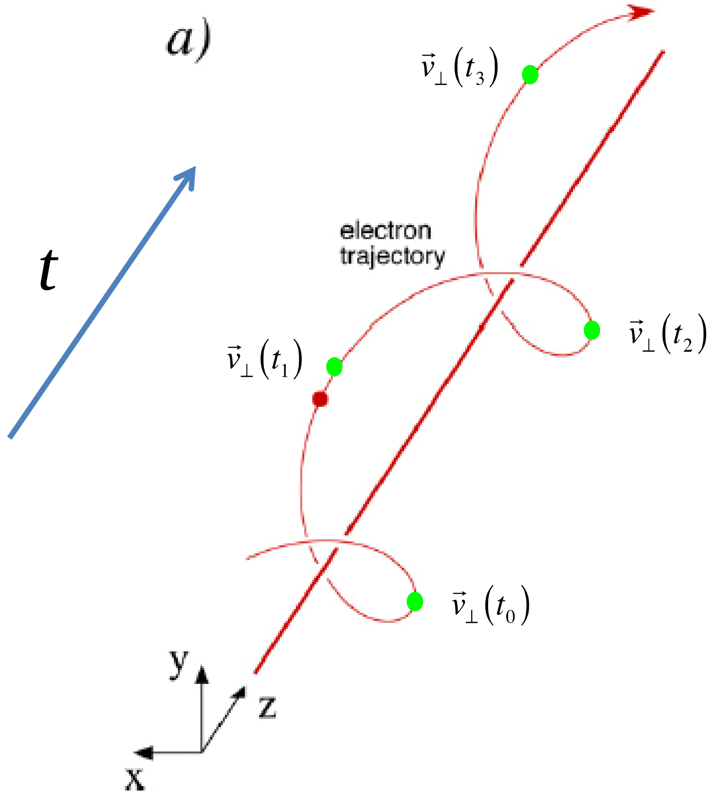
$$K \equiv \frac{eB_w\lambda_w}{2\pi mc}$$

At resonant frequency, the rotation of the electron and the radiation field is synchronized in the x-y plane and hence the energy exchange between them is most efficient.

# Helicity of radiation at synchronization

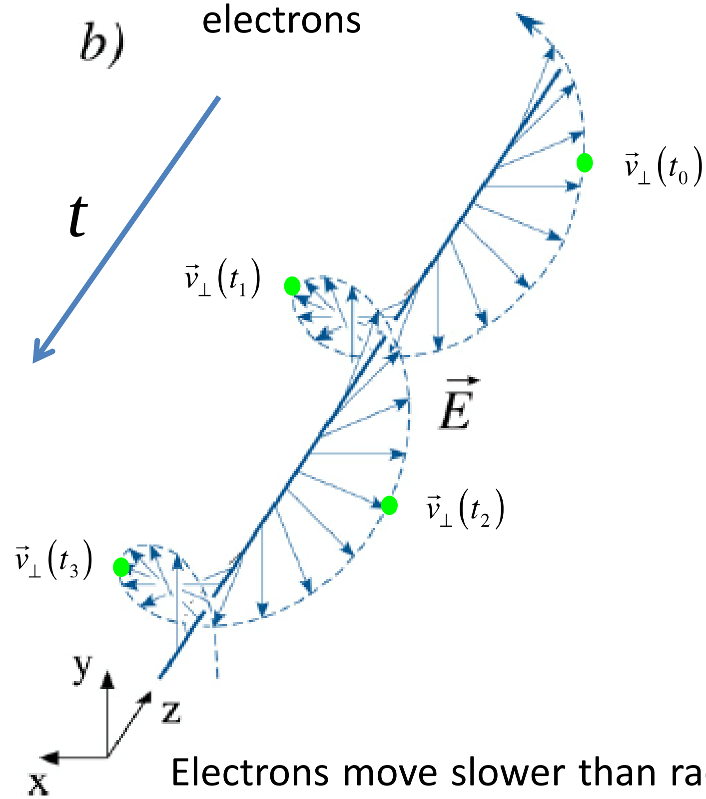
The synchronization requires opposite helicity of radiation with respect to the electrons' trajectories.

Electrons' trajectories




$$t_0 < t_1 < t_2 < t_3$$

Radiation field observed by electrons




Electrons move slower than radiation and hence see the radiation wave slipping ahead. As a result, the rotation direction of the radiation field seen by an electron is the same as its own rotation direction.

# Longitudinal equation of motion

In the presence of the radiation field, the longitudinal equation of motion of an electron read

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \cos(\psi) \qquad \psi = k_w z + k(z - ct)$$

$\mathcal{E}_0$ is the average energy of the beam.

$$\frac{d}{dz}\psi = k_w + k - \frac{\omega}{v_z(\mathcal{E})}$$

$$\approx k_w + k - \omega\left[\frac{1}{v_z(\mathcal{E}_0)} + (\mathcal{E} - \mathcal{E}_0)\frac{d}{d\mathcal{E}}\frac{1}{v_z}\right]$$

$$\approx k_w + k - \frac{\omega}{v_z(\mathcal{E}_0)} + \frac{\omega}{\gamma_z^2 c}\frac{(\mathcal{E} - \mathcal{E}_0)}{\mathcal{E}_0}$$

$$\frac{d}{d\mathcal{E}}\frac{1}{v_z} = \frac{1}{mc^3}\frac{d}{d\gamma}\frac{1}{\beta_z} = \frac{1}{mc^3}\frac{d\gamma_z}{d\gamma}\frac{d}{d\gamma_z}\frac{1}{\beta_z}$$

$$\gamma_z^2 = \frac{\gamma^2}{(1+K^2)} \qquad \frac{d\gamma_z}{d\gamma} = \frac{\gamma}{\gamma_z(1+K^2)}$$

$$\frac{d}{d\gamma_z}\frac{1}{\beta_z} = -\frac{1}{2\beta_z^3}\frac{d}{d\gamma_z}\left(1 - \frac{1}{\gamma_z^2}\right) = -\frac{1}{\beta_z^3\gamma_z^3}$$

$$\Rightarrow \begin{cases} \dfrac{dP}{dz} = -eE\theta_s \cos(\psi) \\[2ex] \dfrac{d}{dz}\psi \approx C + \dfrac{\omega}{\gamma_z^2 c\mathcal{E}_0}P \end{cases}$$

Energy deviation: $P \equiv \mathcal{E} - \mathcal{E}_0$

Detuning parameter: $C \equiv k_w + k - \dfrac{\omega}{v_z(\mathcal{E}_0)}$

# Low Gain Regime: Pendulum Equation

$$\left.\begin{aligned} \frac{dP}{dz} &= -eE\theta_s \cos(\psi) \\ \frac{d}{dz}\psi &= C + \frac{\omega}{\gamma_z^2 c\mathcal{E}_0} P \end{aligned}\right\} \Rightarrow \qquad \frac{d^2}{dz^2}\psi + \frac{eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0}\cos(\psi) = 0$$

We assume that the change of the amplitude of the radiation field, E, is negligible and treat it as a constant over the whole interaction.

$$\frac{d^2}{d\hat{z}^2}\psi + \hat{u}\cos(\psi) = 0 \qquad \hat{u} = \frac{l_w^2 eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0} \qquad \hat{z} = \frac{z}{l_w}$$

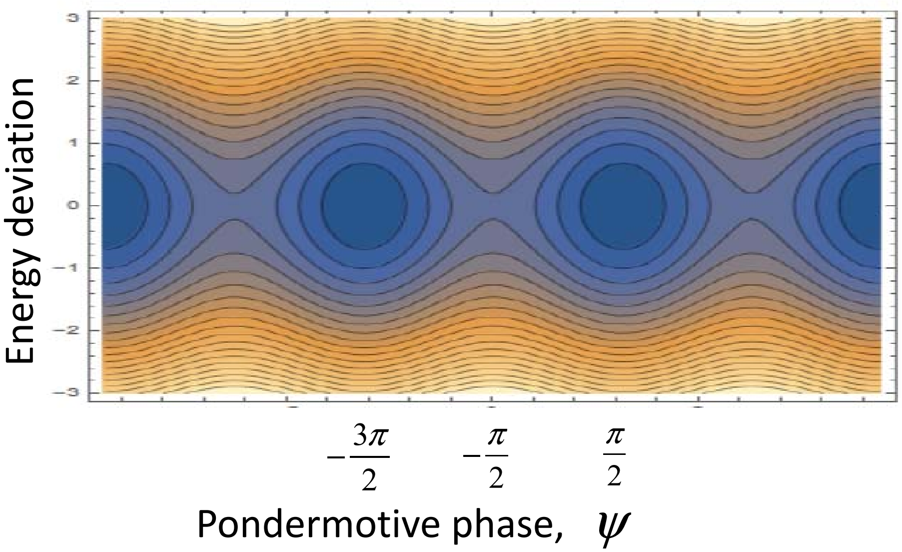
Pendulum equation:

$$\boxed{\frac{d^2}{d\hat{z}^2}\left(\psi + \frac{\pi}{2}\right) + \hat{u}\sin\left(\psi + \frac{\pi}{2}\right) = 0}$$

# Low Gain Regime: Similarity to Synchrotron Oscillation

## FEL

$\psi$ is the angle between the transverse velocity vector and the radiation field vector and hence there is no energy kick for $\psi = \pi/2$

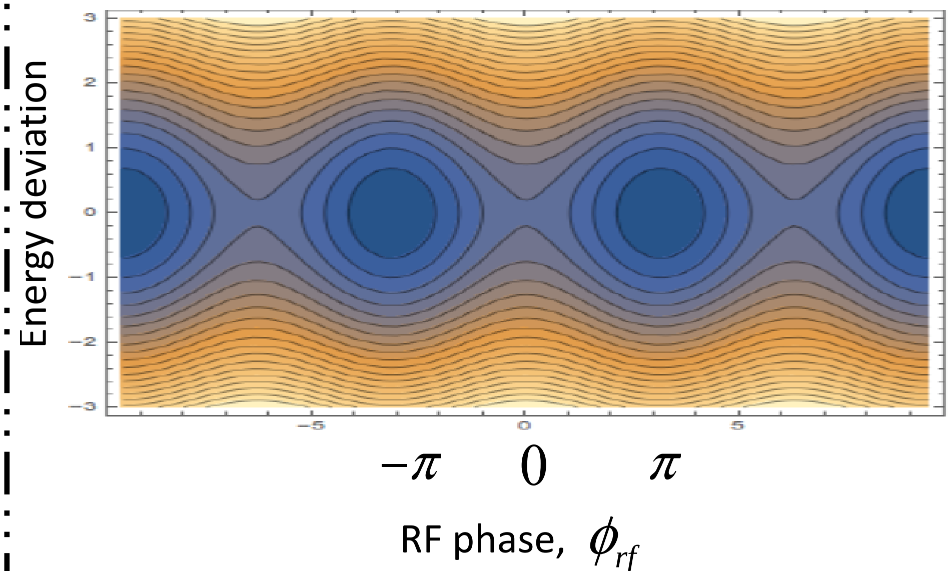



$$\frac{d^2}{d\hat{z}^2}\left(\psi + \frac{\pi}{2}\right) + \hat{u}\sin\left(\psi + \frac{\pi}{2}\right) = 0$$

$$\hat{u} = \frac{l_w^2 eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0} \qquad \psi = k_u z + k(z - ct)$$

## Synchrotron Oscillation

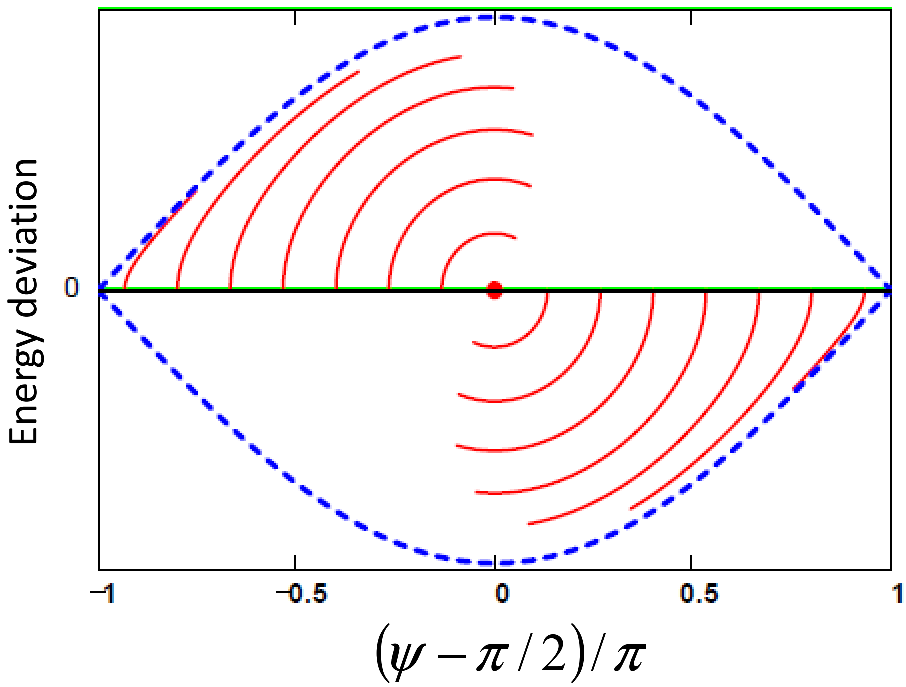
$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau; \quad \frac{d\pi_\tau}{ds} = \frac{1}{C}\frac{eV_{RF}}{p_o c}\sin\left(k_o h_{rf}\tau\right);$$

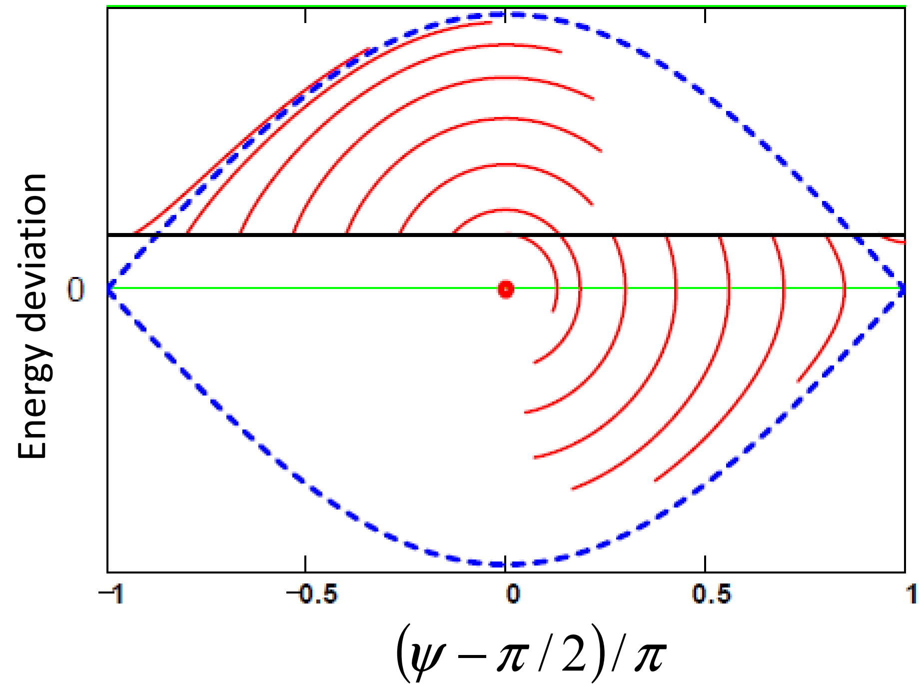



$$\frac{d^2\phi_{rf}}{ds^2} = u_{rf}\sin\phi_{rf}$$

$$u_{rf} = \eta\frac{1}{C}\frac{eV_{RF}k_0 h_{rf}}{p_0 c} \qquad \phi_{rf} = k_0 h_{rf}\tau$$

# Low Gain Regime: Qualitative Observation




The average energy of the electrons is right at resonant energy:

$$\lambda_0 \approx \frac{\lambda_w\left(1+K^2\right)}{2\gamma^2} \Rightarrow \gamma = \gamma_0 = \sqrt{\frac{\lambda_w\left(1+K^2\right)}{2\lambda_0}}$$

*Plots are taken from talk slides by Peter Schmuser.

The average energy of the electrons is slightly above the resonant energy:

$$\gamma = \gamma_0 + \Delta\gamma$$

With positive detuning, there is net energy loss by electrons.

# Low Gain Regime: Derivation of FEL Gain

Change in radiation power density (energy gain per seconds per unit area):

$$\Delta\Pi_r = c\varepsilon_0\left(E_{ext} + \Delta E\right)^2 - c\varepsilon_0 E_{ext}^2 \approx 2c\varepsilon_0 E_{ext}\Delta E$$

Average change rate in electrons' energy per unit beam area:

$$\Delta\Pi_e = \frac{j_0\langle P\rangle}{e}$$

*The average, <…>, is over all electrons in the beam.

Energy deviation at entrance

Pondermotive phase at entrance

$$\langle P(z)\rangle = \int_{-\infty}^{\infty} dP_0 \int_0^{2\pi} d\psi_0\, f\left(P_0,\psi_0\right) P\left(P_0,\psi_0,z\right)$$

Assuming radiation has the same cross section area as the electron beam, we obtain the change in electric field amplitude:

$$\Delta\Pi_r + \Delta\Pi_e = 0 \Rightarrow \boxed{\Delta E = -\frac{j_0\langle P\rangle}{2c\varepsilon_0 E_{ext} e}}$$

$$\left.\begin{aligned} \frac{dP}{dz} &= -eE\theta_s\cos(\psi) \\ \frac{d}{dz}\psi &= C + \frac{\omega}{\gamma_z^2 c\varepsilon_0}P \end{aligned}\right\} \Rightarrow \langle P\rangle = -eEl_w\theta_s\left\langle \int_0^1 \cos\left[\psi\left(\hat{z}\right)\right] d\hat{z} \right\rangle$$

$$\hat{z} = \frac{z}{l_w}$$

# Low Gain Regime: Derivation of FEL Gain

$$\frac{d^2}{d\hat{z}^2}\psi + \hat{u}\cos\psi = 0$$

$$\psi(\hat{z}) = \psi(0) + \psi'(0)\hat{z} - \hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\psi(\hat{z}_2)d\hat{z}_2 \qquad (1)$$

Assuming that all electrons have the same energy and uniformly distributed in the Pondermotive phase at the entrance of FEL: $P_0 = 0$ and $f(\psi_0) = \frac{1}{2\pi}$ .

The zeroth order solution for phase evolution is given by ignoring the effects from FEL interaction:

$$\left.\begin{aligned} &\frac{dP}{dz} = -eE\theta_s\cos(\psi) \\ &\frac{d}{dz}\psi = C + \frac{\omega}{\gamma_z^2 c\mathcal{E}_0}P \end{aligned}\right\} \Rightarrow \frac{d}{d\hat{z}}\psi = \hat{C} \Rightarrow \begin{cases} \psi(\hat{z}) = \psi_0 + \hat{C}\hat{z} \\ \psi'(0) = \hat{C} \end{cases} \qquad \hat{C} \equiv Cl_w$$

Inserting the zeroth order solution back into eq. (1) yields the 1st order solution:

$$\psi(\hat{z}) = \psi_0 + \hat{C}\hat{z} + \Delta\psi(\psi_0, \hat{z}) \qquad \Delta\psi(\psi_0, \hat{z}) \equiv -\hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\left[\psi_0 + \hat{C}\hat{z}_2\right]d\hat{z}_2$$

# Low Gain Regime: Derivation of FEL Gain

$$\Delta\psi(\psi_0,\hat{z}) \equiv -\hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\left[\psi_0 + \hat{C}\hat{z}_2\right] d\hat{z}_2$$

$$= -\frac{\hat{u}}{\hat{C}^2}\left\{\int_0^{\hat{C}\hat{z}} \sin(\psi_0 + x_1)dx_1 - \hat{C}\hat{z}\sin\psi_0\right\} = \frac{\hat{u}}{\hat{C}^2}\left[\cos\left(\psi_0 + \hat{C}\hat{z}\right) - \cos\psi_0 + \hat{C}\hat{z}\sin\psi_0\right]$$

$$\langle P\rangle = -eEl_w\theta_s\left\langle\int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z} + \Delta\psi(\psi_0,\hat{z})\right]d\hat{z}\right\rangle$$

Average energy loss of electrons

$$= eE\theta_s l_w\left\langle\int_0^1 \sin\left[\psi_0 + \hat{C}\hat{z}\right]\sin(\Delta\psi(\psi_0,\hat{z}))d\hat{z}\right\rangle - eE\theta_s l_w\left\langle\int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z}\right]\cos(\Delta\psi(\psi_0,\hat{z}))d\hat{z}\right\rangle$$

$$\approx eE\theta_s l_w\left\langle\int_0^1 \Delta\psi(\psi_0,\hat{z})\sin\left[\psi_0 + \hat{C}\hat{z}\right]d\hat{z}\right\rangle - \cancel{\frac{eE\theta_s l_w}{2\pi}\int_0^1 d\hat{z}\int_0^{2\pi} \cos\left[\psi_0 + \hat{C}\hat{z}\right]d\psi_0}$$

$$\left\langle\int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z}\right]\sin\left[\psi_0 + \hat{C}\hat{z}\right]d\hat{z}\right\rangle = 0$$

$$= \frac{eE\theta_s l_w}{2\pi}\frac{\hat{u}}{\hat{C}^2}\int_0^1 d\hat{z}\left\{\hat{C}\hat{z}\cos\left(\hat{C}\hat{z}\right)\int_0^{2\pi}\sin^2\psi_0 d\psi_0 - \sin\left(\hat{C}\hat{z}\right)\int_0^{2\pi}\cos^2\psi_0 d\psi_0\right\}$$

$$= -eE\theta_s l_w\frac{\hat{u}}{\hat{C}^3}\left(1 - \frac{\hat{C}}{2}\sin\hat{C} - \cos\hat{C}\right)$$

# Low Gain Regime: Derivation of FEL Gain

Growth in the amplitude of radiation field:

$$\Delta E = -\frac{j_0\langle P\rangle}{2c\varepsilon_0 E_{ext} e} = \frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2\gamma}\frac{l_w^3 E_{ext}}{I_A}\frac{2}{\hat{C}^3}\left(1-\frac{\hat{C}}{2}\sin\hat{C}-\cos\hat{C}\right)$$

$$\hat{u} = \frac{l_w^2 e E_{ext}\theta_s \omega}{\gamma_z^2 c \gamma m c^2}$$

$$I_A = \frac{4\pi\varepsilon_0 m c^3}{e}$$

The gain is defined as the relative growth in radiation power:

$$\boxed{g_s = \frac{(E_{ext}+\Delta E)^2 - E_{ext}^2}{E_{ext}^2} \approx \frac{2\Delta E}{E_{ext}} = \tau\cdot f\left(\hat{C}\right)}$$

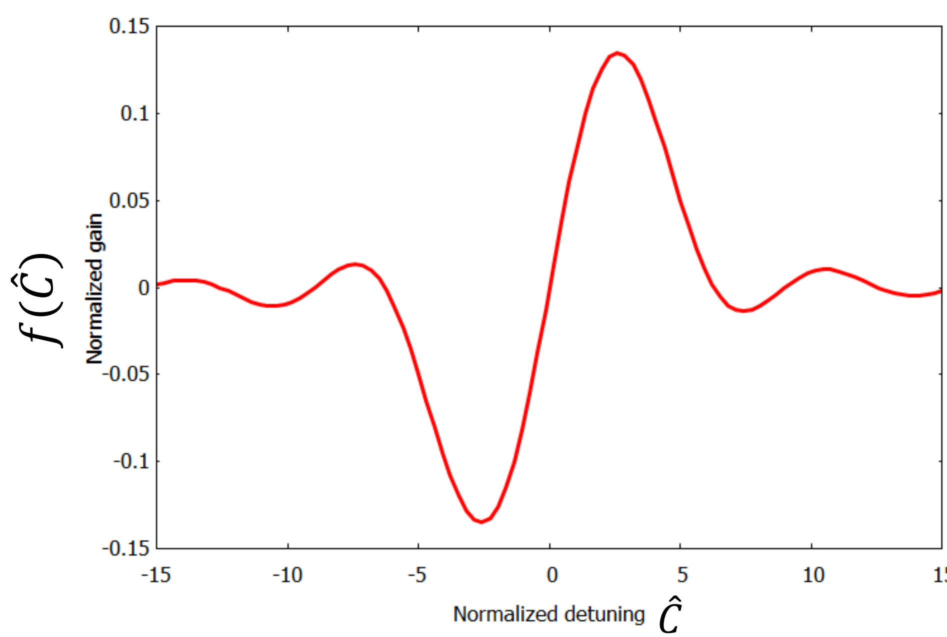
$$\tau \equiv \frac{2\pi j_0 \theta_s^2 \omega}{c\gamma_z^2\gamma}\frac{l_w^3}{I_A}$$

Cubic in FEL length

$$f\left(\hat{C}\right) = \frac{2}{\hat{C}^3}\left(1-\cos\hat{C}-\frac{\hat{C}}{2}\sin\hat{C}\right)$$

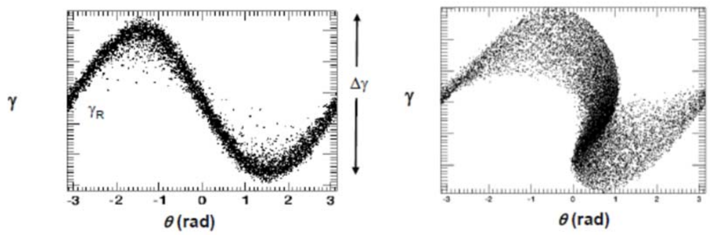
$$= -2\frac{d}{d\hat{C}}\frac{\sin^2\left(\hat{C}/2\right)}{\hat{C}^2}$$

As observed earlier, there is no gain if the electrons has resonant energy.



# High Gain Regime: Concept

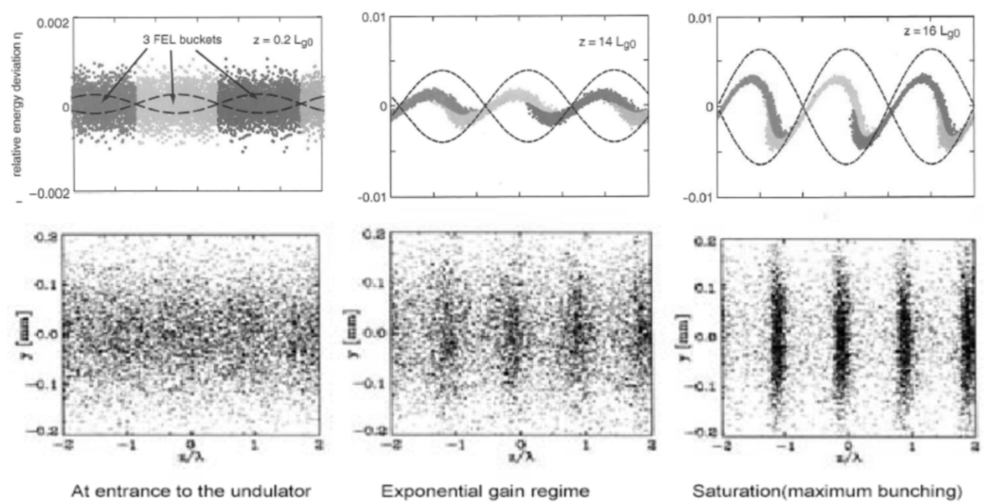
1. Energy kick from radiation field + dispersion/drift -> electron density bunching;

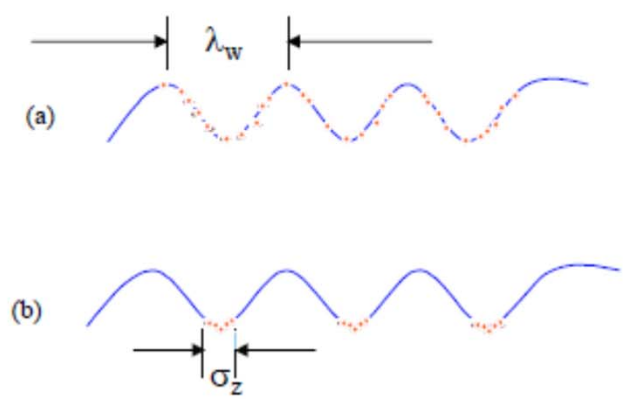


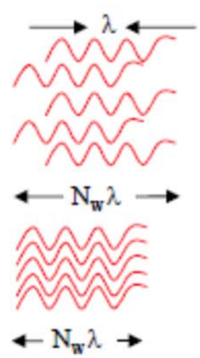

*The plots are for illustration only. The right plot actually shows somewhere close to saturation.

2. Electron density bunching makes more electrons radiates coherently -> higher radiation field;

3. Higher radiation fields leads to more density bunching through 1 and hence closes the positive feedback loop -> FEL instability.




$$|E| \propto \sqrt{N_e}$$

$$I_{incoherent} \propto N_e$$

$$|E| \propto N_e$$

$$I_{coherent} \propto N_e^2$$

The positive feedback loop between radiation field and electron density bunching is the underlying mechanism of high gain FEL regime.