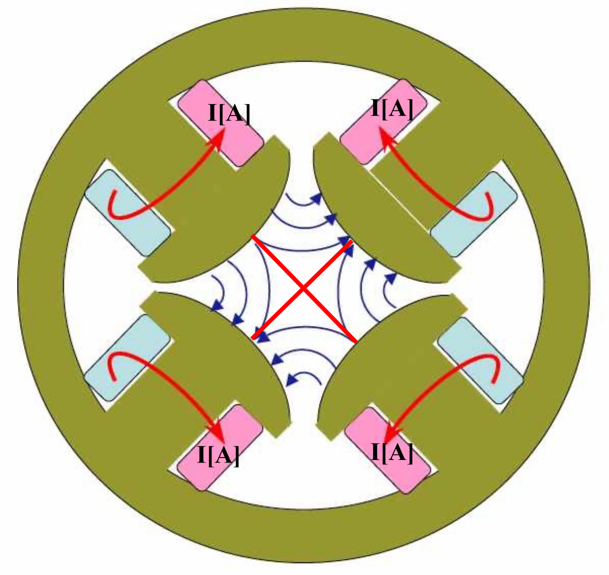
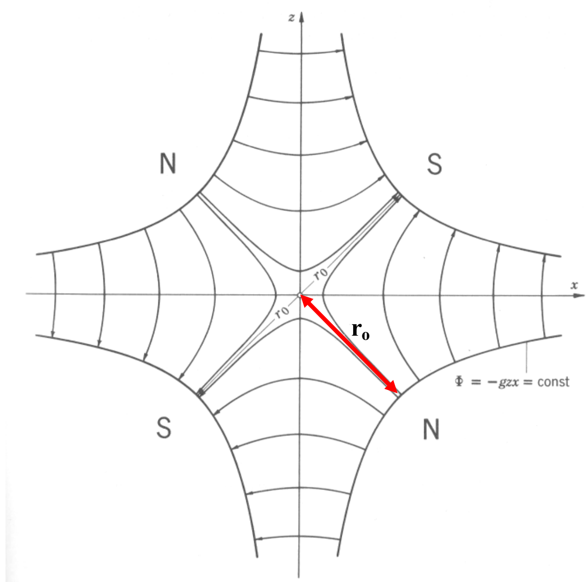
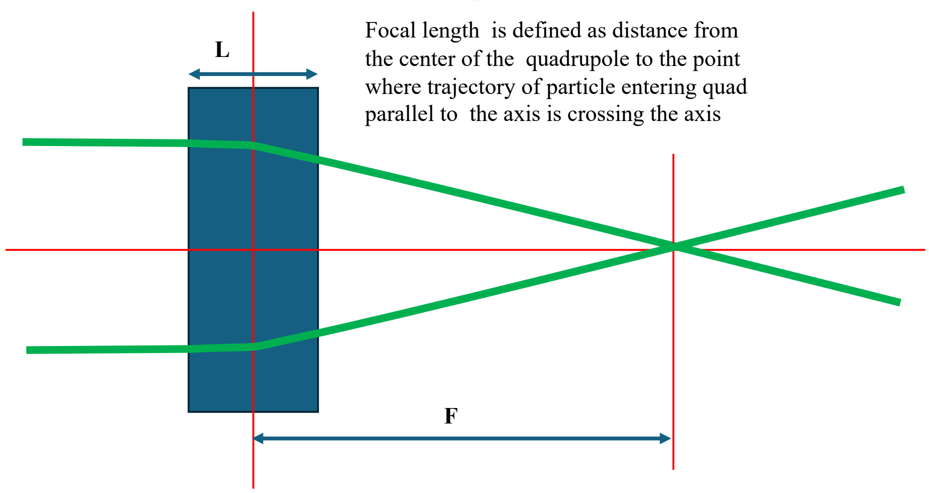
PHY 564 Midterm open-book, take home exam

Sent – evening Monday, March 23, 2026, Due -mid-night, Wednesday 25, 2026

Problem 1. Design a quadrupole for 3 GeV electron beam with pole radius of r_o=*2 cm* (0.*02 m),* field gradient of *5 kGs/cm (50 T/m*) and focal length of *F=1* meter. See the figures and specific questions below. **Total points: 40 points**







a) Fund value of field on the tip of the quadrupole pole (r=2 cm) – **3 points**
b) What is the current in each of 4 quadrupole coils – **4 points**
c) In so-called "short quadrupole" assumption, find necessary length of quadrupole needed for focal length of *F=1* meter – **5 points**
d) For a sick (real) quadrupole white equation for length L needed for *F=1* meter – **10 points**
e) Check what is actual focal length of sick quadrupole with length found in a "short-quad" approximation – **8 points**
f) Attempt to solve (even approximately, or using Mathematica and other numerical tools) what is actual length needed for *F=1* meter. – **10 points**

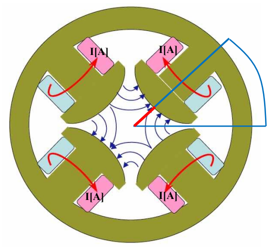
Solution:

a) Magnetic field in quadrupole is

$$\vec{B} = G\cdot(\hat{x}\cdot y + \hat{y}\cdot x);\ x = y = \frac{r}{\sqrt{2}} \rightarrow B_r = G\cdot r$$

moving radially at 45 degrees at radius 2 cm we have radial field of 10 kGs (1 T).

b) Let's consider a contour of integral for Stock's theorem shown in the figure:



$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\cdot I$$

For ideal magnetic steel H=0 in the iron core and outside the magnet. At the horizontal part of the contour the field is perpendicular to the direction of pass. Hence, the only non-zero portion of the integral is coming from integral at 45-degrees from quad center to the pole tip:

$$\oint \vec{B}\cdot d\vec{l} = \int_o^{r_o} G\cdot r\cdot dr = \frac{G\cdot r_o^2}{2} \Rightarrow \frac{G\cdot r_o^2}{2}\left[Gs\cdot cm\right] = 0.4\pi\cdot I[A];$$

$$I[A] = \frac{G\left[\frac{Gs}{cm}\right]\cdot r[cm]_o^2}{0.8\pi} = 7{,}959\ \ A$$

c) Focal length of thin quadrupole is

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ -\frac{eG\cdot l}{pc} & 1 \end{bmatrix};\ F = -\frac{m_{11}}{m_{21}} = \frac{pc}{eG\cdot l};\ l_{sh} = \frac{pc}{eG\cdot F} = \frac{B\varrho}{G\cdot F};$$

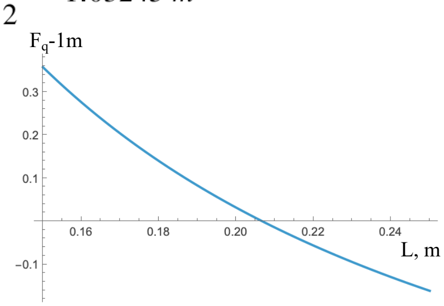
$$B\varrho\left[T\cdot m\right] = \frac{pc\left[GeV\right]}{0.299792458} = 10.0069\ \left[T\cdot m\right];\ \ l_{sh} = 0.200138\ m$$

d) Matrix of real, thick, quadrupole and its focal length are:

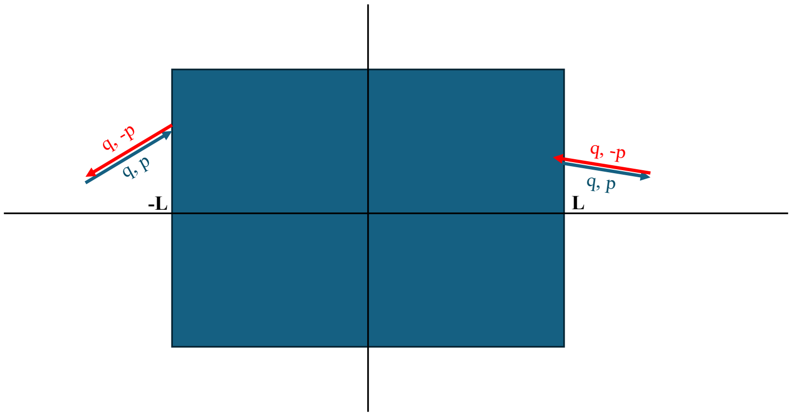
$$\mathbf{M} = \begin{bmatrix} \cos\varphi & \frac{\sin\varphi}{\omega} \\ -\omega\cdot\sin\varphi & \cos\varphi \end{bmatrix};\ \omega = \sqrt{\frac{eG}{pc}} = \sqrt{\frac{G}{B\varrho}} = 2.23529\ m^{-1};\ \varphi = \omega\cdot L;\ \varphi_{test} = \omega\cdot l_{sh} = 0.447368;$$

$$F_q = -\frac{m_{11}}{m_{21}} - \frac{L}{2};\ F_q\left(L = l_{sh}\right) = \frac{\cos\varphi_{test}}{\omega\cdot\sin\varphi_{test}} - \frac{l_{sh}}{2} = 1.03245\ m$$

e,f) Using Mathematica we get L=0.207077 m with a number of other roots like 2.2123 m and 4.0307 m, etc., which are longer than requested focal length. Hence, L=0.207 m is what we need for our design/



Problem 2. Mirror or bilaterally symmetric lattices. Total 45 points
Consider a linear transport system extending from *-L* to *+L* is *s* (value of L is not important!) and that it has a mirror (bilateral) symmetry *H(-s)=H(s)*. Derive condition on the transport matrix coefficient from *-L* to *+L*.



a) Observe (discuss) the fact that reversed trajectory has the same matrix as that in the fromward direction and that $q_{rev}(+L)=q_{fov}(+L)$ and $P_{rev}(+L)=-P_{fov}(+L)$ will result in $q_{rev}(-L)=q_{fov}(-L)$ amd $P_{rev}(-L)=-P_{lof}(-L)$ – **5 points**
b) Consider the fact that reverse motion is described by the same matrix, but we also can use inverse matrix to connect which can be found using symplecticity conditions – **5 points**
c) Derive conditions for 1D case and 2x2 matrix – **15 points**
d) Derive conditions for *N* dimensions and *2N*x*2N* matrices. - **20 points**

Hint: While it does not matter for 1D case, I suggest that you use overscored (Alex Dragt's) convention for 2Nx2N matrices.

Solution:
a) Additional explanation (below) of the problem below explains why reverse motion is described by the same map and therefore the same matrix:

$$\begin{bmatrix} Q \\ P \end{bmatrix}_L = \mathbf{M}\cdot\begin{bmatrix} Q \\ P \end{bmatrix}_{-L} \Rightarrow \begin{bmatrix} Q \\ -P \end{bmatrix}_{-L} = \mathbf{M}\cdot\begin{bmatrix} Q \\ -P \end{bmatrix}_{+L} \qquad (0)$$

Additional explanation. It is important to implicitly specify the meaning of the bilateral summitry. We know the Hamiltonian equations (*Q* and *P* have *N* components):

$$\frac{dQ}{ds} = \frac{\partial H(Q,P,s)}{\partial P};\ \frac{dP}{ds} = -\frac{\partial H(Q,P,s)}{\partial Q}; \qquad (1)$$

Let's now use reverse coordinate *z=-s*:

$$\frac{dQ}{dz} = -\frac{dQ}{ds} = -\frac{\partial H(Q,P,-s)}{\partial P};\ \frac{dP}{dz} = -\frac{dP}{ds} = \frac{\partial H(Q,P,-s)}{\partial Q};$$

Let's also introduce new variable

$$\widehat{P} = -P;$$
$$\frac{dQ}{dz} = \frac{\partial H(Q,-\widehat{P},-z)}{\partial \widehat{P}};\ \frac{d\widehat{P}}{dz} = -\frac{\partial H(Q,-\widehat{P},-z)}{\partial Q}.$$

This set of questions will be identical to eq (1) if

$$H(Q,-P,-s) = H(Q,P,s) \qquad (2)$$

or that in addition to bilateral symmetry the Hamiltonian is an even function of momenta. In this case map from {Q(-L),P(-L)} to {Q(+L),P(+L)} is identical to reverse map from Q(+L),-P(+L)} to {Q(-L),-P(-L)}.

In linear system eq. (2) means that there should be no $h_{qipj}(s)q_iP_j$ terms in the Hamiltonian or there sign is flipping with the sign of s: $h_{qipj}(s)=- h_{qipj}(-s)$. For example, a bilaterally symmetric system that has torsion or solenoidal field should have $\kappa(-s) = -\kappa(s);\ B_s(-s) = -B_s(s)$

I hoped that this would be condition you get in part a) of the problem, but since I already have questions, I decided to add this explanation to the problem.

b) At the same time we know that we can inverse symplectic matrix to connect initial

$$\begin{bmatrix} Q \\ P \end{bmatrix}_{-L} = \mathbf{M}^{-1}\cdot\begin{bmatrix} Q \\ P \end{bmatrix}_{L} = -\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}\cdot\begin{bmatrix} Q \\ P \end{bmatrix}_{L} \qquad (3)$$

and also can rewrite (0) as

$$\begin{bmatrix} Q \\ -P \end{bmatrix}_{-L} = \mathbf{M}\cdot\begin{bmatrix} Q \\ -P \end{bmatrix}_{+L} \Rightarrow \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}\cdot\begin{bmatrix} Q \\ P \end{bmatrix}_{-L} = \mathbf{M}\cdot\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}\cdot\begin{bmatrix} Q \\ P \end{bmatrix}_{+L};\ \begin{bmatrix} Q \\ P \end{bmatrix}_{-L} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}\cdot\mathbf{M}\cdot\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}\cdot\begin{bmatrix} Q \\ P \end{bmatrix}_{+L} \Rightarrow$$
$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}\cdot\mathbf{M}\cdot\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} = -\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S} \Rightarrow \mathbf{M} = -\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}\cdot\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}\cdot\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}. \qquad (4)$$

c) For 2x2 matrix is very simple

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\cdot\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\cdot\begin{bmatrix} a & c \\ b & d \end{bmatrix}\cdot\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot\begin{bmatrix} a & c \\ b & d \end{bmatrix}\cdot\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot\begin{bmatrix} c & a \\ d & b \end{bmatrix} = \begin{bmatrix} d & b \\ c & a \end{bmatrix} \Rightarrow a = d\#$$

resulting in equal diagonal terms.

d) Using overscored conventions, we can simply continue with (4) to derive a bit more conditions:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = -\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{A}^{\mathbf{T}} & \mathbf{C}^{\mathbf{T}} \\ \mathbf{B}^{\mathbf{T}} & \mathbf{D}^{\mathbf{T}} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{A}^{\mathbf{T}} & \mathbf{C}^{\mathbf{T}} \\ \mathbf{B}^{\mathbf{T}} & \mathbf{D}^{\mathbf{T}} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix};$$
$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{D}^{\mathbf{T}} & \mathbf{B}^{\mathbf{T}} \\ \mathbf{C}^{\mathbf{T}} & \mathbf{A}^{\mathbf{T}} \end{bmatrix} \Rightarrow \mathbf{A} = \mathbf{D}^{\mathbf{T}},\ \mathbf{B} = \mathbf{B}^{\mathbf{T}};\ \mathbf{C} = \mathbf{C}^{\mathbf{T}}\#$$

With all block matrices **A,B,C,D** being NxN, the first equation $\mathbf{A}=\mathbf{D}^{\mathbf{T}}$ imposed N^2 conditions and next two symmetry conditions impose $N(N-1)/2$ conditions each, hence the total of $2N^2-N$ conditions: 1 for $N=1$, 6 for $N=2$ and 15 for $N=3$. Hence, such lattices are popular…

Problem 3. Muon collider – **15 points**
Muons are unstable particles with energy of 105.66 MeV and 2.197 microseconds e-fold lifetime in rest frame. Physics community is considering a multi-TeV circular muon collider where muons circulate and collide for a number of turns. Consider a storage ring with average bending magnetic field of 10 T and derive number of turns (collisions) muon can conclude during it e-fold lifetime. Does this number depend on the muon beam energy?
Suggestions/Hints: consider ultra-relativistic muons moving with speed practically indistinguishable from the speed of light (indeed only such colliders are of interest). Try to understand
Solution: Let's conder a circular collider for muons with energy **E** and corresponding momentum

$$p = \beta \cdot \frac{\mathbf{E}}{c};\ \beta = \sqrt{1-\gamma^{-2}};\ \gamma = \frac{\mathbf{E}}{mc^2}.$$

The curvature of the orbit is defined by the value of magnetic field and beam rigidity

$$K_o(s) = \frac{eB_y(s)}{pc}.$$

To make one turn each particle has to turn by 2π angle, i.e. we can connect ring circumference with average bending magnetic field:

$$\int_o^C K_o(s)ds = \frac{e}{pc}\int_o^C B_y(s)ds = \frac{e \cdot \langle B_y \rangle \cdot C}{pc} = 2\pi \Rightarrow C = \frac{2\pi \cdot pc}{e \cdot \langle B_y \rangle} = \frac{2\pi \cdot B\rho}{\langle B_y \rangle}.$$

Time required for particle to make a turn is:

$$T_o = \frac{C}{\mathrm{v}} = \frac{C}{\beta c} = \frac{2\pi \cdot pc}{\beta c \cdot e \cdot \langle B_y \rangle} = \frac{2\pi}{c} \cdot \frac{\mathbf{E}}{e \langle B_y \rangle}.$$

Life time of the moving muon is boosted by factor γ

$$\tau_c = \gamma \cdot \tau_o = \frac{\mathbf{E}}{mc^2} \cdot \tau_o.$$

and number of turns during e-fold reduction of muon beam intensity is

$$N = \frac{\tau_c}{T_o} = c \cdot \tau_o \cdot \frac{\mathbf{E}}{mc^2} \cdot \frac{e\langle B_y \rangle}{2\pi \cdot \mathbf{E}} = \frac{c \cdot \tau_o}{2\pi} \cdot \frac{e \langle B_y \rangle}{mc^2}$$

does not depend on the beam energy but is proportional to the average bending magnetics field. This is why there is a strong drive to develop superconducting dipole magnets with as high field as possible. For 10 T average magnetic field we have

$$N = \frac{c \cdot \tau_o}{2\pi} \cdot \frac{e \langle B_y \rangle}{mc^2} = 2{,}974$$

This number is very small when compared with millions and billions of turns stable particles such as protons and electrons can survive in colliders – hence the problem with generating, cooling and colliding muons is one of the major challenges in modern accelerator science.