High Power RF Engineering -Cavity part 1: RLC model

Binping Xiao

Electron-Ion Collider

BROOKHAVEN
NATIONAL LABORATORY

Jefferson Lab

U.S. DEPARTMENT OF ENERGY | Office of Science

Applications of RF

- Broadcast: Radio (AM/FM), TV, < 1GHz.

- Microwave oven, 2.45GHz.

- Radar, normally between 0.4 and 36 GHz.

- Communications
  - Cellular (from sub-GHz to a few tens of GHz)
  - GPS (1575.42MHz, 1227.6MHz 1176MHz)
  - Bluetooth 2.4GHz ISM (industrial, scientific and medical) band from 2.402GHz to 2.480GHz.
  - WiFi (2.4GHz and 5GHz)
  - And more

- Particle accelerator: use EM field to accelerate charged particles (electrons, protons, heavy ions in the form of beams).

Lorentz force

- $F = q(E + v \times B)$

- To accelerate the beam an RF cavity needs to provide longitudinal force (along the velocity of the beam).

- Longitudinal force can be provided only by E, not B.

- Transverse force can be provided by both E and B.

- Let us focus on longitudinal force first.

Electrostatic vs RF

- DC vs AC, Edison vs Tesla, AC with 50 or 60 Hz wins.
- Electrostatic vs RF
  - Electrostatic: use static high voltage potential *V, E = qV,* can provide up to 30MV. Cannot: stack (requires higher voltage), re-circulate (After one turn voltage must be 0).
  - RF: can stack, can re-circulate.
  - It seems RF is better, but we do use electrostatic in certain applications.

From https://en.wikipedia.org/wiki/Particle_accelerator
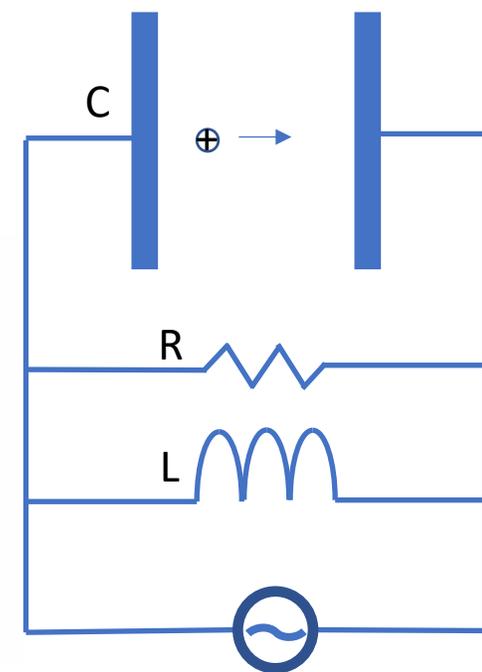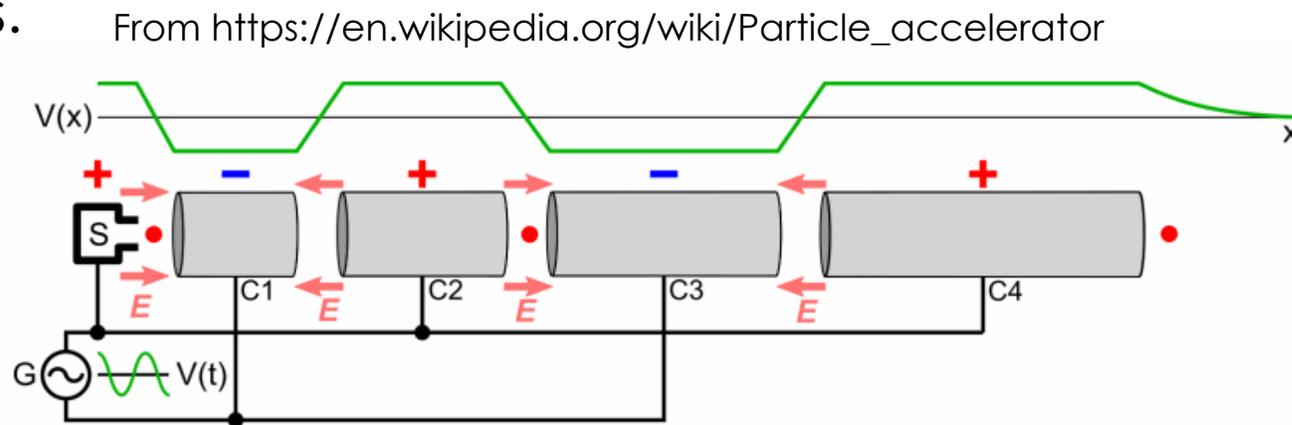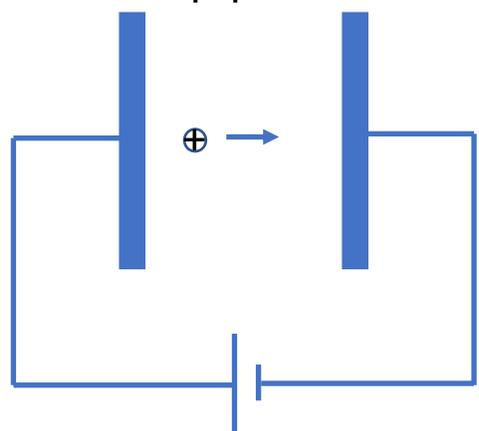
Electrostatic vs RF

- Electrostatic accelerator can be treated as a DC power source V_0 applied to a capacitor with two plates spaced d.

- RF accelerator can be treated as an RF power source $V_0 e^{j\omega t}$ applied to an RLC circuit in parallel.

- With a large size capacitor, the DC field adjacent to ρ=0 is uniform, with:

$$E_z = V_0/d, E_\rho = E_\varphi = H_\rho = H_\varphi = H_z = 0$$

- With RF source, the field pattern in the capacitor is the same as the DC case, with the only difference that the peak field is constant in DC and is in cos form in RF (DC field times $e^{j\omega t}$).

E field pattern

C

R

L
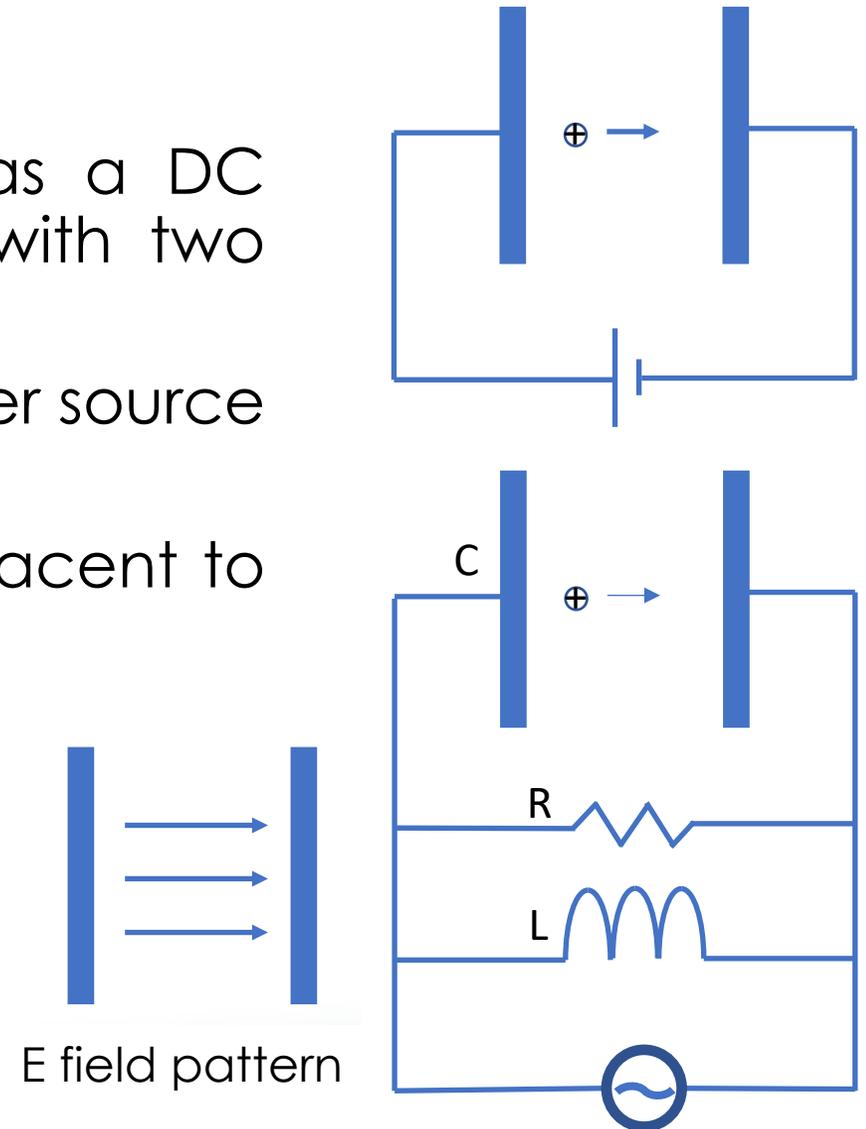
RLC in Parallel and Impedance

- RF source as a function of time: $V(t) = V_0 e^{j\omega t}$
- Current as a function of time:
  - on capacitor $I_c(t) = C\frac{dV(t)}{dt} = j\omega C V(t)$
  - on inductor $I_L(t) = \frac{1}{L}\int V(t)dt = \frac{1}{j\omega L}V(t)$
  - on resistor $I_R(t) = \frac{V(t)}{R}$
  - total current $I(t) = \left(j\omega C + \frac{1}{j\omega L} + \frac{1}{R}\right)V(t)$

- Impedance $Z = \frac{V(t)}{I(t)} = \frac{1}{j\omega C + \frac{1}{j\omega L} + \frac{1}{R}}$, this is the impedance that beam sees.
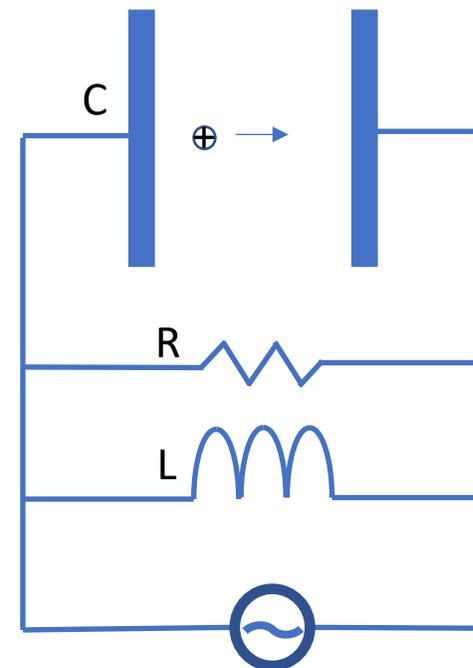
Frequency

- When this system is on resonance, impedance does not have imaginary part, $j\omega_0 C + \dfrac{1}{j\omega_0 L} = 0$, so resonant frequency is

$$f_0 = \frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

- It is in the unit of Hz = 1/second

- 1000Hz=1kHz, 1000kHz=1MHz, 1000MHz=1GHz, 1000GHz=1THz.

- On resonance impedance $Z = R$

- When L and C are scaled up or down, frequency scales down or up. For RF cavity, if the cavity is scaled up (bigger), resonant frequency scales down, and vice versa – frequency is inversely proportional to cavity size in 1D (not volume).

- Please note later-on, when we say a parameter is cavity size dependent, it means this parameter scales (or inversely scales) when we scale the cavity without changing the cavity shape. And when we say a parameter is frequency dependent, it applies to cavity with any shape.

C

R

L
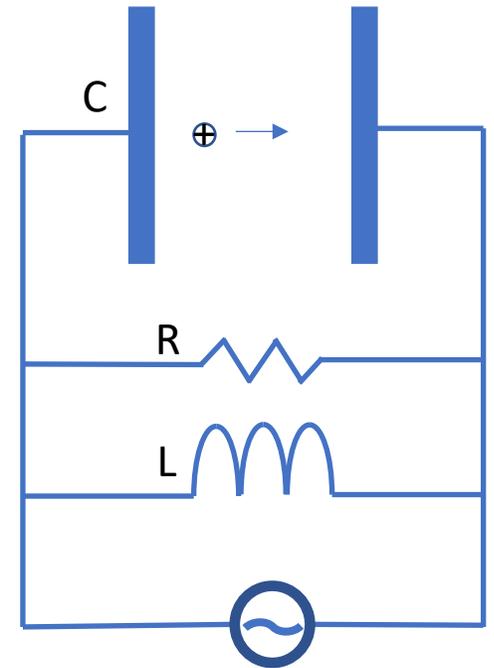
Gap

- For $V(t) = V_0 e^{j\omega t}$, to get the maximum accelerating voltage, the (positively) charged particles/beam (with velocity *v*) enter the capacitor at time *–T/4* and exist at time *T/4*. Accelerating force is at maximum while particles are in the center, so called on-crest. Cavity may also work at off-crest though.

- The length of the capacitor is thus *d=λ/2* for *v* close to *c*, with *λ=c/f$_0$* the wavelength.

- For low β (=*v/c*) cavity, the length is normally less than *λ/2*.

E field pattern

−*T/4*   *T/4*

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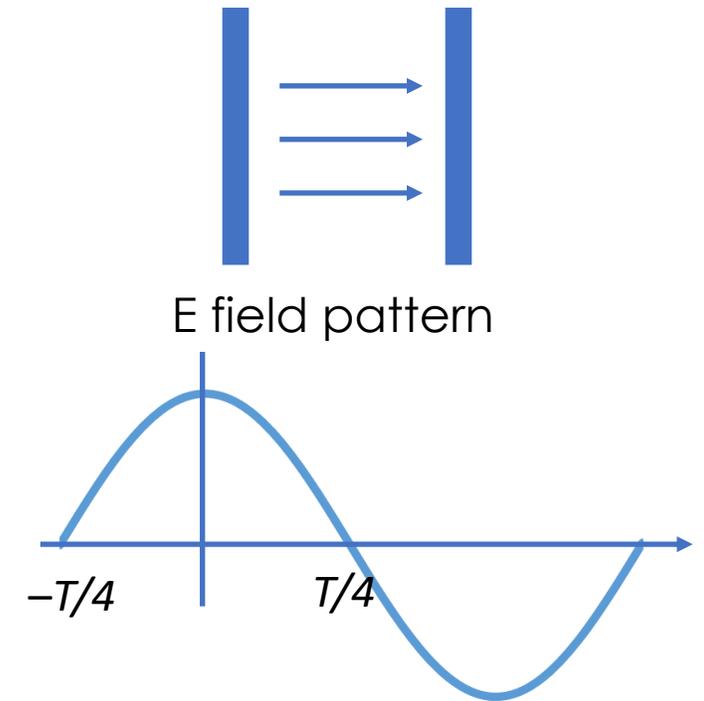
Accelerating Voltage

- Max E field is $E_0 = V_0/d$

- Accelerating voltage is

$$V_c = \left| \int_0^d E_z(\rho = 0, \varphi = 0, z) e^{j\omega t(z)} \, dz \right| = E_0 \left| \int_0^d e^{j\omega z/v} dz \right| =$$
$V_0 \sin\left(\frac{\omega d}{2v}\right) / \left(\frac{\omega d}{2v}\right) = V_0 T$, the change of velocity is ignored here, which is true for high β cavity in which beam is fast enough, or for low β cavity with small d.

- Absolute is used to get the maximum accelerating voltage.

- T is the transit time factor, with $T = \begin{cases} 1, d \to 0 \\ \frac{2}{\pi}, d = \lambda/2 \\ 0, d \to \infty \end{cases}$

- In reality, T is always less than 1 and d should be around λ/2.

E field pattern

$-T/4$ $T/4$
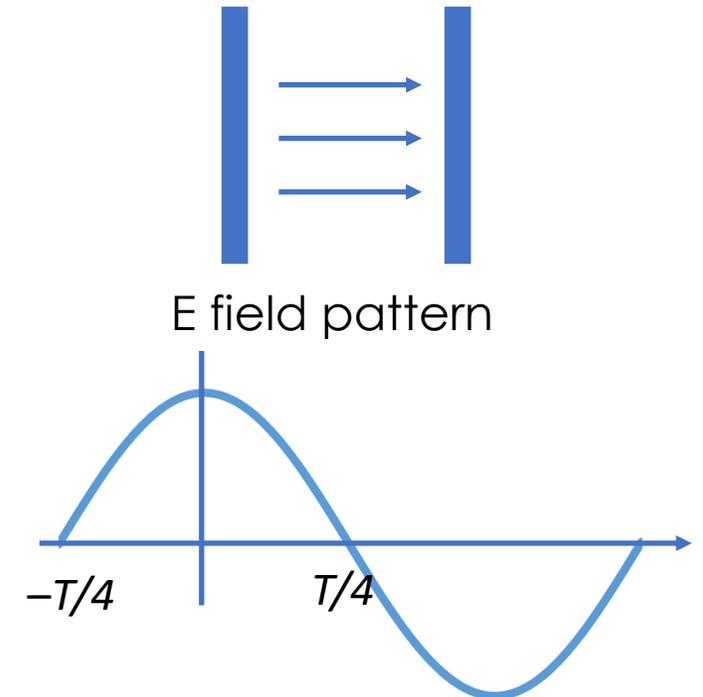
Do not confuse transit time factor with the time for one RF cycle, both are noted as T. T is the later one if not specifically noted.

Stored Energy

- Energy stored (U) in this system
  - In the capacitor $U_c = \int I_c(t)V(t)dt = \int C\frac{dV(t)}{dt}V(t)dt = \int_0^{V_0} CV(t)dV(t) = \frac{1}{2}CV_0^2$, integrated from 0 to maximum energy.
  - In the inductor $U_L = \int I_L(t)V(t)dt = \int I_L(t)L\frac{dI_L(t)}{dt}dt = \int_0^{I_{L0}} LI_L(t)dI_L(t) = \frac{1}{2}LI_{L0}^2 = \frac{V_0^2}{2\omega_0^2 L} = \frac{1}{2}CV_0^2$, should be the same as in the capacitor.
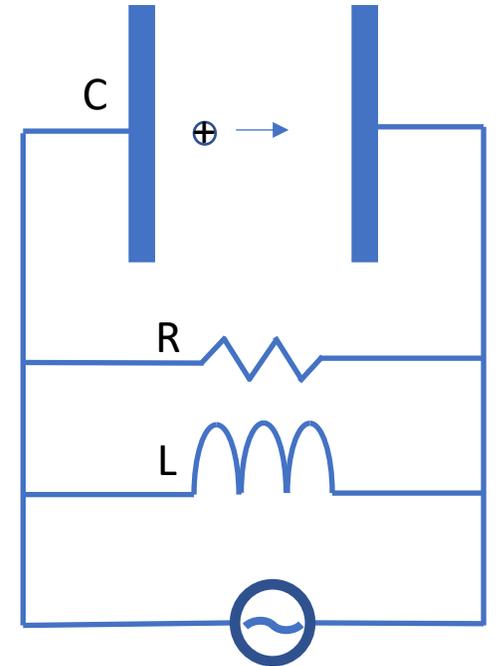
Power Dissipation

- Power required to maintain the voltage needed (or power required to compensate the loss on R) while this system is on resonance: $P = \dfrac{\int_0^T (Re[V(t)])^2 \, dt/T}{R} = \dfrac{V_0^2}{2R}$

- This did not include the power that the beam takes away, which is the beam current I_b times accelerating voltage (V_c).

- It also did not consider other effects.
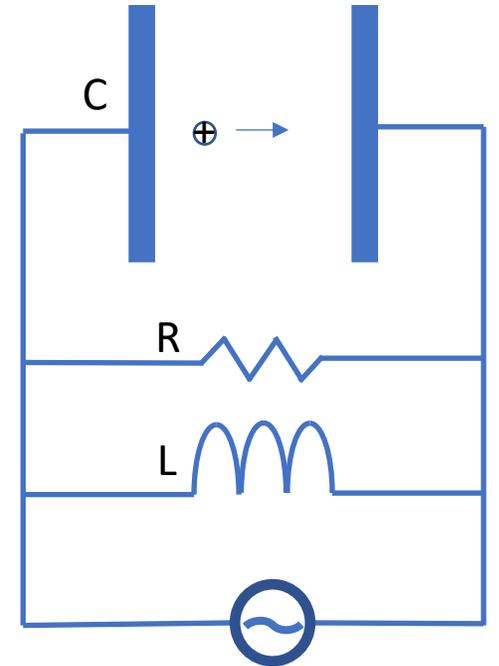
Shunt Impedance

- $P = \frac{V_0^2}{2R}$, for a certain voltage, the higher the R, the lower the P needed to maintain the voltage. (Effective) shunt impedance $R_{sh} = RT^2$ thus $P = \frac{V_c^2}{2R_{sh}}$, with T the transit time factor.

- For simplification, we assume transit time factor $T=1$ so that $R = R_{sh}$ and $V_0 = V_c$ for RLC circuit. For cavities only R_{sh} and V_c are used.

- R_{sh} depends on both cavity shape and size, as well as material properties, we will get back to this point soon.

- To lower the power required, one needs to fine-tune the cavity design, and sometimes, fine-tune the cavity material properties – each of these two can be a series of lectures for a whole semester.
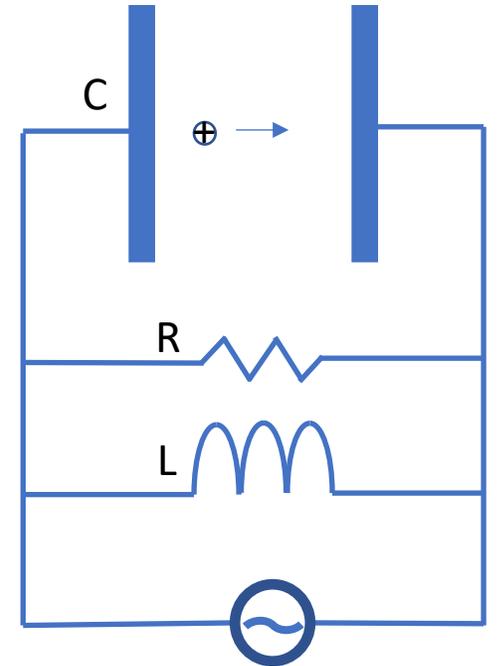
Transient (1)

- $C \frac{dV(t)}{dt} + \frac{1}{L} \int V(t) dt + \frac{V(t)}{R} = 0$ with $V(0) = V_0 \rightarrow$ $\frac{d^2 V(t)}{dt^2} + \frac{1}{RC} \frac{dV(t)}{dt} + \frac{V(t)}{LC} = 0$

- It is a damped oscillation system

- An oscillation system (spring) without damping: $F = m \frac{d^2 s(t)}{dt^2} = -ks(t)$

- Damping force in this system is proportional to $- \frac{ds(t)}{dt}$

- A damped oscillation system: $m \frac{d^2 s(t)}{dt^2} = -ks(t) - p \frac{ds(t)}{dt}$

C

R

L

$t \leq 0, V(t) = V_0 e^{j\omega t}$

C

R

L
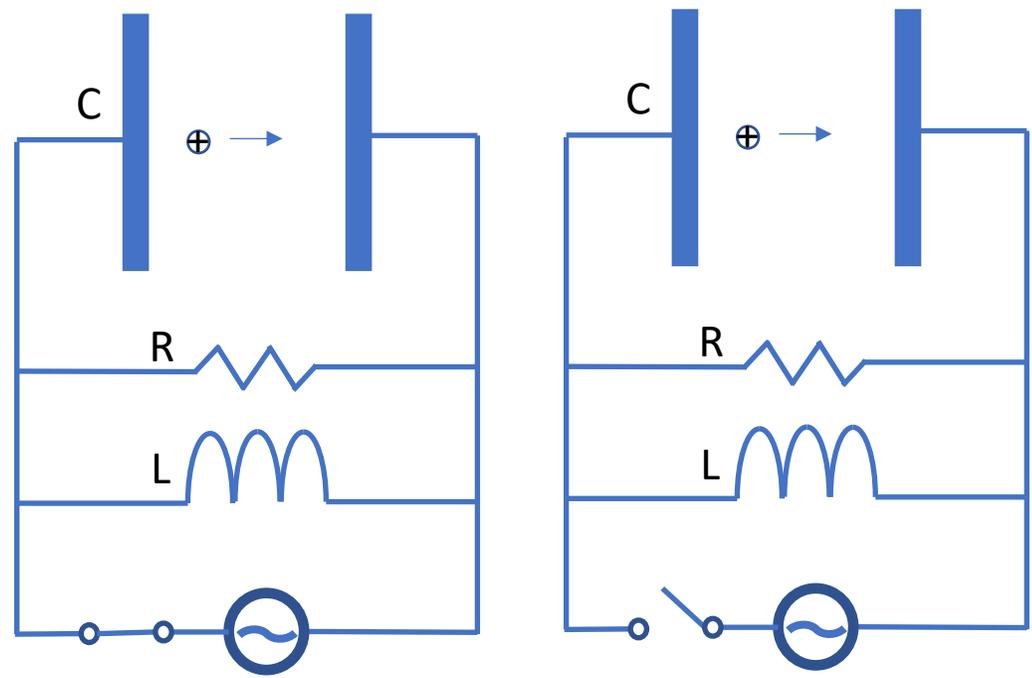
T>0 , $V(t)$ =0
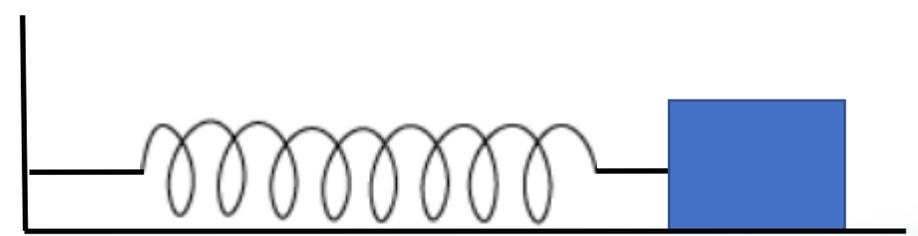
Transient (2)

- $\dfrac{d^2V(t)}{dt^2}+\dfrac{1}{RC}\dfrac{dV(t)}{dt}+\dfrac{V(t)}{LC}=0$

- Solution is in the form $V(t)=V_0e^{j\omega t}e^{-\frac{t}{2\tau}}$, 2τ is used here because we measure the power decay time, and power is proportional to

$$V(t)^2=\left(V_0e^{j\omega t}\right)^2e^{-\frac{t}{\tau}}$$

$t\leq0, V(t)=V_0e^{j\omega t}$

$T>0, V(t)=0$

- $\dfrac{dV(t)}{dt}=\left(j\omega-\dfrac{1}{2\tau}\right)V(t)$ & $\dfrac{d^2V(t)}{dt^2}=\left(j\omega-\dfrac{1}{2\tau}\right)^2V(t)$, thus $\left(j\omega-\dfrac{1}{2\tau}\right)^2+\dfrac{j\omega-\frac{1}{2\tau}}{RC}+\dfrac{1}{LC}=0$

- Imaginary part $-\dfrac{j\omega}{\tau}+\dfrac{j\omega}{RC}=0\rightarrow\tau=CR$

- Real part $-\omega^2+\dfrac{1}{4\tau^2}-\dfrac{1}{2\tau RC}+\dfrac{1}{LC}=0\rightarrow\omega^2=-\dfrac{1}{4(RC)^2}+\dfrac{1}{LC}=-\dfrac{1}{4(RC)^2}+\omega_0^2$

- in an RF oscillation system, typically $\tau\gg T$
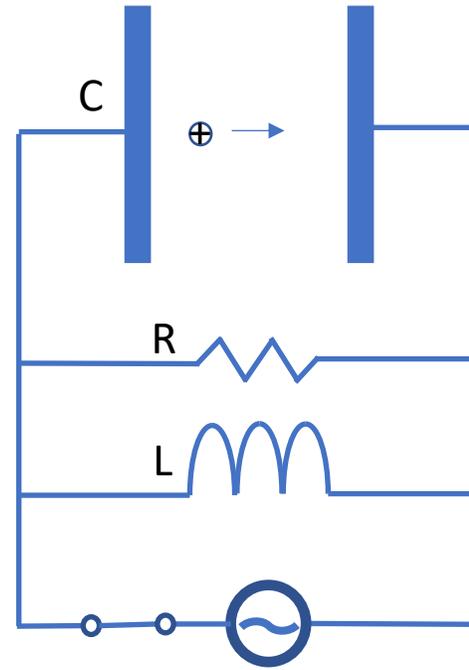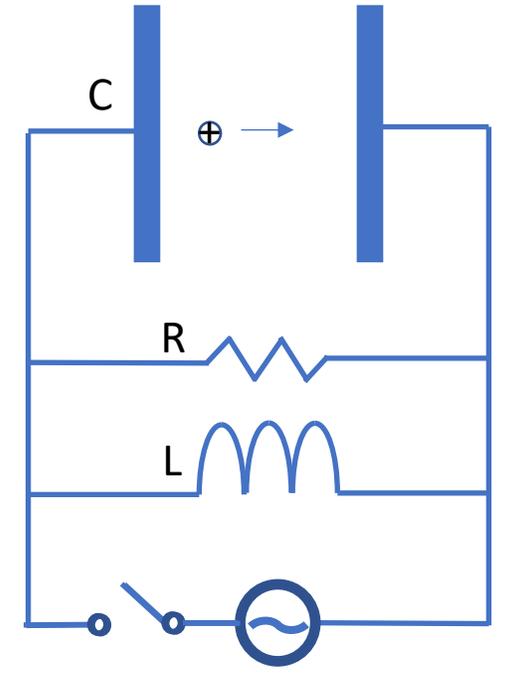
Decay and Quality Factor

- Power decay time $\tau = \dfrac{U}{P} = CR$

- We define quality factor $Q = \dfrac{\omega_0 U}{P}$, thus $Q = \omega_0 \tau = \omega_0 CR = R\sqrt{\dfrac{C}{L}}$

- With RF source disconnected, after time τ, the stored energy decays to 1/e.

- We have $P = \dfrac{\omega_0 U}{Q}$ which also equals to $\dfrac{V_0^2}{2R}$
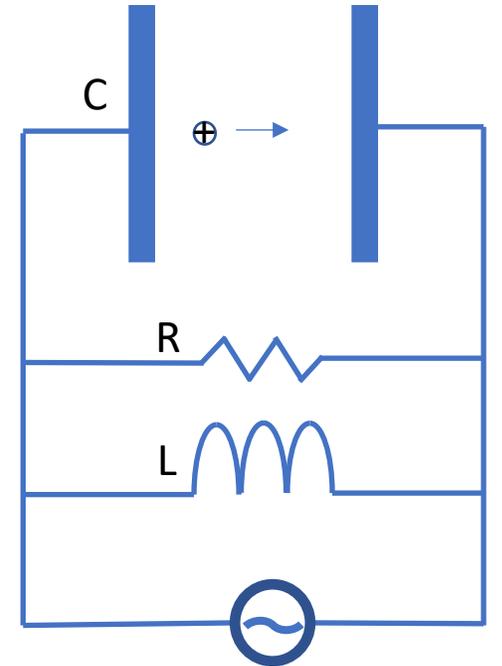
Lorentz Distribution

- $Z(\omega) = \dfrac{1}{j\omega C + \frac{1}{j\omega L} + \frac{1}{R}} = \dfrac{R}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$ $\qquad Q = R\sqrt{\dfrac{C}{L}} \ \& \ \omega_0 = \dfrac{1}{\sqrt{LC}}$

- Define $\omega = \omega_0 + \delta\omega$, with $\delta\omega \ll \omega_0$, $Z(\omega) \simeq \dfrac{R}{1 + jQ2\left(\frac{\delta\omega}{\omega_0}\right)}$

- $Re[Z(\omega)] = \dfrac{R}{1 + \left(Q2\frac{\delta\omega}{\omega_0}\right)^2}$, $Phase[Z(\omega)] = -arctan\left(Q2\dfrac{\delta\omega}{\omega_0}\right)$

- The real part of $Z(\omega)$ is in the form of Lorentz (Cauchy-Lorentz) distribution.

- FWHM $2\Delta\omega$ is defined as the Full Width in frequency that real part of Z lowers to Half of the Maximum (R/2), so $Q2\dfrac{\Delta\omega}{\omega_0} = 1$ and $Q = \dfrac{\omega_0}{2\Delta\omega}$.

Lorentz distribution

- $Re[Z(\omega)] = \dfrac{R}{1+(Q2\frac{\delta\omega}{\omega_0})^2}, \ Phase[Z(\omega)] = -arctan(Q2\frac{\delta\omega}{\omega_0})$

- With $Re[Z(\omega)] = 0.5, \ Phase[Z(\omega)] = 45^o or \ -45^o$

- With frequency small enough or large enough, $Re[Z(\omega)]$ becomes 0 and

$Phase[Z(\omega)] = 90^o \ or \ -90^o$
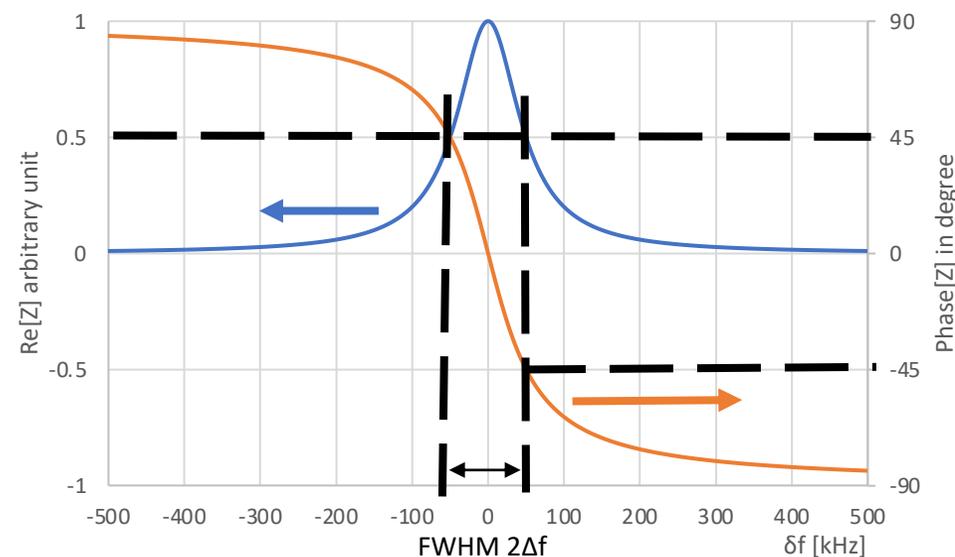
Z(f) real part and phase, with Q=10000 and resonant frequency at 1GHz

Measure the Quality Factor

- We can measure the transmitted S parameter of the cavity, which is proportional to $Z(\omega) = \dfrac{R}{1+jQ2(\frac{\delta\omega}{\omega_0})}$

- The peak (resonant) frequency ω_0 is measured, together with the peak amplitude A. Measure the smaller frequency ω_1 and larger frequency ω_2 where the real part of $Z(\omega)$ has an amplitude of $A/2$. One should have $\Delta\omega = \omega_2 - \omega_0 = \omega_0 - \omega_1$

- $Q = \dfrac{\omega_0}{2\Delta\omega}$

Z(f) real part and phase, with Q=10000 and resonant frequency at 1GHz

Electron-Ion Collider
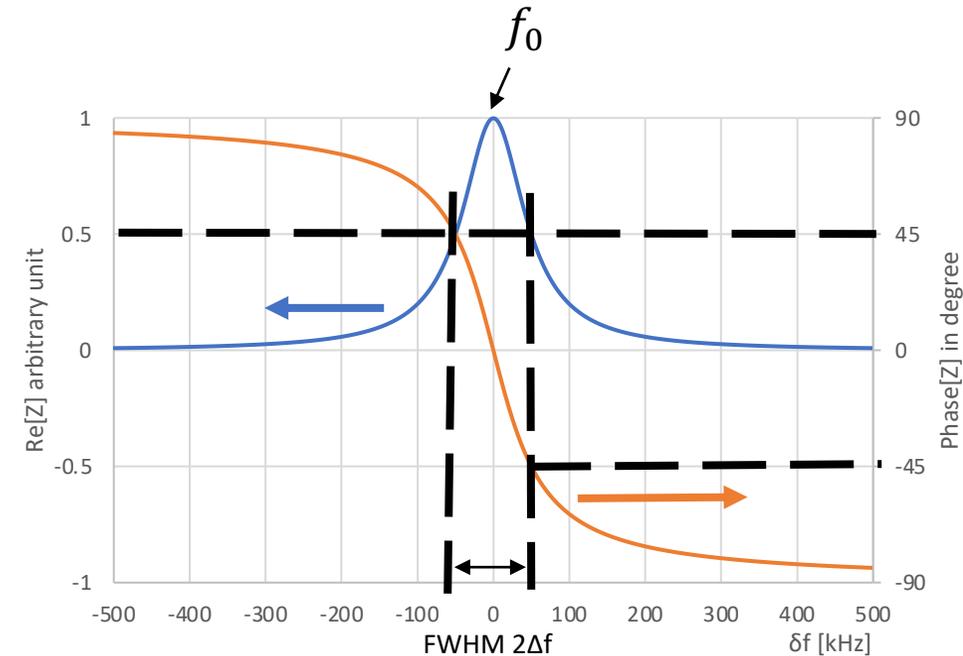
R_s

- $\boldsymbol{J} = \sigma\boldsymbol{E}$ on metal surface with σ the DC conductivity.
- $\nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H}$ & $\nabla \times \boldsymbol{H} = (j\omega\varepsilon+\sigma)\boldsymbol{E} = \sigma\boldsymbol{E}$ for a good conductor at microwave frequency range (true for Cu below 10^{18}Hz) that $\omega\varepsilon \ll \sigma$
- $\nabla \times \nabla \times \boldsymbol{E} = -\nabla^2\boldsymbol{E} = -j\omega\mu \, \nabla \times \boldsymbol{H} = -j\omega\mu\sigma\boldsymbol{E} \rightarrow \nabla^2\boldsymbol{E} = j\omega\mu\sigma\boldsymbol{E}$
- Wave along z (perpendicular to metal surface with metal at z=0 and after), $\boldsymbol{E}$ has x component, H has y component
- $E_x = E_0 e^{-(1+j)\frac{z}{\delta}}$ field along metal thickness z & skin depth $\delta = \sqrt{\dfrac{2}{\omega\mu_0\sigma}}$

  & $H_y = \dfrac{1-j}{\omega\mu_0\delta} E_0 e^{-(1+j)\frac{z}{\delta}}$

- Surface impedance $Z = \dfrac{E_x}{H_y} = (1+j)\sqrt{\dfrac{\mu_0\omega}{2\sigma}}$ with its real part R_s

R_S
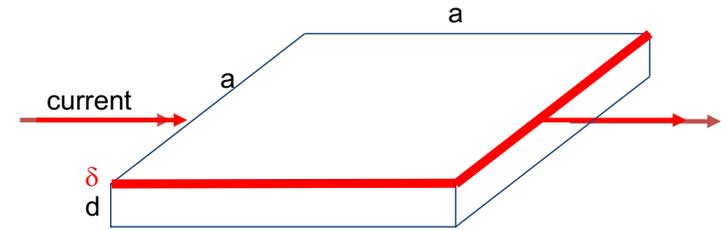
- For DC current passing through a conductor with certain surface area (a x a) and d thickness, with current perpendicular to cross section a x d, $R_{DC} = \frac{\rho a}{da} = \frac{\rho}{d}$

- For RF current, the RF field penetrates into the conductor by a characteristic length called skin depth δ, and $R_s = \sqrt{\frac{\mu_0 \omega}{2\sigma}} = \frac{1}{\sigma\delta} = \frac{\rho}{\delta}$

- R_s is material (and surface treatment of material) dependent and is frequency dependent because skin depth is frequency dependent. It is not related to cavity shape.

- RF loss is then calculated using $P_{rf} = \frac{1}{2} \iint H^2 R_s dS$, with S the metal surface.

R_s and RLC in series
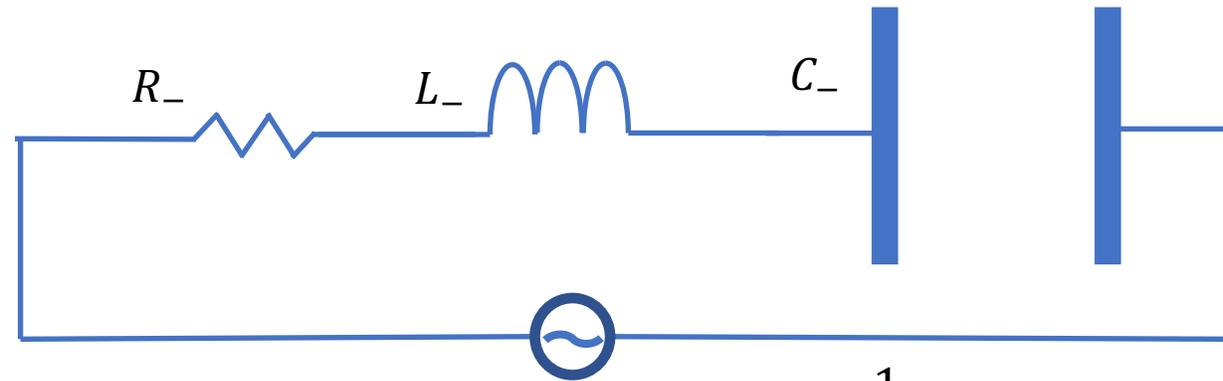
- For an RLC circuit in series, the impedance is $Z_- = R_- + j\omega L_- + \frac{1}{j\omega C_-}$

- It can be expressed as $Z_- = R_s + jX_s + j\omega L_V + \frac{1}{j\omega C_V}$, with $Z_s = R_s + jX_s$ the surface impedance and $R_s = X_s$ for normal conductors, L_V & C_V depend on the cavity design, $X_s \ll \omega L_V$.

- Quality factor $Q = \frac{1}{R_s}\sqrt{\frac{L_-}{C_-}} = \frac{G}{R_s}$, with G geometry factor which depends on the cavity shape. This quality factor is usually noted as Q_0, called intrinsic quality factor, with corresponding $P_0 = \frac{\omega U}{Q_0}$ loss on the metal wall.

- When the RF cavity is scaled up or down, $\frac{L_-}{C_-}$ remains and G keeps the same (cavity size independent). G is not related to the material.

RLC in series and in parallel

- RLC circuit in series, the impedance is $Z_- = R_- + j\omega L_- + \dfrac{1}{j\omega C_-}$

- $\omega_0 = \dfrac{1}{\sqrt{L_- C_-}}$ & $Q = \dfrac{1}{R_S}\sqrt{\dfrac{L_-}{C_-}}$

- RLC circuit in parallel (usually we use // to note elements in parallel, for simplicity we use R L C without // here), the impedance is $Z = \dfrac{1}{j\omega C + \dfrac{1}{j\omega L} + \dfrac{1}{R}}$

- $\omega_0 = \dfrac{1}{\sqrt{LC}}$ & $Q = R\sqrt{\dfrac{C}{L}}$

- Thus we have $L_- = RR_S C$, $C_- = \dfrac{L}{RR_S}$ & $ZZ_- = RR_S$

R_{sh}/Q

- Recall that shunt impedance R_{sh} and quality factor Q satisfies: $Q = R\sqrt{C/L} \rightarrow \frac{R}{Q} = \sqrt{L/C} = \omega_0 L = \frac{1}{\omega_0 C}$ with $\omega_0{}^2 = \frac{1}{LC}$ & $R_{sh} = RT^2$, T the transit time factor.

- L and C are material independent $\rightarrow R_{sh}/Q$ is material independent, $\sqrt{L/C}$ is cavity size independent $\rightarrow R_{sh}/Q$ is independent of cavity size, it depends on cavity design only.

- R_s x R_{sh} = G x R_{sh}/Q.

- A commonly used equation $P = \frac{V_c^2}{2\frac{R_{sh}}{Q}Q} = \frac{\omega_0 U}{Q}$, we also get $\frac{V_c^2}{2\frac{R_{sh}}{Q}} = \omega_0 U$.

- Here we use circuit definition of R_{sh}/Q. Sometimes accelerator definition is used, which is twice the circuit definition.
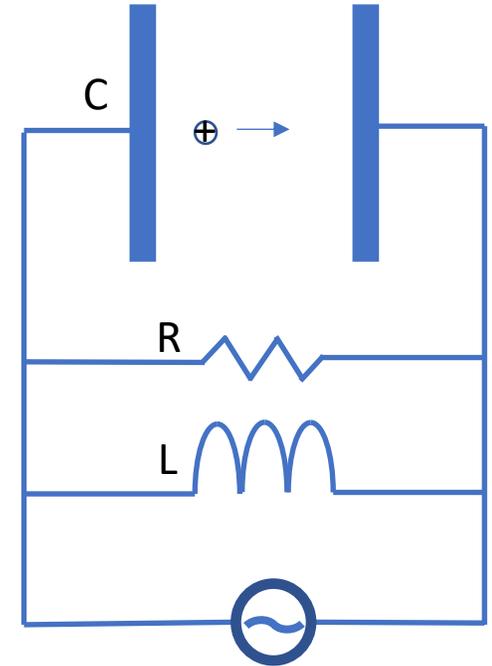
RLC and cavity parameter

- $\omega_0 = \dfrac{1}{\sqrt{LC}}$ and $\dfrac{R_{sh}}{Q} = \sqrt{\dfrac{L}{C}}\,T^2$

- we have $C = \dfrac{T^2}{\omega_0 \frac{R_{sh}}{Q}}$ & $L = \dfrac{1}{\omega_0 T^2}\dfrac{R_{sh}}{Q}$

- and $R_{sh} = RT^2$
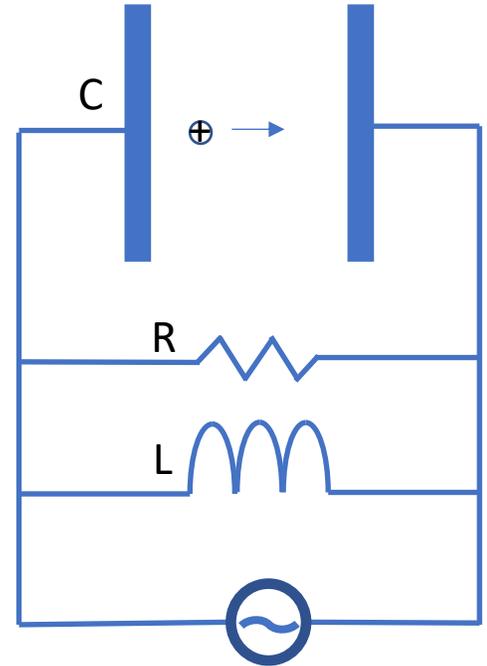
Beam Loading (1)

- For a "power-off" cavity (RLC without RF source and without energy stored), the beam entering in "sees" the shunt impedance and gives energy to the cavity.

- Beam with charge q produces voltage $V_q = \frac{q}{C} = q\omega_0 \frac{R_{sh}}{Q}$, the voltage increase of the cavity.

- Energy transfer from beam to cavity is $U_q = \frac{1}{2}CV_q^2 = \frac{1}{2}qV_q$, **beam "sees" only half of the voltage induced by itself**, which is $\frac{V_q}{2} = q\frac{\omega_0}{2}\frac{R_{sh}}{Q}$.

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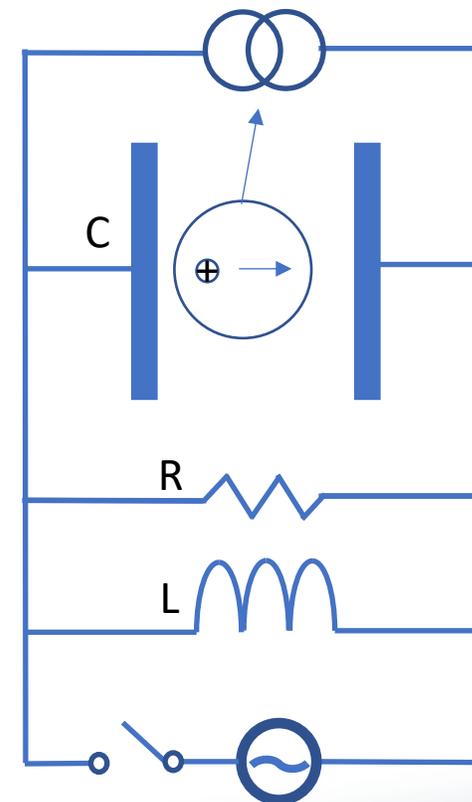
Beam Loading (2)

- For a powered cavity, besides the above effect, the beam is also accelerated by the RF field in the cavity, in another word, it also takes energy (qV_c) away from the cavity.

- For beam "on-crest" the first bunch energy changes by $q(V_c - \frac{V_q}{2})$

- Cavity voltage changes from V_c to $V_c - V_q$, cavity energy change is: $\frac{1}{2}CV_c^2 - \frac{1}{2}C(V_c - V_q)^2 = \frac{1}{2}C(2V_cV_q - V_q^2) = \frac{1}{2}\frac{q}{V_q}(2V_cV_q - V_q^2) = q(V_c - \frac{V_q}{2})$, energy conserved.

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Beam Loading (3)

- The second bunch's energy changes by $q\left((V_c - V_q) - \dfrac{V_q}{2}\right) = q(V_c - \dfrac{3}{2}V_q)$, assuming the cavity voltage decay is small (high Q).

- Cavity voltage changes from $V_c - V_q$ to $V_c - 2V_q$, cavity energy change is: $\dfrac{1}{2}C(V_c - V_q)^2 - \dfrac{1}{2}C(V_c - 2V_q)^2 = q(V_c - \dfrac{3}{2}V_q)$, energy conserved.

Beam is treated as a current source

C

R

L
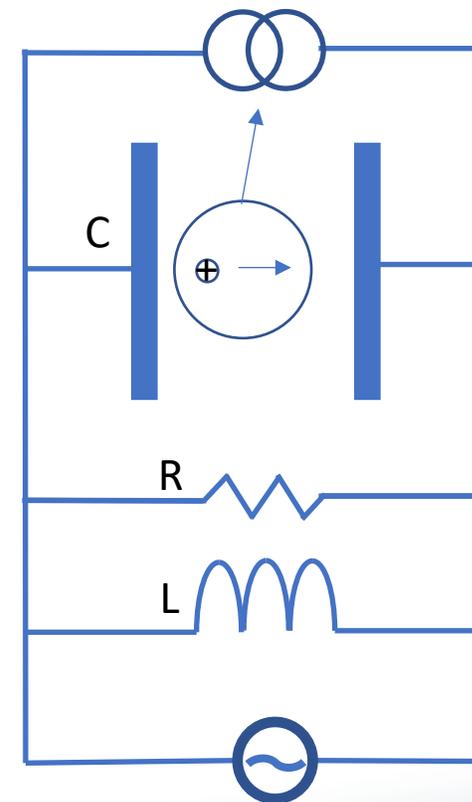
Beam Loading (4)

- In reality, quality factor and phasor need to be considered.

- CW: beam loading causes a cavity voltage reduction of V_q.

- Pulsed: cavity voltage continues to drop till a full train of (n) bunches ends, nV_q voltage reduction.

Beam is treated as a current source

C

R

L
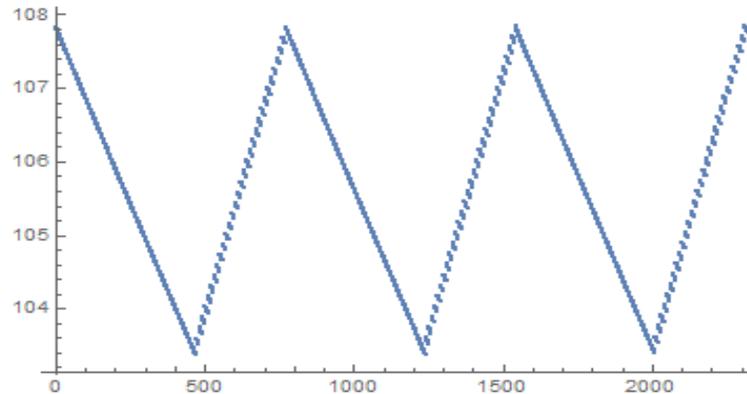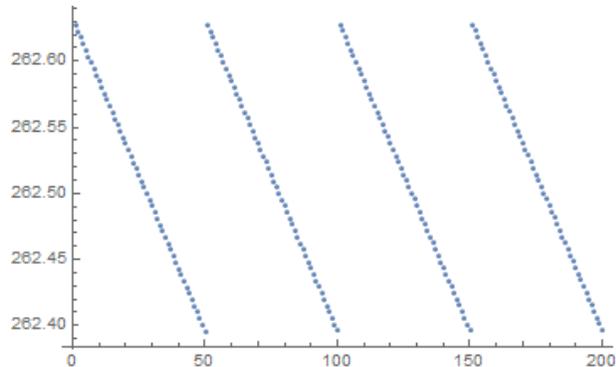
CW vs pulsed (multi bunches in a train)

Electron-Ion Collider
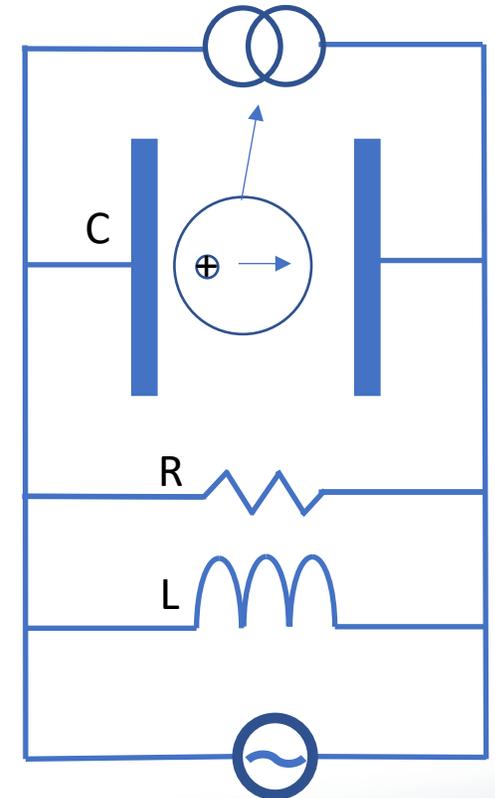
Couplers

- Coupler is an RF component used to conduct RF power into or out of the cavity.

- The coupler conducting power into the RF cavity with the fundamental (or working) mode is called Fundamental Power Coupler (FPC).

- The coupler conducting the fundamental mode power out of the RF cavity is called pickup coupler (PU)

- Some RF cavities need special coupler(s) to couple out the unwanted modes, we will introduce them in the next lecture.

PU in RLC circuit

- Coupler can be treated as a transformer.

- Here PU is represented by a transformer *m:1* together with a load R_{PU}. The voltage on the R_{PU} is $V(t)/n$ and the power is $P_{PU} = \dfrac{V_c^2}{2m^2 R_{PU}} = \dfrac{V_c^2}{2\dfrac{R_{sh}}{Q}Q_{PU}}$ with PU's quality factor $Q_{PU} = \dfrac{\omega_0 U}{P_{PU}} = \omega_0 C m^2 R_{PU}$

- Coupling coefficient $\beta_{PU} = \dfrac{Q_0}{Q_{PU}} = \dfrac{R}{m^2 R_{PU}}$
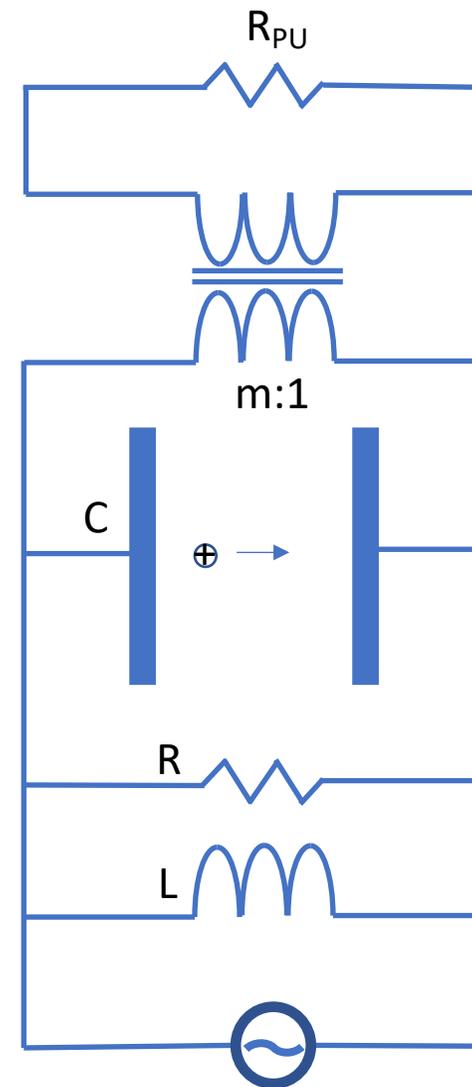
FPC in RLC circuit

- FPC can be dealt in a similar way, assuming a 1:n transformer together with a load R_s. The voltage on the R_s is $nV(t)$ and the power is $P_{FPC} = \dfrac{n^2 V_0^2}{2R_s} = \dfrac{V_0^2}{2\dfrac{R_{sh}}{Q}Q_{FPC}}$ with FPC's quality factor $Q_{FPC} = \dfrac{\omega_0 U}{P_{FPC}} = \dfrac{\omega_0 C R_s}{n^2}$

- Coupling coefficient $\beta_{FPC} = \dfrac{Q_0}{Q_{FPC}} = \dfrac{n^2 R}{R_s}$

- We define loaded quality factor $\dfrac{1}{Q_L} = \dfrac{1}{Q_0} + \dfrac{1}{Q_{FPC}} + \dfrac{1}{Q_{PU}}$ for this 2-port system and $Q_L = \dfrac{Q_0}{1+\beta_{FPC}+\beta_{PU}}$, with $P_L = P_0 + P_{FPC} + P_{PU} = (1 + \beta_{FPC} + \beta_{PU})P_0$, Q_L is what we measure in slide 17

R_{PU}

m:1

C

R

L

1:n

R_s
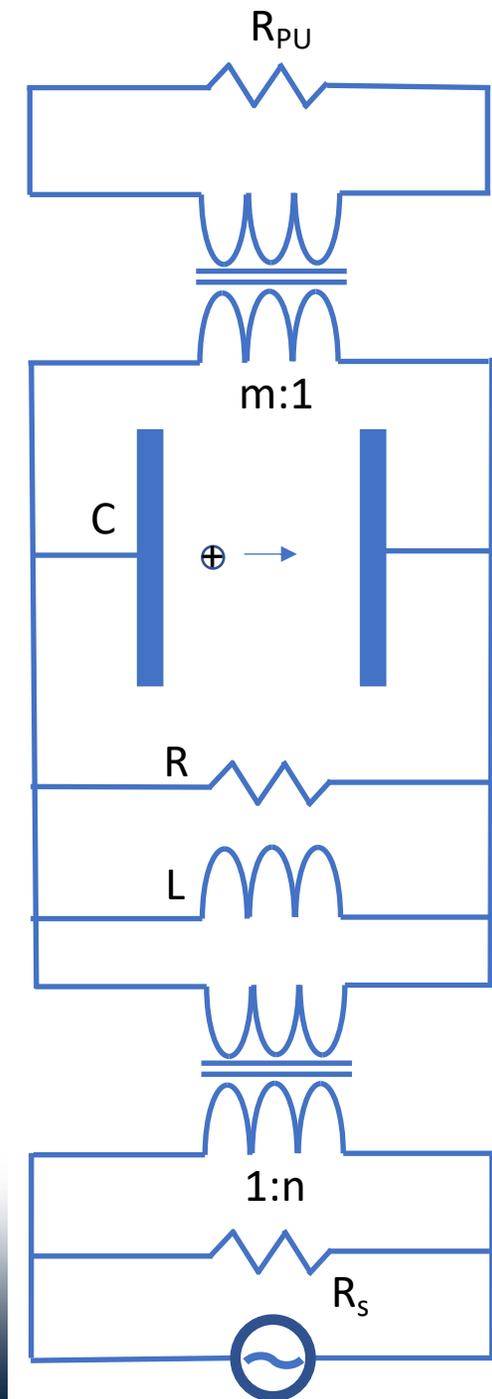
Coupling Coefficient (1)

- P_{PU} is normally in the range of Watts or tens of Watts, and $\beta_{PU} \ll 1$. Sometimes it is safely ignored in the model.

- For normal conducting cavity with beam current I_b, the beam takes power away from the cavity $P_b = V_c \times I_b$.

- P_{FPC} needs to compensate both P_0 and P_b (and more, but let's keep it simple here). When $P_b \ll P_0$, $\beta_{FPC} = 1$.

- For beam-driven (decelerating) cavities, $P_b < 0$.

- For a superconducting accelerating cavity, $P_b \gg P_0$, $\beta_{FPC} \gg 1$.
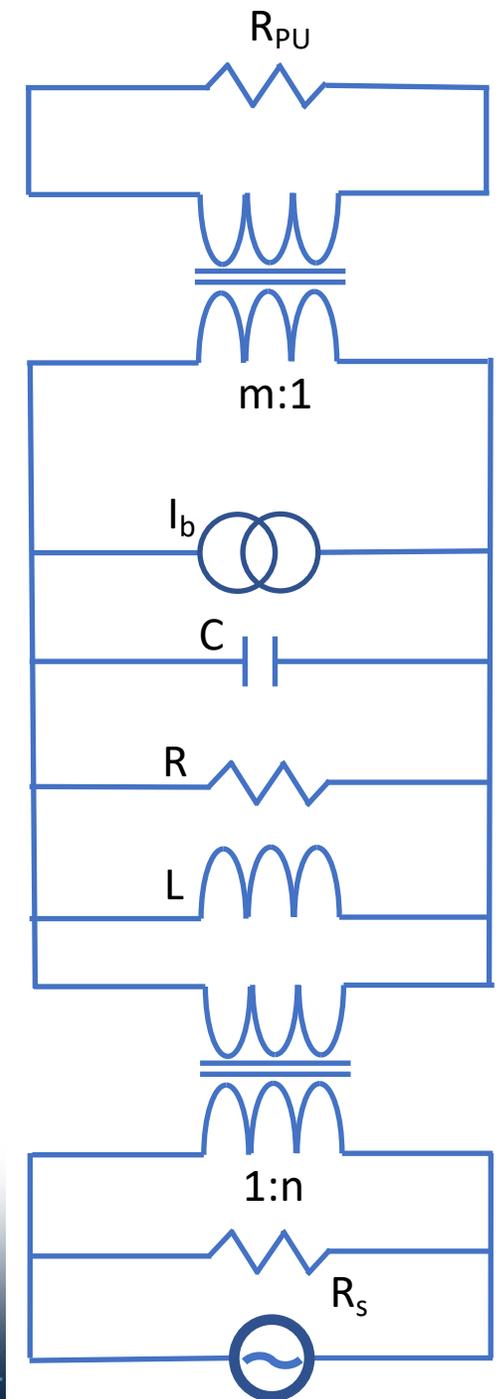
Coupling Coefficient (2)

- Reflection coefficient $\Gamma = \frac{\beta-1}{\beta+1}$ and $VSWR = \frac{1+|\Gamma|}{1-|\Gamma|}$

- $VSWR = \begin{cases} 1/\beta_{FPC}, & 0 < \beta_{FPC} < 1, & -1 < \Gamma < 0, & Under\ coupled \\ 1, & \beta_{FPC} = 1, & \Gamma = 0, & Critical\ coupled \\ \beta_{FPC}, & \beta_{FPC} > 1, & 0 < \Gamma < 1, & Over\ coupled \end{cases}$

- $\beta_{FPC} = 1, Q_L = \frac{Q_0}{2}$

Cavity material, shape and size

- Geometry factor G, R_{sh}/Q depend on cavity shape, but not the size, it is not related to the material.

- Surface resistance R_s is a parameter related to the material, as well as the treatment of the material, it is also frequency dependent.

- Resonant frequency is inversely proportional to the cavity size in 1D.