Homework 9

Problem 1. 4 x 4 points. FODO cell.

Consider a general FODO cell comprised of two quadrupoles F and D separated by two drift sections, e.g. the structure below:

$$F:\ K_F = \frac{e}{pc}\frac{\partial B_y}{\partial x}, l_F;$$
$$O1:\ l_1$$
$$D:\ K_D = \frac{e}{pc}\frac{\partial B_y}{\partial x}, l_D;$$
$$O2:\ l_2$$

(a) write matrix (both x and y or 4x4) of general FODO cell (not assuming any limitations on K F,D).

(b) write stability criteria (for x and y) for periodic lattice built of this FOD cell. Hint – do not try to solve it!

(c,d) make transition to short lens approximation and assume equal strength of

$$l_F K_F = -K_D l_D = \frac{1}{f} = const, l_{F,D} \to 0$$
$$l = l_1 = l_2$$

and

(c) show that both x and y motion can be stable (e.g. prove so called strong focusing: combination of focusing and defocusing length can provide focusing in both directions);

(d) define (e.g solve) the stability criteria for such cell.

Problem 2. 2x5 points. Find not-trivial solution for building an unit 2x2 transport matrix out of repeating cells:

$$M^4 = I; M \neq I$$

(a) show that one of the solutions $trace(M) = 0$; Hint: used $M^2 = -I$;
(b) for a "symmetric" FODO cell and finite length equally strong quadrupoles $K_F = -K_D = K; l_F = l_D = L;\ l_1 = l_2 = l$ write the condition that $M_x^4 = M_y^4 = I$, e.g. the 4x4 transport matrix is unit.

Solutions: Problem 1:

(a) We know already matrices of all these elements and need just multiply them in correct order

$$M_x = O_2 D_x O_1 F_x = \begin{bmatrix} 1 & l_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\varphi_D & \dfrac{\sinh\varphi_D}{\omega_D} \\ \omega_D \sinh\varphi_D & \cosh\varphi_D \end{bmatrix} \begin{bmatrix} 1 & l_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi_F & \dfrac{\sin\varphi_F}{\omega_F} \\ -\omega_F \sin\varphi_F & \cos\varphi_F \end{bmatrix}$$

$$M_y = O_2 D_y O_1 F_y = \begin{bmatrix} 1 & l_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi_D & \dfrac{\sin\varphi_D}{\omega_D} \\ -\omega_D \sin\varphi_D & \cos\varphi_D \end{bmatrix} \begin{bmatrix} 1 & l_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\varphi_F & \dfrac{\sinh\varphi_F}{\omega_F} \\ \omega_F \sinh\varphi_F & \cosh\varphi_F \end{bmatrix}$$

$$\omega_F = \sqrt{|K_F|}; \varphi_F = \omega_F l_F; \omega_D = \sqrt{|K_D|}; \varphi_D = \omega_D l_D$$

$$M_x = O_2 D_x O_1 F_x = \begin{bmatrix} \mathrm{ch}_D + l_2 \omega_D \,\mathrm{sh}\,\varphi_D & \dfrac{\mathrm{sh}_D}{\omega_D} + l_2\,\mathrm{ch}_D \\ \omega_D\,\mathrm{sh}_D & \mathrm{ch}_D \end{bmatrix} \begin{bmatrix} \mathrm{cs}_F - l_1\omega_F\,\mathrm{sn}_F & \dfrac{\mathrm{sn}_F}{\omega_F} + l_1\,\mathrm{cs}_F \\ -\omega_F\,\mathrm{sn}_F & \mathrm{cs}_F \end{bmatrix}$$

$$= \begin{bmatrix} (\mathrm{ch}_D + l_2\omega_D\,\mathrm{sh}_D)(\mathrm{cs}_F - l_1\omega_F\,\mathrm{sn}_F) - \omega_F\,\mathrm{sn}_F\left(\dfrac{\mathrm{sh}_D}{\omega_D} + l_2\,\mathrm{ch}_D\right) & (\mathrm{ch}_D + l_2\omega_D\,\mathrm{sh}\,\varphi_D)\left(\dfrac{\mathrm{sn}_F}{\omega_F} + l_1\,\mathrm{cs}_F\right) + \mathrm{cs}_F\left(\dfrac{\mathrm{sh}_D}{\omega_D} + l_2\,\mathrm{ch}_D\right) \\ \omega_D\,\mathrm{sh}_D(\mathrm{cs}_F - l_1\omega_F\,\mathrm{sn}_F) - \omega_F\,\mathrm{sn}_F\,\mathrm{ch}_D & \omega_D\,\mathrm{sh}_D\left(\dfrac{\mathrm{sn}_F}{\omega_F} + l_1\,\mathrm{cs}_F\right) + \mathrm{cs}_F\,\mathrm{ch}_D \end{bmatrix}$$

and similarly ugly expression for vertical matrix,

$$M_y = O_2 D_y O_1 F_y =$$

$$= \begin{bmatrix} (\mathrm{cs}_D - l_2\omega_D\,\mathrm{sn}_D)(\mathrm{ch}_F + l_1\omega_F\,\mathrm{sh}_F) + \omega_F\,\mathrm{sh}_F\left(\dfrac{\mathrm{sn}_D}{\omega_D} + l_2\,\mathrm{cs}_D\right) & (\mathrm{cs}_D + l_2\omega_D\,\mathrm{sn}\,\varphi_D)\left(\dfrac{\mathrm{sh}_F}{\omega_F} + l_1\,\mathrm{ch}_F\right) + \mathrm{ch}_F\left(\dfrac{\mathrm{sn}_D}{\omega_D} + l_2\,\mathrm{cs}_D\right) \\ -\omega_D\,\mathrm{sn}_D(\mathrm{ch}_F + l_1\omega_F\,\mathrm{sh}_F) + \omega_F\,\mathrm{sh}_F\,\mathrm{cs}_D & -\omega_D\,\mathrm{sn}_D\left(\dfrac{\mathrm{sh}_F}{\omega_F} + l_1\,\mathrm{ch}_F\right) + \mathrm{ch}_F\,\mathrm{cs}_D \end{bmatrix}$$

(b) stability criteria are:

$$\left|Trace[M_{x,y}]\right| < 2$$

$$-2 < (\mathrm{ch}_D + l_2\omega_D\,\mathrm{sh}_D)(\mathrm{cs}_F - l_1\omega_F\,\mathrm{sn}_F) - \omega_F\,\mathrm{sn}_F\left(\frac{\mathrm{sh}_D}{\omega_D} + l_2\,\mathrm{ch}_D\right) + \omega_D\,\mathrm{sh}_D\left(\frac{\mathrm{sn}_F}{\omega_F} + l_1\,\mathrm{cs}_F\right) + \mathrm{cs}_F\,\mathrm{ch}_D < 2$$

$$-2 < (\mathrm{ch}_F + l_1\omega_F\,\mathrm{sh}_F)(\mathrm{cs}_D - l_2\omega_D\,\mathrm{sn}_D) - \omega_D\,\mathrm{sn}_D\left(\frac{\mathrm{sh}_F}{\omega_F} + l_1\,\mathrm{ch}_F\right) + \omega_F\,\mathrm{sh}_F\left(\frac{\mathrm{sn}_F}{\omega_F} + l_2\,\mathrm{cs}_D\right) + \mathrm{cs}_D\,\mathrm{ch}_F < 2$$

(c,d)

$$\begin{bmatrix} \cosh\varphi & \dfrac{\sinh\varphi}{\omega} \\ \omega\sinh\varphi & \cosh\varphi \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ \dfrac{1}{f} & 1 \end{bmatrix}; \begin{bmatrix} \cos\varphi & \dfrac{\text{sih}\varphi}{\omega} \\ -\omega\sin\varphi & \cos\varphi \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$$

$$M_x = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1+\dfrac{l}{f} & l \\ \dfrac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1-\dfrac{l}{f} & l \\ -\dfrac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1-\left(\dfrac{l}{f}\right)^2 - \dfrac{l}{f} & 2l+\dfrac{l^2}{f} \\ -\dfrac{l}{f^2} & 1+\dfrac{l}{f} \end{bmatrix}$$

$$M_y = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1-\left(\dfrac{l}{f}\right)^2 + \dfrac{l}{f} & 2l-\dfrac{l^2}{f} \\ -\dfrac{l}{f^2} & 1-\dfrac{l}{f} \end{bmatrix}$$

Stability criteria is

$$\left|Trace[M_{x,y}]\right| < 2 \;\; \to \;\; -2 < 2-\left(\frac{l}{f}\right)^2 < 2$$

$$\left(\frac{l}{f}\right)^2 < 4; \left|\frac{l}{f}\right| < 2$$

can be satisfied for both directions.

Problem 2. Ignoring trivial solution and complications imposed by $M^2 = I$

$$M^4 = I \to M^2 = \pm I \;\; pick \;\; M^2 = -I; \;\; ad - bc = 1$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & a^2+bc \end{bmatrix} \begin{bmatrix} a(a+d)-1 & b(a+d) \\ c(a+d) & d(a+d)-1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

has obvious solution

$$TraceM = a + d = 0$$

Using previous problem, we can write

$$Trace[M_{x,y}] = 0$$

$$(\cosh\varphi + l\omega\sinh\varphi)(\cos\varphi - l\omega\sin\varphi) - \omega\sin\varphi\left(\frac{\sinh\varphi}{\omega} + l\cosh\varphi\right)$$

$$+\omega\sinh\varphi\left(\frac{\sin\varphi}{\omega} + l\cos\varphi\right) + \cos\varphi\cosh\varphi = 0$$

This a transcendental equation, which has solution which can be found numerically.