PHY 564
Advanced Accelerator Physics
Lecture 25

Nonlinear dynamics. Part III

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Brief recapture of last lecture

$$X=\{x_i, i=1,2n\}=\{\{q_k,P^k\}k=1,n\};$$

$$f=f(X,s)\equiv f(q_k,P^k,s); g=g(X,s)\equiv g(q_k,P^k,s);$$

$$[f,g]_{def}=\sum_{k=1}^{n}\left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial P^k}-\frac{\partial g}{\partial q_k}\frac{\partial f}{\partial P^k}\right)=\sum_{i,j=1}^{2n}\left(\frac{\partial f}{\partial x_i}S_{ij}\frac{\partial g}{\partial x_j}\right)= \tag{15}$$

$$(\partial_X f, \mathbf{S}\cdot\partial_X g)=(\partial_X f)^T\,\mathbf{S}\cdot(\partial_X g).$$

Let's now introduce one more object, **Lie operator** $:f:$ defined as :

$$:f:g=[f,g];$$

$$:f:^0 g=g;\quad :f:^2 g=[f[f,g]];\quad :f:^{n+1}g=[f,:f:^n g]. \tag{16}$$

$$:f:^n(a\cdot g+b\cdot h)=(a\cdot :f:^n g+b\cdot :f:^n h) \tag{17}$$

$$:f:(g\cdot h)=(:f:g)\cdot h+g\cdot(:f:h) \tag{18}$$

$$:f:[g,h]=[:f:g,h]+[g,:f:h] \tag{21}$$

Product (algebraic, not simple multiplication) of Lie operators:

$$\{:f:,:g:\}=:f::g:-:g::f: \tag{22}$$

$$\{:f:,:g:\}h=(:f::g:-:g::f:)h=[:f:,:g:]h=:[f,g]:h \tag{23}$$

with $:[f,g]:$ being a compact form of the product of two operators.

$$\exp(:f:) = \sum_{n=0}^{\infty} \frac{:f:^n}{n!} \tag{24}$$

$$\exp(:f:)(gh) = \left(\exp(:f:)g\right)\left(\exp(:f:)h\right) \tag{25}$$

$$\exp(:f:)x^n = \left(\exp(:f:)x\right)^n;$$

$$g(X) = \sum_{n=0}^{\infty} g_n X^n \rightarrow \exp(:f:)g(X) = \sum_{n=0}^{\infty} g_n \left(\exp(:f:)X\right)^n = g\left(\exp(:f:)X\right). \tag{26}$$

$$\exp(:f:)[g,h] = \left[\exp(:f:)g, \exp(:f:)h\right] \tag{27}$$

Any analytical symplectic map can be presented as a product of linear (Gaussian optics) Lie transformation and product of Lie transformations comprising homogeneous polynomials of increasing power:

$$\mathtt{M} = \overbrace{\exp(:f_2:)}^{Gaussian\,optics} \bullet \overbrace{\exp(:f_3:)\exp(:f_4:)\exp(:f_5:)\ldots}^{Abberations,\,Nonlinear\ effects} \tag{44}$$

Lecture 25: Nonlinear beam dynamics, Part III

We firefly introduced adjoin Lie operator in last lecture but did not had enough time to discuss

This important ratio can be derived by introducing adjoin Lie operator

$$
\begin{gathered}
def: \quad \#f\#:g: = \{:f:,:g:\};\ \#f\#^2:g: = \{:f:,\{:f:,:g:\}\}; \\
\#f\#^0:g: = :g:; \quad \#f\#^{n+1}:g: = \{:f:,\#f\#^n:g:\}; \\
\{\#f\#,\#g\#\} = \#f\#\#g\# - \#g\#\#f\#; \\
\{\#f\#,\#g\#\} = \#\{f,g\}\# = \#\{f,g\}\#; \\
\exp(\#:f:\#) = \sum_{n=0} \frac{\#:f:\#^n}{n!}.
\end{gathered} \tag{1}
$$

These definition and rules are similar to Lie operators, but not they act on Lie operators themselves. The most non-trivial difference results is:

$$
\begin{gathered}
\exp(\#:f:\#):g: = \exp(:f:):g:\exp(-:f:)\ ; \\
exp(\#:f:\#):g:^n = \exp(:f:):g:^n\exp(-:f:); \\
exp(\#:f:\#)exp(:g:) = \exp(:f:)\exp(:g:)\exp(-:f:) = \exp(\exp(:f:):g:).
\end{gathered} \tag{2}
$$

We specifically plan to use the last relation

$$\exp(:f:)\exp(:g:)\exp(-:f:) = \exp(\exp(:f:):g:)$$

which worth a proof. Let's consider symplectic map

$$\bar{x} = exp(:f(x):)\ x$$

and apply it to an arbitrary function *h*:

$$exp(:f(x):)\ h(x) = h(\bar{x})$$

and define

$$v(x) = exp(:g(x):)\ h(x)$$

Then we can use already proven relations to derive that

$$\begin{aligned}
&exp(:f(x):)\cdot exp(:g(x):)\cdot exp(-:f(x):)\cdot h(\bar{x}) = \\
&exp(:f(x):)\cdot exp(:g(x):)\cdot h(x) = exp(:f(x):)\cdot v(x) = v(\bar{x}) \\
&= exp(:g(\bar{x}):)\cdot h(\bar{x}) = exp(:exp(:f(x):)\cdot g(x):)\cdot h(\bar{\bar{x}}),
\end{aligned}$$

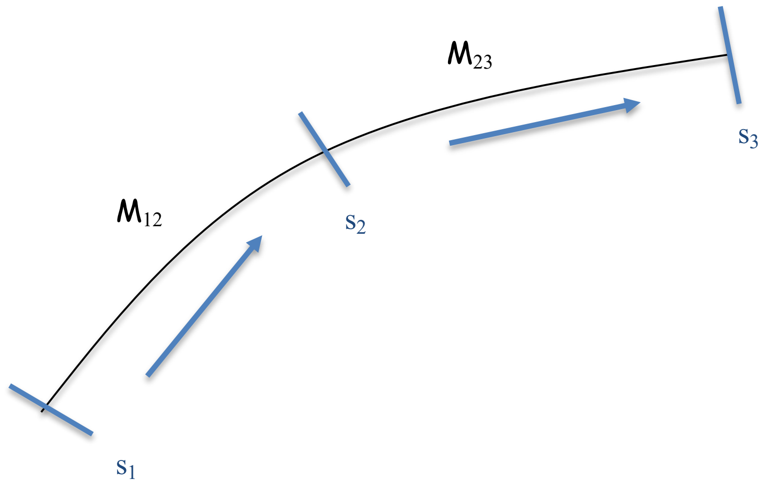
for an arbitrary function in the phase space. Here we used previously discussed:

$$g(\bar{x}) = exp(:f(x):)\cdot g(x)\ \Rightarrow :g(\bar{x}): = :exp(:f(x):)\cdot g(x):.$$

Hence, we have a proof of the magic feature of the Lie transformations:

$$\exp(:h:)\exp(:g:)\exp(-:h:) = \exp(\exp(:h:):g:) \quad (3)$$

Let's not consider how this important feature is applied to the maps by considering two consequent maps:



$$x(s_2) = \mathbb{M}_{12}(x(s_1))\cdot x(s_1);\; x(s_3) = \mathbb{M}_{23}(x(s_2))\cdot x(s_2) \equiv \mathbb{M}_{12}(M_{12}x(s_1))\cdot \mathbb{M}_{12}(x(s_1))\cdot x(s_1) \quad (4)$$

And also write is in form of Lie operators

$$x(s_2) = exp(:f_{12}(x(s_1)):)x(s_1);\; x(s_3) = exp(:f_{32}(x(s_2)):)x(s_2) =$$
$$exp(:f_{32}(exp(:f_{12}(x(s_1)):)x(s_1)):)\cdot exp(:f_{12}(x(s_1)):)\cdot x(s_1);$$
$$exp(:f_{32}(exp(:f_{12}(x(s_1)):)\cdot x(s_1)):) = exp(:f_{12}(x(s_1)):)\cdot exp(:f_{32}(x(s_1)):)\cdot exp(-:f_{12}(x(s_1)):)$$
$$x(s_3) = exp(:f_{12}(x(s_1)):)\cdot exp(:f_{32}(x(s_1)):)\, x(s_1)$$
$$(5)$$

It means that Lie operators (using phase space coordinates at initial position) are applied in opposite order we used for matrix multiplications:

$$x(s_3) = \mathbb{M}_{12}(x(s_1)) : \mathbb{M}_{23}(x(s_1)) : x(s_1) \text{ vs } X(s_3) = \mathbf{M}_{23}\cdot \mathbf{M}_{23}\cdot X(s_1). \quad (6)$$

This is an additional significant difference between using differential Lie operators vs matrix multiplications. Naturally, for linear transformation, both left and right side of (6) generate identical transformation, but application of differential Lie operator for the last element of the lattice and going backwards is indeed confusing.

The rules for Lie transformation/Symplectic maps $\mathcal{A}$, $\mathcal{M}$ can be formalized as following:

$$\mathcal{A}\,(gh) = (\mathcal{A}\,g)\,(\mathcal{A}\,h)$$

$$\mathcal{A}\,[g,h] = [\mathcal{A}\,g, \mathcal{A}\,h]$$

$$\mathcal{A}\,(g(z) = g\,(\mathcal{A}\,z)$$

$$\mathscr{A} = :f:,\ \mathscr{B} = :g:,\ exp(\alpha \cdot \mathscr{A}) \cdot exp(\beta \cdot \mathscr{B}) = exp(\mathscr{G});$$

$$\mathscr{G} = \alpha \cdot \mathscr{A} + \beta \cdot \mathscr{B} + \frac{\alpha \cdot \beta}{2}[\mathscr{A},\mathscr{B}] + \frac{\alpha^2 \cdot \beta}{12}[\mathscr{A},[\mathscr{A},\mathscr{B}]] + \frac{\alpha \cdot \beta^2}{12}[\mathscr{B},[\mathscr{B},\mathscr{A}]] + O(\varepsilon^4)$$

$$\mathcal{A} :g: \mathcal{A}^{-1} = :(\mathcal{A}\,g):$$

$$\mathcal{A}\,\mathcal{M}\,(z)\,\mathcal{A}^{-1} = \mathcal{M}\,(\mathcal{A}\,z)$$

These are the same rules we discussed before but written in a compact form. The last relation has a special importance – it has appearance of similarity transformation for matrices and allows us to consider a possibility of transforming complex map $\mathcal{M}$ to a simpler form

$$\mathcal{N}(\underline{z}) = \mathcal{A}\,\mathcal{M}\,(z)\,\mathcal{A}^{-1},\ \underline{z} = \mathcal{A}\,z \tag{7}$$

CBH theorem (analog of Campbel-Baker-Hausdorff theorem in quantum mechanics)

$$\mathscr{A} = :f:;\ \mathscr{B} = :g:;\ exp(\alpha \cdot \mathscr{A}) \cdot exp(\beta \cdot \mathscr{B}) = exp(\mathscr{C});$$

$$\mathscr{C} = \alpha \cdot \mathscr{A} + \beta \cdot \mathscr{B} + \frac{\alpha \cdot \beta}{2}[\mathscr{A},\mathscr{B}] + \frac{\alpha^2 \cdot \beta}{12}[\mathscr{A},[\mathscr{A},\mathscr{B}]] + \frac{\alpha \cdot \beta^2}{12}[\mathscr{B},[\mathscr{B},\mathscr{A}]].. \tag{8}$$

is a very important practical tool for understanding of the symplectic map's properties. What is also important form:

$$\mathscr{C} = \alpha \cdot \mathscr{A} + \alpha \cdot \beta \cdot \left[\frac{\mathscr{A}}{1 - exp(-\alpha \cdot \mathscr{A})}, \mathscr{B}\right] + O(\beta^2).. \tag{9}$$

with proof to be found in A.J. Dragt, J.M. Finn, Lie Series and In variant Functions for Analytical Symplectic Maps, *J. Math. Phys.* **17,** 2215, 1976 (https://wiki.classe.cornell.edu/pub/CBB/RDTProject/1.522868.pdf)

Equation (8) allows us to expand further map factorization into concatenation formula:

$$\mathcal{M}_f = exp(:f_1:)exp(:f_2:)exp(:f_3:)\ldots$$
$$\mathcal{M}_g = exp(:g_1:)exp(:g_2:)exp(:g_3:)\ldots$$
$$\mathcal{M}_h = \mathcal{M}_f \mathcal{M}_g = exp(:h_1:)exp(:h_2:)exp(:h_3:)$$

with

$$h_1 = f_1 + exp(:f_2:)\left(g_1 + \frac{1}{2}:g_2:^2 f_3 - \frac{1}{6}:g_2:^4 f_4 + \frac{1}{4}\left[:g_1:f_3,:g_1:^2 f_3\right]\right)\ldots$$

It is quite natural to let computers calculating h_j … as well as calculating nonlinear transfer maps:

$$\mathcal{M}(s_i,s_f) = \lim_{\Delta s -> 0} exp(\Delta s \cdot :H(s_1):)exp(\Delta s \cdot :H(s_2):)\ldots exp(\Delta s \cdot :H(s_N):);$$
$$s_k \in \{s_i + (k-1)\Delta s, s_i + k\Delta s\}; \ \Delta s = \frac{s_f - s_i}{N}. \qquad (10)$$

Here we are using similar method as in calculating transfer matrices assuming that coefficients in the Hamiltonian are smooth functions of *s* and with sufficiently small *Δs* we can achieve necessary accuracy for the map.

Important note: the maps listed from left to right with increase of their index (increase in s) !

Invariants of motion are important for understanding of the particles dynamics. Invariant functions of a symplectic map are preserved the map, i.e.

$$\mathcal{M} f(x) = f(x) \Leftrightarrow f(\mathcal{M}x) = f(x)$$

With map expressed as Lie operator

$$\mathcal{M} = exp(:h(x,s):)$$

h is an invariant of motion

$$\mathcal{M}\ h(x,s) = exp(:h:)h = :h:^0 h + \sum_{n=1} \frac{:h:^n}{n!} h = h; \quad :h:^n h = [h,...[h,h]] \equiv 0$$

Conjugate maps are defined as related to each other by symplectic map $\mathcal{A}$

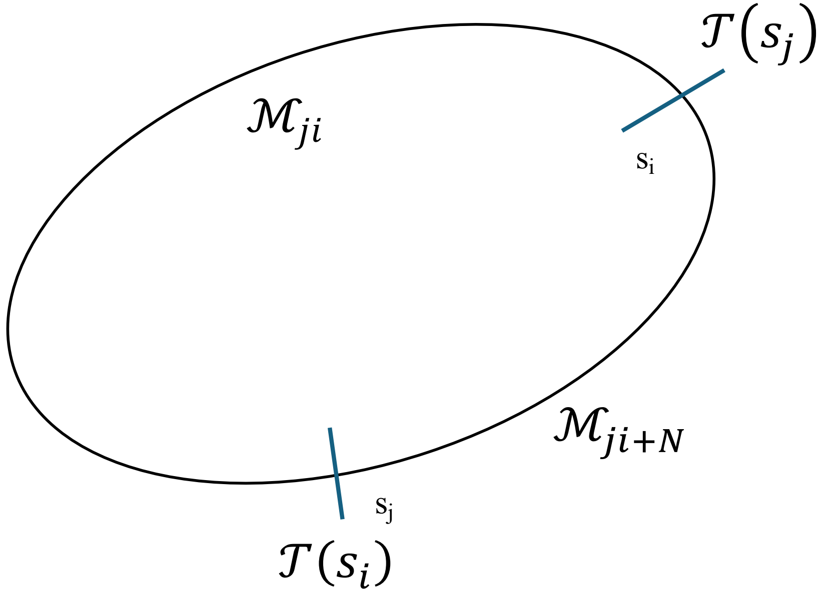
$$\mathfrak{N} = \mathcal{A}\mathcal{M}\mathcal{A}^{-1}$$

and they share the same set of invariant functions:

$$\begin{gathered} \forall h(x,s):\ \mathcal{M}\ h(x,s) = h(\mathcal{M}x,s) = h(x,s)\ ; \\ \exists\ u(x,s) = \mathcal{A}h(x,s) = h(\mathcal{A}x,s) \Rightarrow \\ \mathcal{N}u(x,s) = \mathcal{A}\mathcal{M}\mathcal{A}^{-1}\mathcal{A}h(x,s) = \mathcal{A}\mathcal{M}\ h(x,s) = \mathcal{A}\ h(x,s) = u(x,s). \end{gathered} \tag{11}$$

Hence, there is one to one correspondence of the set invariants for motion between conjugate maps, and reducing map to a simpler (called normal) form provides for easier understanding of the beam dynamics.

Maps for circular machines (or any periodic lattices with period C!) are of special interest because the transfer maps from s to $s+C$ are conjugate:



$$\mathcal{T}(s+C)=\mathcal{T}(s) \Leftrightarrow \mathcal{M}_{ji}=\mathcal{M}_{ji+N}$$

$$\mathcal{T}(s_i)=\mathcal{M}_{ij}\mathcal{M}_{ji+N}=\mathcal{M}_{ij}\mathcal{M}_{ji};$$

$$\mathcal{T}(s_j)=\mathcal{M}_{ji}\mathcal{M}_{ij}=\mathcal{M}_{ji}\mathcal{T}(s_i)\mathcal{M}_{ji}^{-1}$$

It means that the set of invariants of motion is function of the entire ring and invariant functions

Since maps can be already non-linear, there is no reasons to stick with x's and p's.

We can make Canonical transformation to action and phase variables:

$$\boldsymbol{\varphi}=(\varphi_1,..,\varphi_n);\ \mathbf{I}=(I_1,..,I_n);\ \mathbf{I}\cdot\boldsymbol{\varphi}=\sum_i \varphi_i\cdot I_i$$

The linear motion preserves actions and we can present the Lie map by a simple phase advance

$$\mathcal{R}=exp(-:\Delta\boldsymbol{\mu}\cdot\boldsymbol{I}:);\ [\varphi_i,I_k]=-[I_k,\varphi_i]=\delta_{ik};$$

$$\mathcal{R}:\varphi_i=\varphi_i-\Delta\boldsymbol{\mu}\cdot[\boldsymbol{I},\varphi_i]=\varphi_i+\Delta\mu_i;\ \ [I,[I,\varphi_i]]=0..$$

$$\mathcal{R}:I_i=I_i-\Delta\boldsymbol{\mu}\cdot e-[\mathbf{I},I_i]+\ldots=I_i.$$

In. the ring (or periodic cell), the phases are advancing for those of the eigen modes. Naturally, nonlinearity of the maps changing form of the actions – they are no longer just as quadratic form of x's and p's. But we can assume that a map can be represented by a Lie transformation and factorized as

$$\mathfrak{N}=\mathcal{A}\mathcal{M}\ \mathcal{A}^{-1}=exp(:-\boldsymbol{\mu}\cdot\boldsymbol{I}+D(\boldsymbol{I}))$$

with new (and unknow in general case) actions. It is easy to show that in this case action does not changes (commutator is still zero) while tune advance started depending on actions:

$$\mathcal{R}:I_i==I_i$$

$$\mathcal{R}:\varphi_i=\varphi_i+\mu_i-\frac{\partial D(\mathbf{I})}{\partial I_i}$$

Let's start from first step beyond linear map with 1D case:

$$\mathscr{M} = \mathscr{R}\ exp\left(:f_3:\right)exp\left(:f_4:\right)\dots$$

And construct a conjugate map:

$$\mathscr{U} = e^{:F_3:}\mathscr{M}e^{-:F_3:} = e^{:F_3:}\mathscr{R}\ e^{:f_3:}e^{:f_4:}\dots e^{-:F_3:}\dots =$$

$$\mathscr{R}\mathscr{R}^{-1}e^{:F_3:}\mathscr{R}\ e^{:f_3:}e^{-:F_3:}e^{:F_3:}e^{:f_3:}e^{-:F_3:}..$$

$$e^{:h:}e^{:g:}e^{-:h:} = e^{:e^{-:h:}g:} \Rightarrow \mathscr{U} = \mathscr{R}e^{:\mathscr{R}^{-1}F_3:}e^{:f_3:}e^{-:F_3:}e^{:e^{:F_3:}f_3:}$$

Using first order in CBH formular

$$e^{:A:}e^{:B:} = e^{:\left(A+B+\frac{1}{2}[A,B]\right):}\dots$$

$$\mathscr{U} = \mathscr{R}e^{:\mathscr{R}^{-1}F_3+f_3-F_3:}e^{:f_4^{(1)}:};\ f_3^{(1)} = \mathscr{R}^{-1}F_3+f_3-F_3$$

We dan define F_3 to remove all angle variables

$$F_3 = \frac{f_3 - f_3^{(1)}}{I-\mathscr{R}^{-1}};\ f_3 = \sum_{m=1}^{3} f_{3,i}(\mathrm{I})e^{im\varphi};\quad f_3^{(1)} = f_{3,0}(I);\ F_3 = \frac{\sum_{m=1}^{3} f_{3,i}(I)e^{im\varphi}}{I-\mathscr{R}^{-1}}.$$

Similar treatment can be applied to a single octupole

Let's start from first step beyond linear map with 1D case:

$$\mathscr{M} = \mathscr{R}\ exp\left(:f_4:\right)\ldots$$

with

$$x = \sqrt{2\beta \cdot \mathbf{I}}\cos\varphi;\ f_4 = -O\frac{x^4}{4!} = -\frac{O}{48}\beta^2 \cdot \mathbf{I}^2(3 + 4\cos 2\varphi + \cos 4\varphi)$$

$$f_{4,0} = -\frac{O}{16}\beta^2 \cdot \mathbf{I}^2;\ \mathbf{I} \to \mathbf{I};\ \varphi \to \varphi + \mu + \frac{\mathrm{O}}{8} \cdot \beta^2 \cdot \mathbf{I}$$

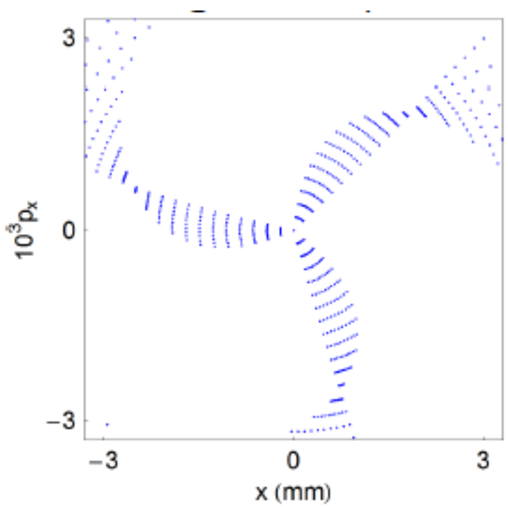
We dan define F_3 to remove all angle variables

$$F_4 = \frac{\sum_{m=1}^{4} f_{4,i}(\mathbf{I})\, e^{im\varphi}}{I - e^{-im\mu}} =$$

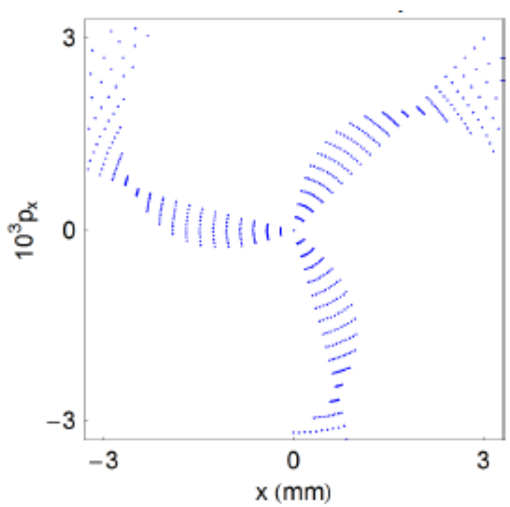
$$-\frac{O}{96}\beta^2\mathbf{I}^2\left(\frac{4(\cos 2\varphi - \cos 2(\varphi + \mu))}{1 - \cos 2\mu} + \frac{\cos 4\varphi - \cos 4(\varphi + \mu)}{1 - \cos 4\mu}\right)$$

The normalized map now contains only action variable (easy to integrate) while all the phase information has been pushed to higher order.

From the generator F_4, we see the octupole drives half integer and quarter integer resonances. We can track the Poincare map using exact map and the normalized map respectively (assuming O= 4800 m^{-3} and β=1 m). Assuming the tune μ is 0.33×2π far from resonances



exact map

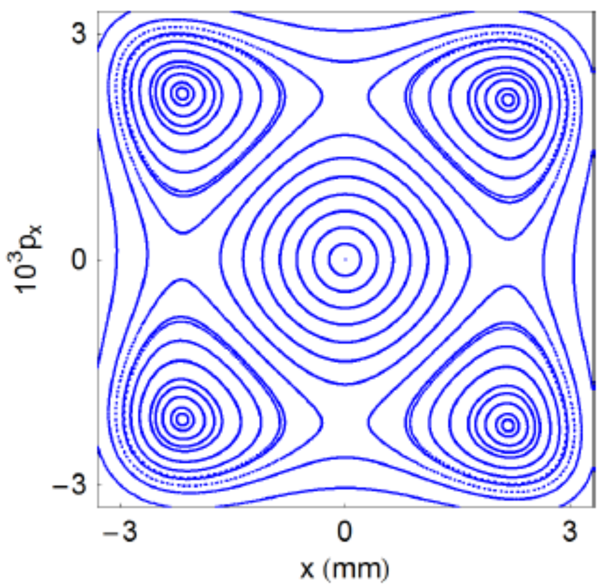


normalized map

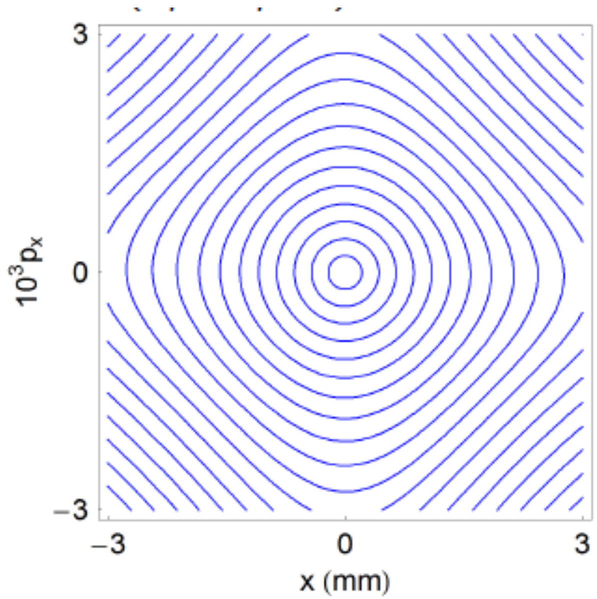
30 turns

Tune shift with amplitude!!

For a tune less than quarter integer, i.e., μ is $0.248\times 2\pi$, we see strong resonances from exact tracking while for the normalized map, we only see a rotation in phase space.



exact map



normalized map

2500 turns

Normal form of a one turn map preserves the information on tune amplitude dependence while loses the key phase information (when close to resonances). Need to retain higher order terms!

What we learned today

- Lie operators apply to the phase space coordinates in opposite sequence that we are multiplying maps: i.e. the maps of the last element is applied fist followed backwards towards first element in the transport line
- Conjugate map have identical set of the invariants, which are transferred from location to location by local maps
- There is a process of brining maps to a normal form, but these maps do not include resonances. At the resonances, F_k terms become infinite with one of denominators turning into zero.
- We will discuss resonances and resonant terms in next lecture