Home Work PHY 554 #10

HW 1 (2 points): Calculate relations between three dimensionless infinitesimal parameters:

$$\frac{dE}{E} \equiv \frac{d\gamma}{\gamma}; \frac{dp}{p} \equiv \frac{d(\beta\gamma)}{\beta\gamma}; \frac{d\mathrm{v}}{\mathrm{v}} \equiv \frac{d\beta}{\beta}$$

where E is energy, p is momentum and v is velocity of a particle. Hint: use relativistic relations between β and γ.

Solution: The easiest way is to use the following

$$\beta^2 = 1 - \gamma^{-2} \rightarrow \beta d\beta = \gamma^{-3} d\gamma$$

$$\frac{d\mathrm{v}}{\mathrm{v}} = \frac{d\beta}{\beta} = \frac{1}{\beta^2\gamma^2}\frac{d\gamma}{\gamma} = \frac{1}{\beta^2\gamma^2}\frac{dE}{E};$$

$$E^2 = \mathrm{p}^2c^2 + m^2c^4 \rightarrow EdE = c^2\mathrm{pdp};$$

$$\frac{dp}{p} = \frac{E^2}{\mathrm{p}^2c^2}\frac{dE}{E} \equiv \frac{1}{\beta^2}\frac{dE}{E};$$

$$\frac{d\mathrm{v}}{\mathrm{v}} = \frac{1}{\gamma^2}\frac{dp}{p}.$$

HW 2 (5 points): In class we introduced the map of longitudinal motion in a storage ring

$$\delta_{n+1} = \delta_n + \frac{eV_{rf}}{\beta^2 E_o}\left(\sin\phi_n - \sin\phi_s\right); \qquad (1)$$

$$\phi_{n+1} = \phi_n + 2\pi h\eta \cdot \delta_{n+1},$$

1. For small oscillation variations of the RF phase about the synchronous phase

$$\varphi = \phi - \phi_s; |\varphi| << 1$$

linearize the map (1) by keeping only first order on φ and find one turn transport matrix M for longitudinal motion:

$$\begin{pmatrix} \varphi \\ \delta \end{pmatrix}_{n+1} = M \begin{pmatrix} \varphi \\ \delta \end{pmatrix}_n$$

Solution: linearization is straight forward

$$\sin\phi_n = \sin\phi_s + \varphi_n \cos\phi_s + O\left(\varphi_n^{\ 2}\right)$$

$$\delta_{n+1} \cong \delta_n + \frac{eV_{rf}\cos\phi_s}{\beta^2 E_o}\varphi_n; \qquad (1)$$

$$\varphi_{n+1} = \varphi_n + 2\pi h\eta \cdot \delta_{n+1} = 2\pi h\eta\left(\delta_n + \frac{eV_{rf}\cos\phi_s}{\beta^2 E_o}\varphi_n\right);$$

and giving us desirable one turn matrix

$$\begin{pmatrix} \varphi \\ \delta \end{pmatrix}_{n+1} = \begin{pmatrix} 1+2\pi h\eta \dfrac{eV_{rf}\cos\phi_s}{\beta^2 E_o} & 2\pi h\eta \\ \dfrac{eV_{rf}\cos\phi_s}{\beta^2 E_o} & 1 \end{pmatrix} \begin{pmatrix} \varphi \\ \delta \end{pmatrix}_n ;$$

$$M = \begin{pmatrix} 1+2\pi h\eta \dfrac{eV_{rf}\cos\phi_s}{\beta^2 E_o} & 2\pi h\eta \\ \dfrac{eV_{rf}\cos\phi_s}{\beta^2 E_o} & 1 \end{pmatrix} \qquad (2)$$

2. Using Courant-Snyder parametrization we used for transverse motion find value of $\cos\mu_s, \beta_s, \alpha_s$ in parametric form (e.g. using $\sin\mu_s = \sqrt{1-\cos^2\mu_s}$, $\mu_s = 2\pi Q_s = \cos^{-1}(\cos\mu_s)$).

Solution:

$$M = \begin{pmatrix} 1+2\pi h\eta \dfrac{eV_{rf}\cos\phi_s}{\beta^2 E_o} & 2\pi h\eta \\ \dfrac{eV_{rf}\cos\phi_s}{\beta^2 E_o} & 1 \end{pmatrix} = \begin{pmatrix} \cos\mu_s + \alpha_s \sin\mu_s & \beta_s \sin\mu_s \\ \gamma_s \sin\mu_s & \cos\mu_s - \alpha_s \sin\mu_s \end{pmatrix};$$

$$A = \pi h\eta \frac{eV_{rf}\cos\phi_s}{\beta^2 E_o}; \cos\mu_s = \frac{1}{2} TraceM = 1 + A; \qquad (2)$$

$$\sin\mu_s = \sqrt{-2A - A^2}; \alpha_s = \frac{A}{\sqrt{-2A-A^2}}; \beta_s = \frac{2\pi h\eta}{\sqrt{-2A-A^2}}$$

Motion is stable when $A(2+A)<0;\ -2<A<0$. We derived first A<0 condition in our class. Second condition A>-2 ;protest from never observed over-focusing by RF cavity…

3. Assuming that μ_s<<1, find analytical expression for synchrotron tune and compare it with that we found in Lecture 12:

Solution: for $\mu_s << 1$

$$\cos\mu_s = 1 - \frac{\mu_s{}^2}{2} + O\left(\mu_s{}^4\right) = 1 + \pi h\eta \frac{eV_{rf}\cos\phi_s}{\beta^2 E_o};$$

$$\mu_s = \sqrt{-2\pi h\eta \frac{eV_{rf}\cos\phi_s}{\beta^2 E_o}}; Q_s = \frac{\mu_s}{2\pi} = \sqrt{-\frac{h\eta}{2\pi}\frac{eV_{rf}\cos\phi_s}{\beta^2 E_o}}$$

with synchrotron tune, naturally, identical to that derived in Lecture 12 when $\eta\cos\phi_s < 0$.

HW 3 (3 points): For our example in lecture 12, find the synchrotron tunes for 100 GeV and 15 GeV protons in a storage ring for the following parameters (similar to RHIC collider at BNL):

RF voltage, V= 500 kV

Depending on the sing of the slip facor the synonymous phase is zero or 180 degrees,

$\phi_s = 0, \pi$ **- it is also called zero crossing**

Harmonic number, h=360

Compaction factor, $\alpha_c = 0.002$

Solution: for zero crossing $\sin\phi_s = 0$ and proper sign $\eta\cos\phi_s < 0$ we haev

$$Q_s == \sqrt{\frac{h|\eta|}{2\pi}\frac{eV_{rf}}{\beta^2 E_o}}$$

where we should add relation between compaction and slip factors:

$$\eta = \alpha_c - \frac{1}{\gamma^2}$$

The rest requires to know the rest mass of proton which is 0.9382720 GeV. The rest is just crunching the numbers:

Energy, GeV	15	100
γ	15.987	106.579
β	0.9980	1.0000
V, GeV	0.0005	0.0005
h	360	360
αc	0.002	0.002
η	-0.00191	0.00191
Qs	0.00191	0.00074