HW II.1 (4 points): Pillbox cavity

(a) Calculate the RF surface resistance and skin depth of room-temperature copper at 500 MHz. Use DC resistivity $\rho = 1.7\cdot10^{-8}$ Ohm·m.

$$\delta = \frac{1}{\sqrt{\pi f \mu_0 \sigma}} = \sqrt{\frac{\rho}{\pi f \mu_0}} = \sqrt{\frac{1.7\cdot10^{-8}}{\pi\cdot500\cdot10^{6}\cdot4\pi\cdot10^{-7}}} = 2.93\cdot10^{-6}\text{ m} = 2.93\ \mu\text{m}$$

$$R_s = \frac{1}{\sigma\cdot\delta} = \frac{\rho}{\delta} = \frac{1.7\cdot10^{-8}}{2.93\cdot10^{-6}} = 5.8\cdot10^{-3}\text{ Ohm} = 5.8\text{ mOhm}$$

(b) Calculate the RF surface resistance of superconducting niobium at 500 MHz at T = 4.2 K and T = 2 K. Assume a residual resistance $R_{res} = 10\cdot10^{-9}$ Ohm. What is the ratio of superconducting niobium to that of room-temperature copper?

$$R_s(T) = R_{BCS} + R_{res} \approx 2\cdot10^{-4}\left(\frac{f[\text{MHz}]}{1500}\right)^2 \frac{1}{T} e^{-\frac{17.67}{T}} + 10\cdot10^{-9}$$

R_s (4.2 K) = 78.8 nOhm + 10 nOhm = 88.8 nOhm

R_s (2 K) = 1.61 nOhm + 10 nOhm = 11.6 nOhm

R_{s_Cu} / R_{s_Nb} (4.2 K) = $1.53\cdot10^{-5}$

R_{s_Cu} / R_{s_Nb} (2 K) = $2\cdot10^{-6}$

(c) Design a cylindrical (pillbox) cavity that operates in the TM_{010} mode at 500 MHz with an axial electric field of E_0 = 1 MV/m, and a length $l = \lambda/2$, where λ is the RF wavelength in free space. Calculate the length and diameter of the cavity. Calculate the maximum H and E fields on the cavity wall. Where do they occur? Calculate the electromagnetic stored energy in the cavity.

$$\lambda_{010} = c/f \approx 3\cdot10^{8}/500\cdot10^{6} = 0.6\text{ m} = 60\text{ cm}$$

$$\lambda_{010} = 2.61\cdot R \rightarrow D = 2R = \frac{2\lambda_{010}}{2.61} = \frac{2\cdot0.6}{2.61} = 0.46\text{ m} = 46\text{ cm}$$

$l = \lambda/2 = 0.3$ m = 30 cm

The maximum surface electric field occurs on axis, on both end walls ⇒

$E_{max} = E_0\cdot J_0(0) = 1\cdot10^{6}\cdot1 = 1\cdot10^{6}$ V/m = 1 MV/m

The maximum surface magnetic field occurs when $J_1(x)$ reaches maximum at $x_{max} = 1.8412 \Rightarrow r_{max} = 1.8412\cdot R/2.405 = 0.176$ m = 17.6 cm ⇒

$H_{max} = E_0/\eta\cdot J_1(1.8412) = 1\cdot10^{6}/377\cdot0.582 = 1{,}544$ A/m

$$U = \frac{V_c^2}{\omega\cdot R/Q},$$

R/Q for the pillbox cavity is 196 Ohm,

$V_c = E_{acc}\cdot l = E_0\cdot T\cdot l = E_0\cdot 2/\pi\cdot\lambda/2 = E_0\cdot\lambda/\pi = 1\cdot10^{6}\cdot0.6/3.14 = 191\cdot10^{3}$ V = 191 kV,

then $U = (1.91\cdot10^{5})^2/(2\pi\cdot500\cdot10^{6}\cdot196) = 0.0592$ J = 59.2 mJ

(d) Calculate the power loss P_c, the quality factor Q_0, and the decay time τ for a room-temperature copper surface and for niobium surface at 4.2 K.

For copper at room temperature $Q_0 = \frac{G}{R_s} = \frac{257}{0.0058} = 44{,}300$,

then $P_c = \frac{V_c^2}{R/Q \cdot Q_0} = \frac{\left(1.91 \cdot 10^5\right)}{196 \cdot 44{,}300} = 4{,}200\ \text{W} = 4.2\ \text{kW}$,

the decay time $\tau = Q_0/\omega = 44{,}300/(2\pi \cdot 500 \cdot 10^6) = 14.1 \cdot 10^{-6}\ \text{s} = 14.1\ \text{ms}$

For niobium at 4.2 K $Q_0 = \frac{G}{R_s} = \frac{257}{88.8 \cdot 10^{-9}} = 2.89 \cdot 10^9$,

then $P_c = \frac{V_c^2}{R/Q \cdot Q_0} = \frac{\left(1.91 \cdot 10^5\right)}{196 \cdot 2.89 \cdot 10^9} = 0.064\ \text{W} = 64\ \text{mW}$,

the decay time $\tau = Q_0/\omega = 2.89 \cdot 10^9/(2\pi \cdot 500 \cdot 10^6) = 0.92\ \text{s}$

HW II.2 (1 point): Equivalent RLC circuit

A superconducting cavity with residual resistivity of 10 nOhm operates at a frequency of 1300 MHz. The geometry factor is 267 Ohm, and R/Q = 900 Ohm (*accelerator definition*). Calculate parameters of the equivalent parallel *RLC* lumped-*circuit* model of this cavity.

As some parameters are not specified, let us assume that temperature is 2 K and the cavity is made of niobium. Then

$$R_s(T) = R_{BCS} + R_{res} \approx 2 \cdot 10^{-4} \left(\frac{f[\text{MHz}]}{1500} \right)^2 \frac{1}{T} e^{-\frac{17.67}{T}} + 10 \cdot 10^{-9} = 20.9\ \text{nOhm},$$

$Q_0 = G/R_s = 257/(20.9 \cdot 10^{-9}) = 1.23 \cdot 10^{10}$

$R = R/Q \cdot Q_0/2 = 900 \cdot 1.23 \cdot 10^{10}/2 = 5.54 \cdot 10^{12}$ Ohm

$L = (R/Q)/2\omega = 900/(2 \cdot 2\pi \cdot 1300 \cdot 10^6) = 0.055\ \mu\text{H}$

$C = 2/(R/Q \cdot \omega) = 2/(900 \cdot 2\pi \cdot 1300 \cdot 10^6) = 0.272\ \text{pF}$

HW II.3 (2 points): Anomalous skin effect

(a) Determine the improvement factor that can be expected for the Q_0 of a 500 MHz copper cavity if it is cooled down from room temperature to liquid helium temperature (4.2 K). What is the quality factor of the pillbox cavity from problem *HW II.1* with copper walls cooled to 4.2 K? The ρl product of copper is $6.8 \cdot 10^{-16}$ Ohm·m^2. The resistivity of copper at room temperature is $1.7 \cdot 10^{-8}$ Ohm·m. A Residual Resistivity Ratio (RRR) of 100 was supposed to be specified!

$\rho_{4K} = \frac{\rho_{300K}}{RRR} = 1.7 \times 10^{-10}$ Ohm·m

$l = \frac{\rho l}{\rho_{4K}} = 4\ \mu\text{m}$

$\alpha_s = \frac{3}{4} \frac{\mu_0 \omega}{\rho l} l^3 = 279 > 0.016$, anomalous skin effect should be considered

$$R_s(l \to \infty) = \left(\sqrt{3}\pi\left(\frac{\mu_0}{4\pi}\right)^2 \omega^2 \rho l\right)^{1/3} = 0.715 \text{ mOhm}$$

$$R_{4K}(l) = R_s(\infty)(1 + 1.157 \cdot \alpha_s^{-0.2757}) = 0.89 \text{ mOhm}$$

$$R_{300K} = \sqrt{\frac{\omega\mu_0\rho}{2}} = 5.9 \text{ mOhm}$$

$$\frac{Q_{4K}}{Q_{300K}} = \frac{R_{300K}}{R_{4K}} = 6.63$$

(b) Calculate the surface resistivity of niobium of RRR = 30 (reactor grade niobium) and RRR = 250 (high RRR niobium) at 500 MHz in the normal conducting state at 10 K (assume that RRR is given for this temperature.) What is the Q_0 of the pillbox cavity from problem *HW II.1* with niobium walls at room temperature? What is the improvement factor for a niobium cavity on cooling from room temperature to 10 K? The resistivity of niobium at room temperature is $15\cdot10^{-8}$ Ohm·m, and the ρl product of niobium is $6\cdot10^{-16}$ Ohm·m^2.

$$\rho_{RRR30}(10K) = \frac{\rho(300K)}{30} = 5\cdot10^{-9} \text{ Ohm·m}$$

$$\rho_{RRR250}(10K) = \frac{\rho(300K)}{250} = 6\cdot10^{-10} \text{ Ohm·m}$$

$$l_{RRR30}(10K) = \frac{\rho l}{\rho_{RRR30}(10K)} = 1.27\cdot10^{-7} \text{ m}$$

$$l_{RRR250}(10K) = \frac{\rho l}{\rho_{RRR250}(10K)} = 1\cdot10^{-6} \text{ m}$$

$$\alpha_s = \frac{3}{4}\mu_0\omega\left(\frac{1}{\rho l}\right)l^3$$

$\alpha_{s\_RRR30} = 0.0085$ - classical expression is valid

$$\delta_{RRR30} = \sqrt{\frac{\rho_{RRR30}}{\pi f \mu_0}} = 1.59\cdot10^{-6} \text{ m}$$

$$R_{s\_RRR30} = \frac{\rho_{RRR30}}{\delta_{RRR30}} = 3.1\cdot10^{-3} \text{ Ohm}$$

$\alpha_{s\_RRR250} = 4.93$

$$R_s(l \to \infty) = 3.789\cdot10^{-5}\omega^{2/3}(\rho l)^{1/3} = 6.85\cdot10^{-4} \text{ Ohm}$$

$$R_s(l_{RRR250}) = R_s(\infty)\cdot\left(1 + 1.157\alpha_s^{-0.2757}\right) = 1.2\cdot10^{-3} \text{ Ohm}$$

assuming G = 257 Ohm (pillbox cavity)

$$Q_{RRR30} = G/R_{s\_RRR30} = 81{,}806$$

$Q_{RRR250} = G/R_{s\_RRR250} = 214,840$

at room temperature R_s = 0.0172 Ohm

Q_{RT} = 14,942

The improvement factor is 5.475 for RRR30 and 14.38 for RRR250