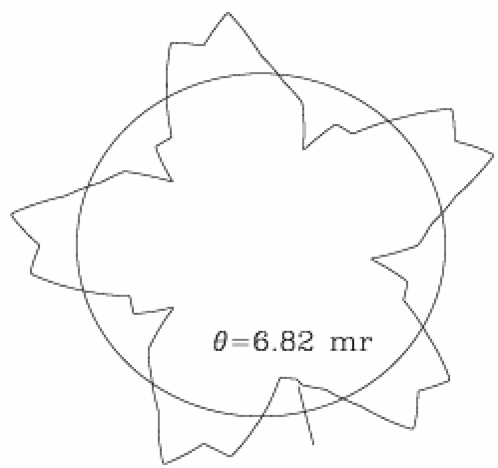
PHY 554

Fundamentals of Accelerator Physics

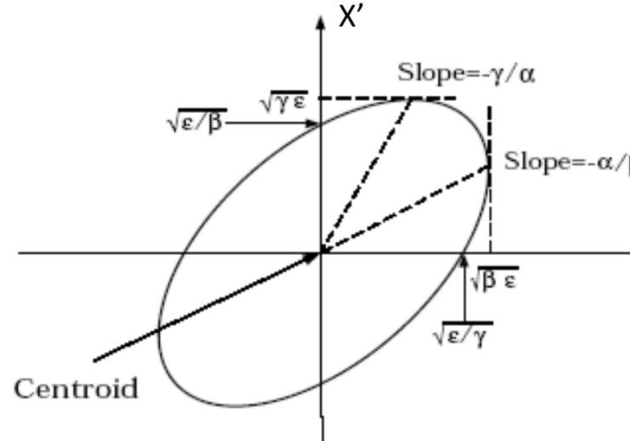
Lecture 6: Emittance and Orbit Errors

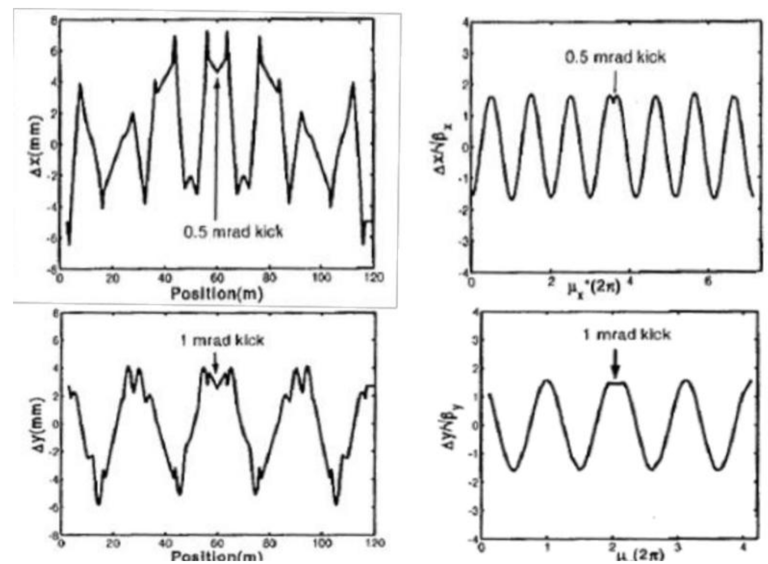


From single-particle invariants to beam emittance, Gaussian beams, normalized emittance, and closed-orbit distortion.

Today’s class: from single particle to real beams

- Review Floquet motion and the Courant-Snyder invariant.
- Define beam emittance from second moments of a distribution.
- Show why rms emittance is invariant in linear transport.
- Use the sigma matrix to connect emittance and Twiss functions.
- Introduce Gaussian beam distributions, admittance and dynamic aperture.
- Discuss normalized emittance, adiabatic damping and simple lattice examples.
- Start linear magnetic errors: dipole kicks and closed-orbit distortion.





Design question: how do local errors and beam distributions become machine-wide performance limits?

What we learned:

Floquet Theorem $$X'' + K(s)X = 0 \qquad K(s) = K(s+L)$$

$$X(s) = aw(s)e^{j\psi(s)}, \quad w(s) = w(s+L), \quad \psi(s+L) - \psi(s) = 2\pi\mu$$

$$\beta(s) = w^2, \quad \alpha = -\frac{1}{2}\beta', \quad \gamma = \frac{1+\alpha^2}{\beta}, \qquad w(s) = \sqrt{\beta(s)}, \quad \psi(s) = \int_{s_0}^{s} \frac{1}{\beta} ds$$

Beta tells the local amplitude scale; tune tells the accumulated phase per turn.

$$\begin{pmatrix} X(s_2) \\ X'(s_2) \end{pmatrix} = M(s_2, s_1) \begin{pmatrix} X(s_1) \\ X'(s_1) \end{pmatrix}$$

$$M(s_2, s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1 \sin\mu) & \sqrt{\beta_1\beta_2}\sin\mu \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu - \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1\beta_2}}\cos\mu & \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu - \alpha_1 \sin\mu) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

The values of the Courant–Snyder parameters α_2, β_2, γ_2 at s_2 are related to α_1, β_1, γ_1 at s_1 by

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1$$

The evolution of the betatron amplitude function in a drift space is

$$\beta_2 = \frac{1}{\gamma_1} + \gamma_1 (s - \frac{\alpha_1}{\gamma_1})^2 = \beta^* + \frac{(s-s^*)^2}{\beta^*},$$

$$\alpha_2 = \alpha_1 - \gamma_1 s = -\frac{(s-s^*)}{\beta^*}, \quad \gamma_2 = \gamma_1 = \frac{1}{\beta^*}$$

Passing through a thin-lens quadrupole, the evolution of betatron function is

$$\beta_2 = \beta_1, \quad \alpha_2 = \alpha_1 + \frac{\beta_1}{f}, \quad \gamma_2 = \gamma_1 + \frac{2\alpha_1}{f} + \frac{\beta_1}{f^2}$$

The combination of quads and drifts (dipole with thin lens approx.) can reshape the phase space ellipse

$$X=\sqrt{2\beta J}\cos\psi, \quad X'=-\sqrt{\frac{2J}{\beta}}(\sin\psi+\alpha\cos\psi)$$

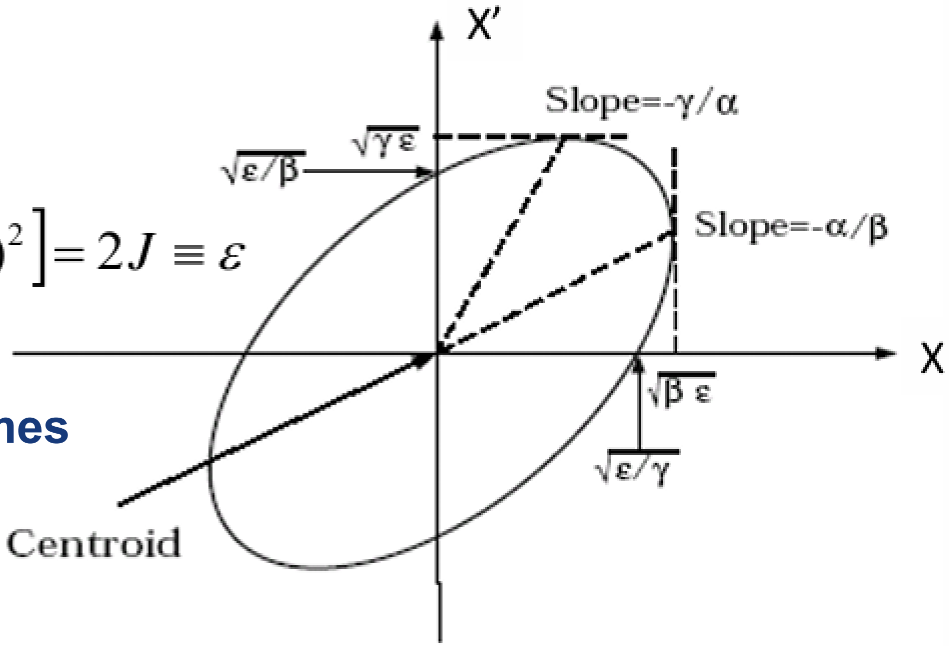
$$P_X=\beta X'+\alpha X=-\sqrt{2\beta J}\sin\psi$$

(X,P_X) form a normalized phase space coordinates with $X^2+P_X^2=2\beta J$, here J is called **action**.

Courant-Snyder Invariant

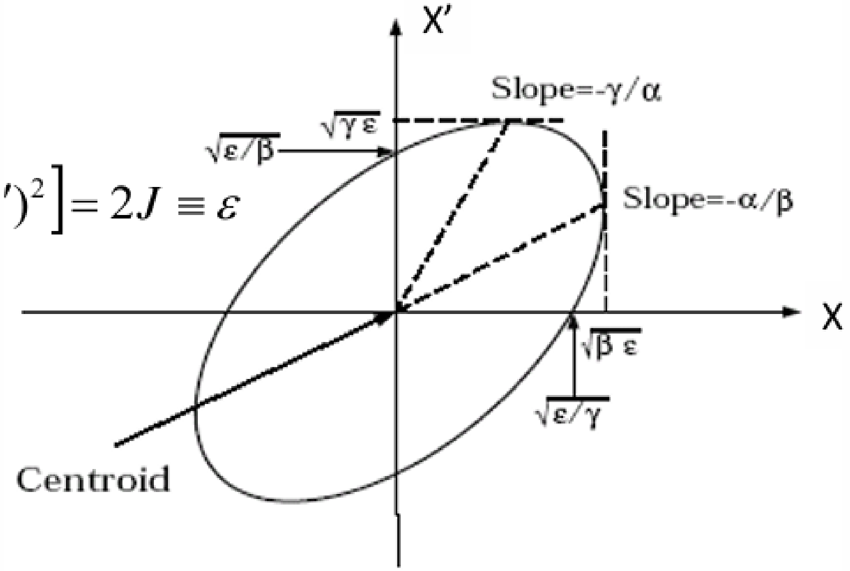
$$\gamma X^2+2\alpha XX'+\beta X'^2=\frac{1}{\beta}\left[X^2+(\alpha X+\beta X')^2\right]=2J\equiv\varepsilon$$

The ellipse in physical coordinates becomes a circle after the Twiss normalization.



Courant-Snyder Invariant

$$\gamma X^2 + 2\alpha X X' + \beta X'^2 = \frac{1}{\beta}\left[X^2 + (\alpha X + \beta X')^2\right] = 2J \equiv \varepsilon$$



A real beam is a distribution in (X, X'), not one trajectory..

Emittance of a beam

Given a normalized distribution function ρ(X, X′) with ∫ρ(X, X′)dXdX′ = 1, the moments of the beam distribution are

$$\langle X \rangle = \int X\rho(X,X')dXdX', \quad \langle X' \rangle = \int X'\rho(X,X')dXdX',$$

$$\sigma_X^2 = \int (X - \langle X \rangle)^2 \rho(X,X')dXdX', \quad \sigma_{X'}^2 = \int (X' - \langle X' \rangle)^2 \rho(X,X')dXdX',$$

$$\sigma_{XX'} = \int (X - \langle X \rangle)(X' - \langle X' \rangle)\rho(X,X')dXdX' = r\sigma_X \sigma_{X'}$$

Where σ_X and $\sigma_{X'}$ are the rms beam widths, $\sigma_{XX'}$ is the correlation, and r is the correlation coefficient. The rms beam emittance is then defined as

$$\varepsilon_{rms} = \sqrt{\sigma_X^2 \sigma_{X'}^2 - \sigma_{XX'}^2} = \sigma_X \sigma_{X'} \sqrt{1 - r^2}$$

The rms emittance is the area scale of the rms phase-space ellipse.

$$\sigma_X^2 = \beta \varepsilon_{rms}$$

The rms emittance is invariant in linear transport:

$$\varepsilon^2 = \sigma_X^2 \sigma_{X'}^2 - \sigma_{XX'}^2$$

$$\sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2, \quad \sigma_{X'}^2 = \langle X'^2 \rangle - \langle X' \rangle^2, \quad \sigma_{XX'} = \langle XX' \rangle - \langle X \rangle \langle X' \rangle$$

we find

$$\frac{d\sigma_X^2}{ds} = 2\langle XX' \rangle - 2\langle X \rangle \langle X' \rangle$$

$$\frac{d\sigma_{X'}^2}{ds} = 2\langle X'X'' \rangle - 2\langle X' \rangle \langle X'' \rangle$$

$$\frac{d\sigma_{XX'}}{ds} = \langle X'^2 \rangle - \langle X' \rangle^2 - \langle X \rangle \langle X'' \rangle + \langle XX'' \rangle$$

$$X'' + KX = 0$$

$$\frac{d\varepsilon^2}{ds} = \sigma_X^2 \frac{d\sigma_{X'}^2}{ds} + \sigma_{X'}^2 \frac{d\sigma_X^2}{ds} - 2\sigma_{XX'} \frac{d\sigma_{XX'}}{ds} = 0$$

A linear lattice can change the ellipse shape, but not its area.

The Gaussian distribution function

The equilibrium beam distribution in the linearized betatron phase space may be any function of the invariant action. However, the Gaussian distribution function is commonly used to evaluate the beam properties. Expressing the normalized Gaussian distribution in the normalized phase space, we obtain

$$\rho(X, P_X) = \frac{1}{2\pi\sigma_X^2} e^{-(X^2+P_X^2)/2\sigma_X^2}$$

where $\langle X^2 \rangle = \langle P_X{}^2 \rangle = \sigma_X{}^2 = \beta_X \varepsilon_{rms}$ with an rms emittance ε_{rms}. Transforming (X,P_X) into the action-angle variables (J, ψ) with

$$X = \sqrt{2\beta J}\cos\psi, \quad P_X = -\sqrt{2\beta J}\sin\psi$$

The Jacobian of the transformation is β_X, and the distribution function becomes

$$\rho(J) = \frac{1}{\varepsilon_{rms}} e^{-J/\varepsilon_{rms}}, \quad \rho(\varepsilon) = \frac{1}{2\varepsilon_{rms}} e^{-\varepsilon/2\varepsilon_{rms}}$$

The percentage of particles contained within $\varepsilon = n\varepsilon_{rms}$ is $1 - e^{-n/2}$

$\epsilon/\epsilon_{\mathrm{rms}}$	2	4	6	8
Percentage in 1D [%]	63	86	95	98
Percentage in 2D [%]	40	74	90	96

When talking about emittance, be explicit about what you mean: RMS, 95%, 98%, etc

Adiabatic damping and the normalized emittance: $\varepsilon_n = \varepsilon\beta\gamma$

The Courant–Snyder invariant, derived from the phase-space coordinate X, X′, is not invariant when the energy is changed. To obtain the Liouville invariant phase-space area, we should use the conjugate phase-space coordinates (X, P_X) in Hamiltonian. Since $p_X = p_X' = mc\beta\gamma X'$, where m is the particle's mass, p is its momentum, and $\beta\gamma$ is the Lorentz relativistic factor, the *normalized emittance* defined by $\boldsymbol{\varepsilon_n = \varepsilon\beta\gamma}$ is invariant. The beam emittance decreases with increasing beam momentum, i.e. $\varepsilon = \varepsilon_n/\beta\gamma$. This is called ***adiabatic damping***. Since the transverse velocity of a particle does not change during acceleration, the transverse angle $X' = p_X/p$ becomes smaller at a higher particle momentum. Thus the beam emittance $\varepsilon = \varepsilon_n/\beta\gamma$ decreases with energy. The adiabatic damping also applies to beam emittance in proton or electron **linacs**.

Because of the quantum fluctuation, The beam emittance in electron storage rings **increases** with energy ($\sim\gamma^2$). The corresponding normalized emittance is proportional to γ^3.

Geometric emittance can shrink during acceleration without violating phase-space conservation.

Linear magnetic field errors: what changes the equation?

$$x'' + K_x(s)x = \frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = -\frac{\Delta B_x}{B\rho}$$

$$\Delta B_y + j\Delta B_x = B_0 \sum_n (b_n + ja_n)(x + jy)^n,$$

$$B_y = B_0 b_0, \quad B_x = B_0 a_0, \qquad \text{Dipole field error}$$

$$B_y = B_0 b_1 x, \quad B_x = B_0 b_1 y, \qquad \text{Quadrupole field error}$$

$$\cancel{B_y = -B_0 a_1 y, \quad B_x = B_0 a_1 x, \qquad \text{Skew Quadrupole field error}}$$

$$B_y = B_0 b_2 (x^2 - y^2), \quad B_x = 2B_0 b_2 xy,$$

Sextupole field error

$$B_y = -2B_0 a_2 xy, \quad B_x = B_0 a_2 (x^2 - y^2),$$

$$x'' + [K_x(s) + k(s)]x = \frac{b_0}{\rho}, \qquad y'' + [K_y(s) - k(s)]y = -\frac{a_0}{\rho}$$

Effect of dipole field error:

We consider a single localized dipole error with the kick angle given by θ=ΔBℓ/Bρ. Because of the dipole field error, the reference orbit is perturbed! The idea is to find a new closed orbit that include the dipole field error.

$$X'' + K_X(s)X = \theta\delta(s - s_0)$$

The closed orbit is given by the following condition:

$$\begin{pmatrix} X_0 \\ X_0' - \theta \end{pmatrix} = M \begin{pmatrix} X_0 \\ X_0' \end{pmatrix} = \begin{pmatrix} \cos\Phi + \alpha_0 \sin\Phi & \beta_0 \sin\Phi \\ -\gamma_0 \sin\Phi & \cos\Phi - \alpha_0 \sin\Phi \end{pmatrix} \begin{pmatrix} X_0 \\ X_0' \end{pmatrix}$$

Where Φ=2πν, ν is the betatron tune, the parameters α_0, β_0, and γ_0 are values of the Courant-Snyder parameters at the kicker location. The solution is

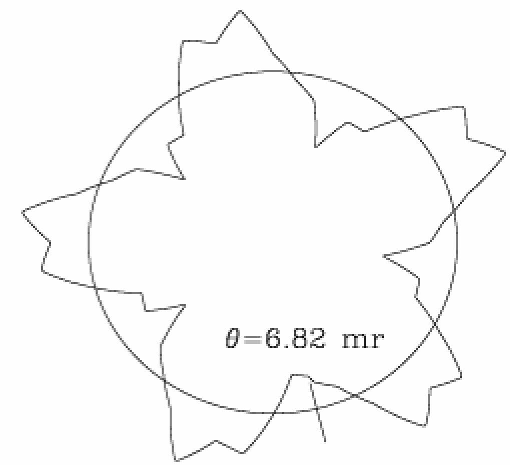
$$X_0 = \frac{\beta_0\theta}{2\sin\pi\nu}\cos\pi\nu,$$

$$X_0' = \frac{\theta}{2\sin\pi\nu}(\sin\pi\nu - \alpha_0\cos\pi\nu)$$

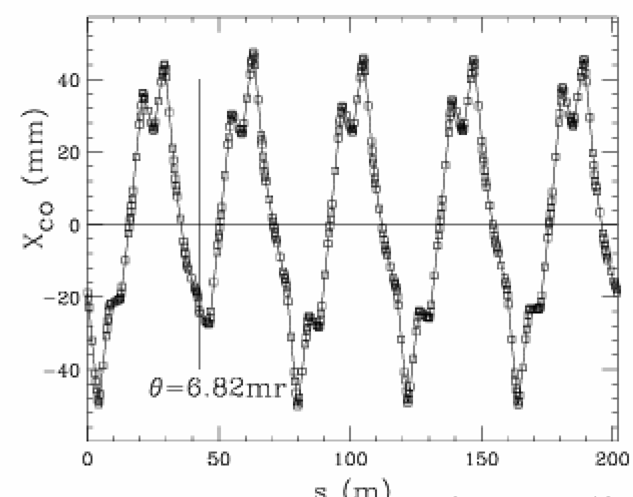
We have solved the closed orbit at one point s_0. The closed orbit of the accelerator can be obtained by making mapping matrix:

$$\begin{pmatrix} X(s) \\ X'(s) \end{pmatrix}_{\rm co} = M(s,s_0)\begin{pmatrix} X_0 \\ X_0' \end{pmatrix} \qquad X_{\rm co}(s) = G(s,s_0)\theta$$

$$G(s,s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin\pi\nu}\cos[\pi\nu - |\psi(s)-\psi(s_0)|]$$

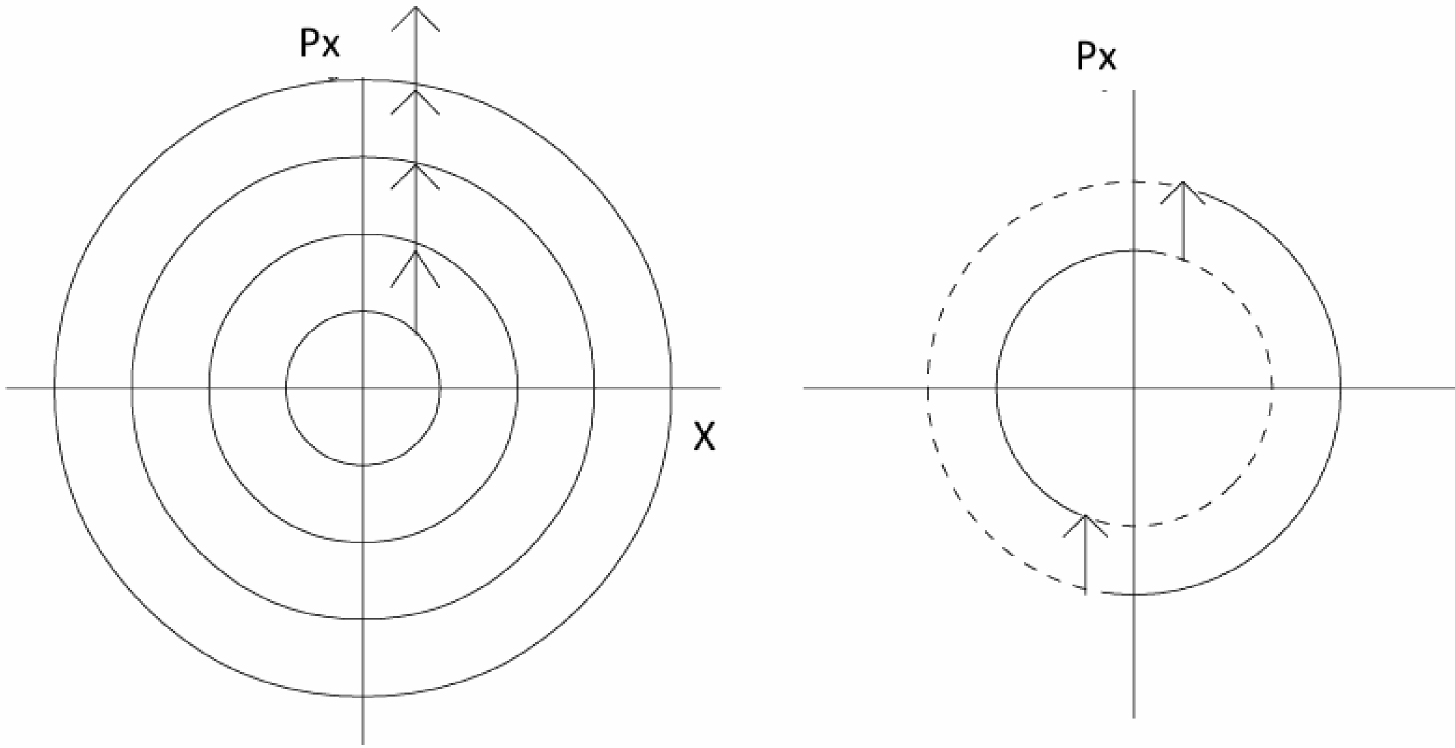


Note that the closed orbit is described by Green's function. When the betatron tune is an integer, the closed orbit diverges. Each time, when the particle arrives the same location will receive a coherent kick and the particle becomes unstable.



A dipole error does not just make a local mistake; it excites a ring-wide orbit response.

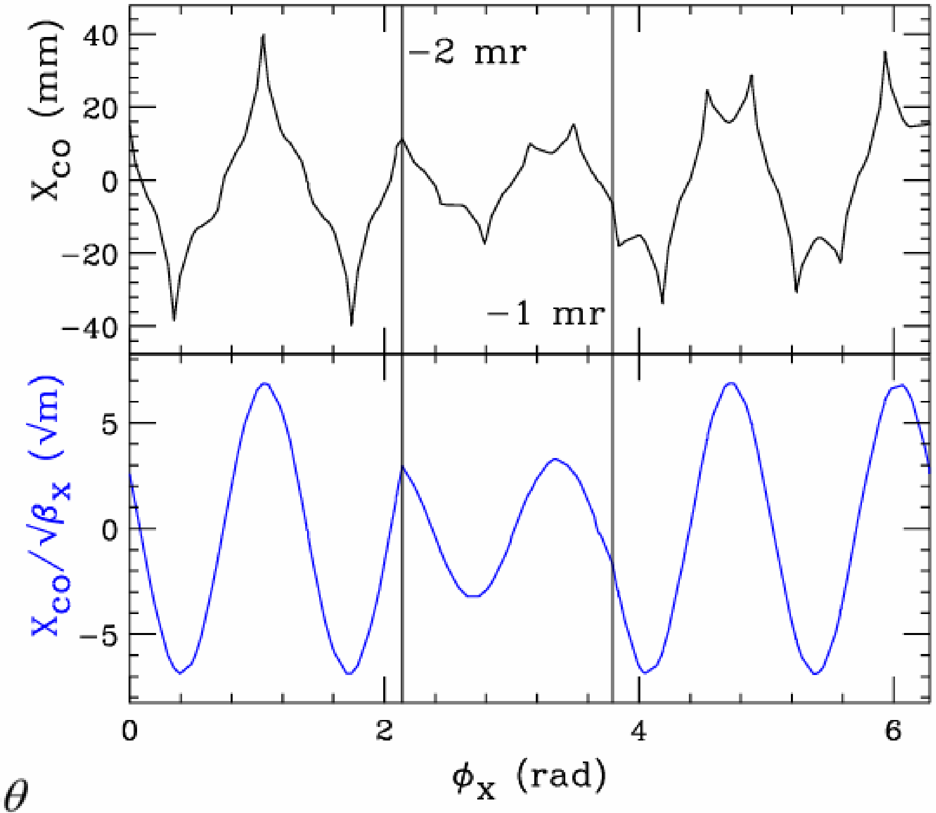
Why integer tune is dangerous



- At integer tune, the particle returns to the error with the same betatron phase each turn.
- The kick is coherent: each turn adds in the same direction in normalized phase space.
- The closed-orbit formula diverges because the lattice cannot absorb the repeated kick.
- At half-integer tune, idealized kicks can cancel in this simple picture.

Multiple dipole errors

• Example: circumference 360 m, 18 FODO cells and horizontal tune nu_x = 4.8.
• Two dipole errors produce an orbit distortion distributed around the ring.
• The raw orbit follows both the beta beating and phase advance of the lattice.
• Normalizing by sqrt(beta) makes the phase response easier to see.
• Linear superposition lets many errors be combined into an orbit response model.



$$\begin{pmatrix} X(s) \\ X'(s) \end{pmatrix}_{co} = M(s,s_0)\begin{pmatrix} X_0 \\ X_0' \end{pmatrix} \qquad X_{co}(s) = G(s,s_0)\theta$$

$$G(s,s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin\pi\nu}\cos[\pi\nu - |\psi(s)-\psi(s_0)|]$$

Orbit correction is a global problem, even when the error is local.

What to remember from Lecture 6

- Courant-Snyder action is the single-particle invariant for linear betatron motion.
- RMS emittance is the beam version of phase-space area and is preserved by ideal linear transport.
- The sigma matrix connects measured beam moments to Twiss parameters and emittance.
- Gaussian beam conventions require care: rms, 95%, geometric and normalized have different meanings.
- Normalized emittance stays approximately invariant during acceleration; geometric emittance damps as 1/(beta gamma).
- Dipole field errors create closed-orbit distortion described by a Green's function response.
- Integer tunes are dangerous because kicks repeat coherently.

Questions to think about

- Why can focusing shrink beam size without reducing emittance?
- Which emittance convention is appropriate for a machine aperture limit?
- Why does a local dipole kick produce a global closed-orbit response?

Homework #2: Lab report II, due next Monday 9/21/26

Use the Elegant files from Lab I. Answer following questions (20 pts, each 4):

1. What are the betatron tunes for the entire ring? What is the phase advance evolution along the ring? Are they consistent with each other?
2. What are the maximum/minimum beta/alpha functions for both x and y? Where are their locations? (**Hint**, you might need to divide the element to see what's in each element, e.g., QF: Quad, L=1 can be divided into QF: quad, L=0.5 but replace the QF in the line section by QF, QF)
3. What can you say about these TWISS functions, do they agree with your estimation using thin lens approximation? Try stronger quads and see how thin lens approximation holds.
4. What are the emittances for the beam? Are they RMS, 95%, or 98%?
5. Can you use what was learned in the class and calculate what is RMS beam sizes? Do they agree with what you got from Lab I?