PHY 564
Advanced Accelerator Physics
Lectures 24

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Collective Effect and Instabilities

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Outline

- Wakefields and Impedances
  - Wake functions
  - Panofsky-Wenzel theorem
  - Cylindrical symmetric structure
  - Wake potentials, loss factor and kick factor
  - Impedances
- Longitudinal Microwave Instabilities
  - Dispersion relation
  - Instability in Cold beam
  - Instability in warm beam, Keil-Schnell criterion

Many pictures and derivations used in the slides are taken from the following references:

[1] 'Wake and Impedance' by G.V. Stupakov, SLAC-PUB-8683;

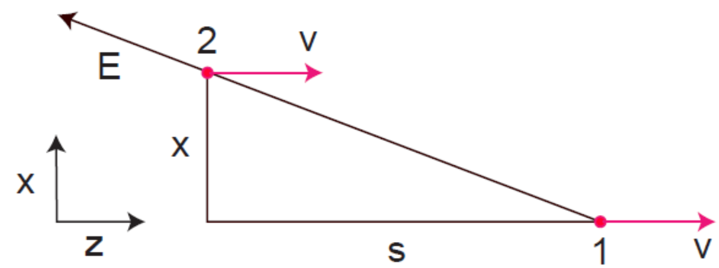
[2] 'Physics of Intensity Dependent Instabilities' by K.Y. Ng, Lecture Notes in USPAS 2002;

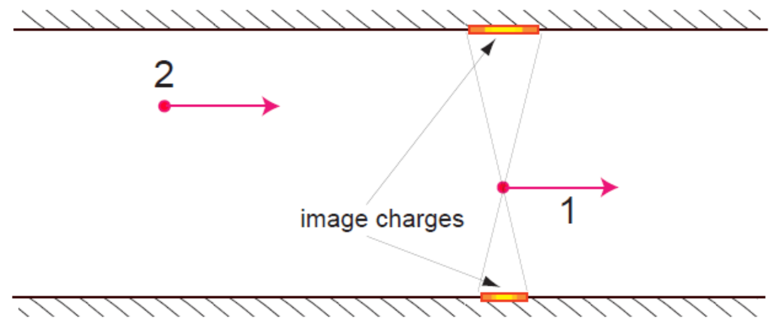
[3] 'Accelerator Physics' by S.Y. Lee;

[4] 'Physics of Collective Beam Instabilities in High Energy Accelerators' by A. Chao;

[5] 'Impedances and Wakes in High-Energy Particle Accelerators' by B. Zotter and S. Kheifets.

Ultra-relativistic beam and cylindrical perfect conducting beam pipe





$$\boldsymbol{E} = \frac{q\boldsymbol{R}}{\gamma^2 R^{*3}}, \qquad \boldsymbol{H} = \frac{1}{c}\boldsymbol{v} \times \boldsymbol{E},$$

$$R^{*2} = s^2 + x^2/\gamma^2$$

$$F_l = E_z = -\frac{qs}{\gamma^2(s^2 + x^2/\gamma^2)^{3/2}},$$

$$F_t = E_x - \frac{v}{c}B_y = \frac{qx}{\gamma^4(s^2 + x^2/\gamma^2)^{3/2}}.$$

At the limit of $\gamma \to \infty$

$$\boldsymbol{E} = \frac{2q\boldsymbol{r}}{r^2}\delta(z - ct)$$

$$\boldsymbol{H} = \hat{\boldsymbol{z}} \times \boldsymbol{E},$$

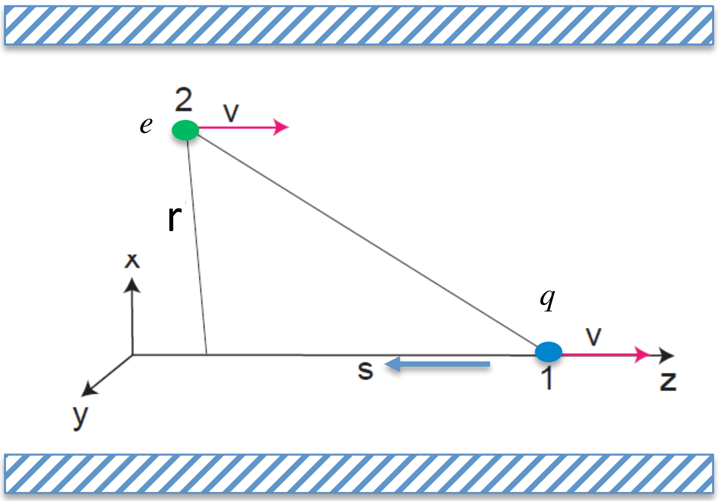
For $\gamma \to \infty$, interaction among the particles and their images from the wall vanishes if

1. the wall is perfectly conducting, and
2. there are no discontinuities (cavities, bpms, bellows…).

(It is also assumed that particles go straight, i.e. no radiations from particles)

Wake Functions

- Rigid bunch approximation
- Impulse approximation



$$\Delta\vec{p}(x,y,s) = e\int_{-\infty}^{\infty} dt\left[\vec{E}(x,y,z,t) + c\hat{z}\times\vec{B}(x,y,z,t)\right]_{z=ct-s}$$

*Particle 2 has charge e

Longitudinal wake function*: $$w_l(x,y,s) = -\frac{c}{qe}\Delta p_z = -\frac{c}{q}\int_{-\infty}^{\infty} E_z(x,y,ct-s,t)\,dt$$

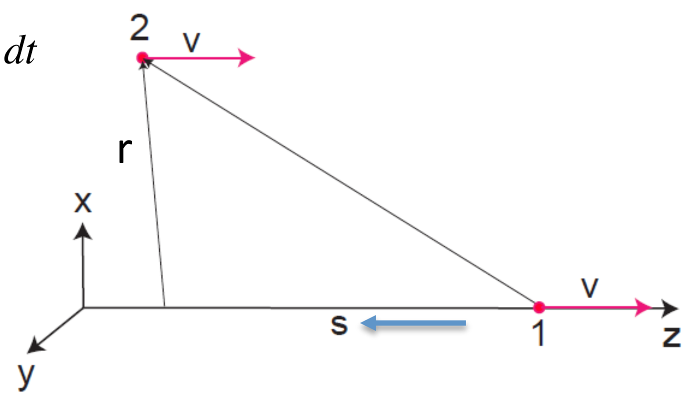
Transverse wake function*: $$\vec{w}_t(x,y,s) = \frac{c}{qe}\Delta\vec{p}_\perp = \frac{c}{q}\int_{-\infty}^{\infty}\left[\vec{E}_\perp(x,y,z,t) + c\hat{z}\times\vec{B}(x,y,z,t)\right]_{z=ct-s} dt$$

* These definition follow from ‘Impedances and Wakes in High-Energy Particle Accelerators by B. Zotter, which is different from those in ‘ Physics of Collective Beam Instabilities in High Energy Accelerators’ by A. Chao.

Panofsky-Wenzel Theorem

We want to find relation between longitudinal wake function and transverse wake function due to a structure (a piece of beam pipe, bpm, bellow, cavity....)

$$\nabla_s \times \Delta\vec{p}(x,y,s) = \nabla_s \times \int_{-\infty}^{\infty} \vec{F}(x,y,z,t)\Big|_{z=vt-s} dt = \int_{-\infty}^{\infty} \left[\nabla \times \vec{F}(x,y,z,t)\right]\Big|_{z=vt-s} dt$$



$$\nabla_s = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} - \hat{z}\frac{\partial}{\partial s} \qquad \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$\nabla\times\vec{F} = q\nabla\times\vec{E} + q\nabla\times\left(\vec{v}\times\vec{B}\right)$$

$$= -q\frac{\partial\vec{B}}{\partial t} + q\vec{v}\left(\nabla\cdot\vec{B}\right) - q\left(\vec{v}\cdot\nabla\right)\vec{B}$$

$$= -q\frac{\partial\vec{B}}{\partial t} - qv\frac{\partial}{\partial z}\vec{B}$$

$$\boxed{\vec{v} = v\hat{z} \qquad \nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

We assume the B field due to the structure has limited spatial range, i.e. it is localized.

$$\nabla_s \times \Delta\vec{p}(x,y,s) = -q\int_{-\infty}^{\infty}\left[\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}\right)\vec{B}(x,y,z,t)\right]_{z=vt-s} = -q\int_{-\infty}^{\infty}\frac{d}{dt}\vec{B}(x,y,vt-s,t)dt = 0$$

$$\frac{\partial}{\partial s}\Delta\vec{p}_\perp = -\vec{\nabla}_\perp \Delta p_z \quad \Longrightarrow \quad \boxed{\frac{\partial}{\partial s}w_t(x,y,s) = \vec{\nabla}_\perp w_l(x,y,s)}$$

This is called Panofsky-Wenzel theorem

$$\left[\nabla_\perp \times \Delta\vec{p}_\perp(x,y,s)\right]\cdot\hat{z} = 0$$

*The derivation follows from USPAS note by K.Y. Ng.

Another Relation at $\beta \to 1$

$$\nabla_s \cdot \Delta\vec{p}(x,y,s) = \nabla_s \cdot \int_{-\infty}^{\infty} \vec{F}(x,y,z,t)\Big|_{z=vt-s} dt = \int_{-\infty}^{\infty} \left[\nabla \cdot \vec{F}(x,y,z,t)\right]\Big|_{z=vt-s} dt$$

$$\nabla_s \cdot \Delta\vec{p}(x,y,s) = -\frac{q}{c}\int_{-\infty}^{\infty} \left[\frac{\partial}{\partial t} E_z(x,y,z,t)\right]\Bigg|_{z=vt-s} dt$$

$$\begin{aligned}
\nabla \cdot \vec{F} &= q\nabla \cdot \vec{E} + q\nabla \cdot \left(\vec{v} \times \vec{B}\right) \\
&= q\frac{\rho}{\varepsilon_0} - q\vec{v} \cdot \left(\nabla \times \vec{B}\right) \\
&= q\frac{\rho}{\varepsilon_0} - q\mu_0\vec{v} \cdot \left(\rho v\hat{z} + \varepsilon_0 \frac{\partial}{\partial t}\vec{E}\right) \\
&= q\frac{\rho}{\varepsilon_0\gamma^2} - \frac{q}{c}\beta\frac{\partial}{\partial t}E_z \\
&\approx -\frac{q}{c}\frac{\partial}{\partial t}E_z
\end{aligned}$$

$$\begin{aligned}
&= -\frac{q}{c}\int_{-\infty}^{\infty} \left\{\frac{d}{dt}E_z(x,y,vt-s,t) - \left[v\frac{\partial}{\partial z}E_z(x,y,z,t)\right]\Bigg|_{z=vt-s}\right\} dt \\
&= \frac{qv}{c}\int_{-\infty}^{\infty} \left[\frac{\partial}{\partial z}E_z(x,y,z,t)\right]\Bigg|_{z=vt-s} dt \\
&= -\frac{qv}{c}\frac{\partial}{\partial s}\int_{-\infty}^{\infty} E_z(x,y,vt-s,t)\,dt \\
&\approx -\frac{\partial}{\partial s}\Delta p_z(x,y,s)
\end{aligned}$$

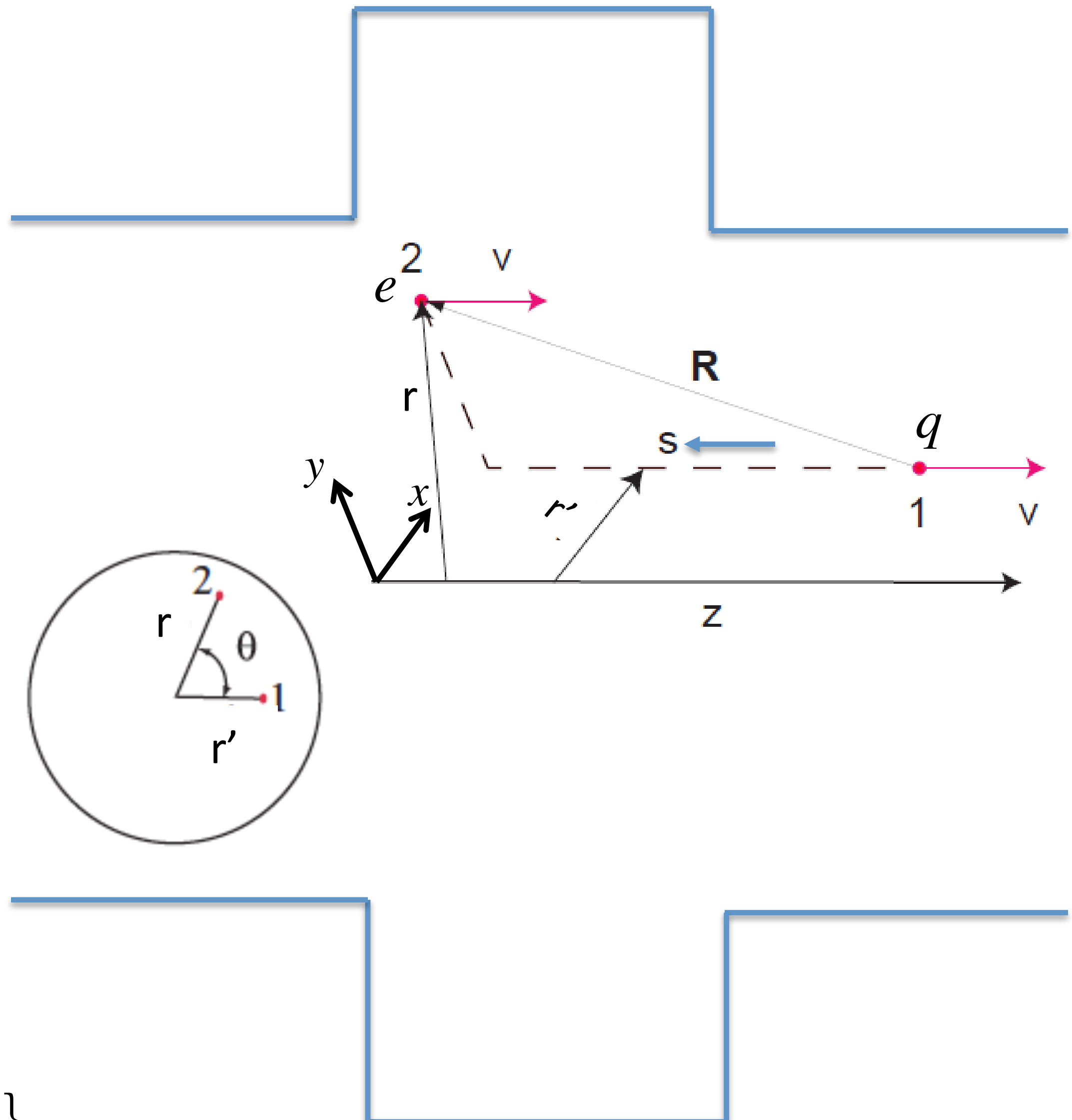
$$\boxed{\nabla_\perp \cdot \Delta\vec{p}(x,y,s) = 0}$$

Cylindrical Symmetric Structure

$$\nabla_s \times \Delta\vec{p}(r',r,\theta,z) = 0$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial s}\Delta\vec{p}_\perp = -\vec{\nabla}_\perp \Delta p_z \\ \\ [\nabla_\perp \times \Delta\vec{p}_\perp]\cdot\hat{z} = 0 \end{cases}$$

$$\nabla_\perp \cdot \Delta\vec{p}(r',r,\theta,s) = 0$$



For a system with cylindrical symmetry, it is usually more convenient to decompose quantities into azimuthal modes:

$$\Delta\vec{p}(r',r,\theta,s) \sim \{\cos(m\theta),\sin(m\theta)\}$$

Cylindrical Symmetric Structure I

$$\frac{\partial}{\partial s}\Delta\vec{p}_\perp = -\vec{\nabla}_\perp \Delta p_z \quad \Rightarrow \begin{cases} \dfrac{\partial}{\partial s}\Delta p_\theta = -\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\Delta p_z \\ \dfrac{\partial}{\partial s}\Delta p_r = -\dfrac{\partial}{\partial r}\Delta p_z \end{cases}$$

$$\Delta p_r(r',r,\theta,s) = \sum_{m=0}^{\infty}\Delta\tilde{p}_{m,r}(r',r,s)\cos(m\theta)$$

$$\Longrightarrow \Delta p_z(r',r,\theta,s) = \sum_{m=0}^{\infty}\Delta\tilde{p}_{m,z}(r',r,s)\cos(m\theta)$$

$$\left[\nabla_\perp \times \Delta\vec{p}_\perp(r',r,\theta,s)\right]\cdot\hat{z} = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial r}(r\Delta p_\theta) = \frac{\partial}{\partial\theta}\Delta p_r \quad \Longrightarrow \quad \Delta p_\theta(r',r,\theta,s) = \sum_{m=0}^{\infty}\Delta\tilde{p}_{m,\theta}(r',r,s)\sin(m\theta)$$

$$\vec{\nabla}_\perp \cdot \Delta\vec{p}(r',r,\theta,s) = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial r}(r\Delta p_r) = -\frac{\partial}{\partial\theta}\Delta p_\theta$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial s}\Delta\tilde{p}_{m,\theta} = \dfrac{m}{r}\Delta\tilde{p}_{m,z} & \dfrac{\partial}{\partial s}\Delta\tilde{p}_{m,r} = -\dfrac{\partial}{\partial r}\Delta\tilde{p}_{m,z} \\ \dfrac{\partial}{\partial r}(r\Delta\tilde{p}_{m,\theta}) = -m\Delta\tilde{p}_{m,r} & \dfrac{\partial}{\partial r}(r\Delta\tilde{p}_{m,r}) = -m\Delta\tilde{p}_{m,\theta} \end{cases} \quad \Longrightarrow$$

$$\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(r\Delta\tilde{p}_{m,r}(r',r,s)\right)\right] - m^2\Delta\tilde{p}_{m,r}(r',r,s) = 0$$

$$\Delta\tilde{p}_{m,r}(r,s) = A_m(r',s)r^{m-1} + \cancel{B_m(r',s)r^{-m-1}}$$

$$= \frac{qe}{v}r'^m W_m(s)mr^{m-1}$$

$$\Delta\tilde{p}_{m,\theta}(r',r,s) = -\frac{qe}{v}Q_m(r')W_m(s)mr^{m-1}$$

$$\Delta\tilde{p}_{m,z}(r',r,s) = -\frac{qe}{v}Q_m(r')W'_m(s)r^m$$

*Reference: K. Bane et al, AIP conference proceedings 127, pp. 875-928

Cylindrical Symmetric Structure II

$$w_l(r',r,\theta,s) = -\frac{c}{qe}\Delta p_z(r',r,\theta,s) = \sum_{m=0}^{\infty} W'_m(s)\, r'^m r^m \cos(m\theta)$$

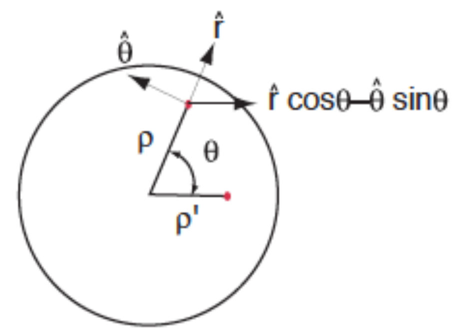
$$\vec{w}_t(r',r,\theta,s) = \frac{c}{qe}\Delta \vec{p}_\perp(r',r,\theta,s) = \sum_{m=0}^{\infty} W_m(s)\, m r'^m r^{m-1}\left[\cos(m\theta)\hat{r} - \sin(m\theta)\hat{\theta}\right]$$

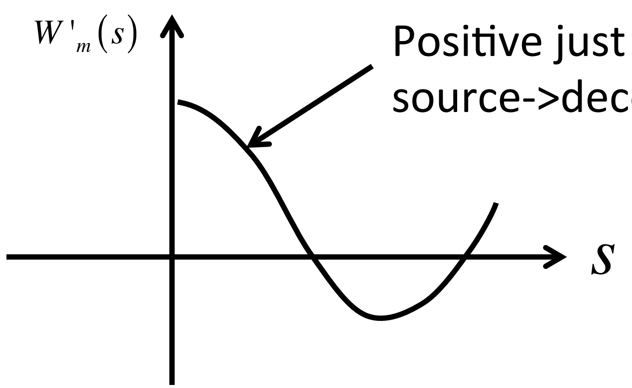
* In many references (by A. Chao, K.Y. Ng ...), $W_m(s)$ and $W'_m(s)$ are called wake functions.

$$m=0 \quad w_l(r',r,\theta,s) = W'_0(s) \qquad w_t(r',r,\theta,s) = 0$$

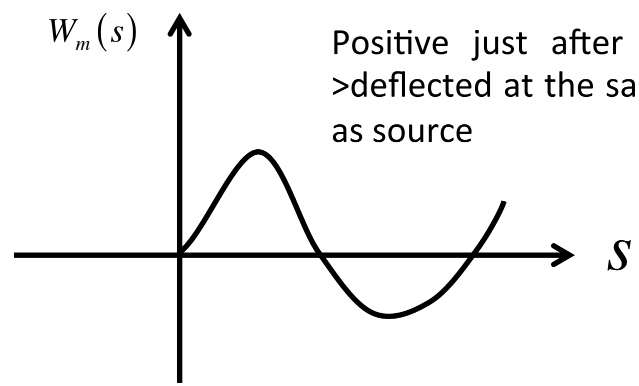
$$m=1 \quad \vec{w}_t(r',r,\theta,s) = W_1(s)\, r'\left[\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}\right]$$

$$w_l(r',r,\theta,s) = W'_1(s)\, r' r \cos(\theta)$$





Positive just after the source->deceleration



Positive just after the source->deflected at the same direction as source

Wake Potential

- In practice, usually only monopole mode (m=0) wake is considered for longitudinal wake field and only dipole mode (m=1) is considered for transverse mode.

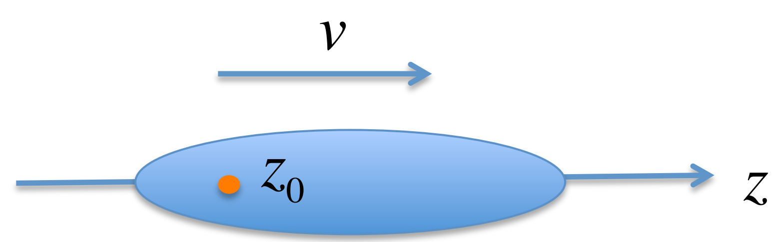
monopole longitudinal wake: $w_{//}(s) = W'_0(s)$ V/C

dipole transverse wake: $w_\perp(s) = W_1(s)$ V/(C*m)

- Wake potentials are defined to describe the momentum change induced by all particles in a bunch to a test unit charge:

$$V_{//}(z_0) = \frac{-c\Delta p_z(z_0)}{eQ_e} = \int_{z_0}^{\infty} \lambda(z_1) w_{//}(z_1 - z_0) dz_1 \quad \text{[V/C]}$$

$$\vec{V}_\perp(z_0) = \frac{c\Delta \vec{p}_\perp(z_0)}{eQ_e} = \int_{z_0}^{\infty} \langle \vec{x}_\perp(z_1) \rangle \lambda(z_1) w_\perp(z_1 - z_0) dz_1 \quad \text{[V/C]}$$



$\lambda(z)$ is line number density of a bunch

* If we observe at $z = z^*$ and use arriving time, $t = \frac{1}{c}(z^* - z)$ as longitudinal variables, above definition become

$$V_{//}(t_0) = \int_{-\infty}^{t_0} \lambda(t_1) w_{//}(t_0 - t_1) dt_1 \qquad \vec{V}_\perp(t_0) = \int_{-\infty}^{t_0} \langle \vec{x}_\perp(t_1) \rangle \lambda(t_1) w_\perp(t_0 - t_1) dt_1$$

Loss Factor and Kick Factor

- Once the longitudinal wake potential is known, the total energy change of a bunch to the wakefields is given by

$$\Delta U = -\int_{-\infty}^{\infty} \left[Q_e V_{//}(z) \right] \left[Q_e \lambda(z) \right] dz$$

Potential at slice (z,z+dz)

Charge in slice (z,z+dz)

Definition of Loss Factor:

$$\boxed{\kappa_{//} \equiv \frac{-\Delta U}{Q_e^2} = \int_{-\infty}^{\infty} V_{//}(z) \lambda(z) dz}$$

[V/C]

- Similarly, the total transverse momentum change of a bunch to the wakefields is given by

$$\Delta \vec{P}_{\perp} = \int_{-\infty}^{\infty} \left[e Q_e \frac{\vec{V}_{\perp}(z)}{c} \right] \left[\frac{Q_e}{e} \lambda(z) \right] dz$$

$$\boxed{\vec{\kappa}_{\perp} = \frac{c \Delta \vec{P}_{\perp}}{Q_e^2} = \int_{-\infty}^{\infty} V_x(z) \lambda(z) dz}$$

[V/C]

Transverse momentum change of a particle at slice (z,z+dz).

Particle number in slice (z,z+dz)

Impedances

- Although the time domain description of particle-enviroment interaction, the wake fields, contains all informations, it is often more convinient to describe the interaction in frequency domain (convolution vs multiplication, calculate wakes in frequency domain can be easier some times, solving beam instability problems...), i.e. the impedances

$$Z_{//}(\omega)=\frac{1}{c}\int_0^{\infty} w_{//}(s)e^{i\omega s/c}\,ds \qquad \text{[s*V/C]=[Ohm]}$$

$$Z_{\perp}(\omega)=-\frac{i}{c}\int_0^{\infty} w_{\perp}(s)e^{i\omega s/c}\,ds \qquad \text{[s*V/(C*m)]=[Ohm/m]}$$

*The frequency ω is frequently allowed to have an imaginary part, in that case the transformation is actually Laplace transform, which is only defined for $\mathrm{Im}(\omega)\geq 0$

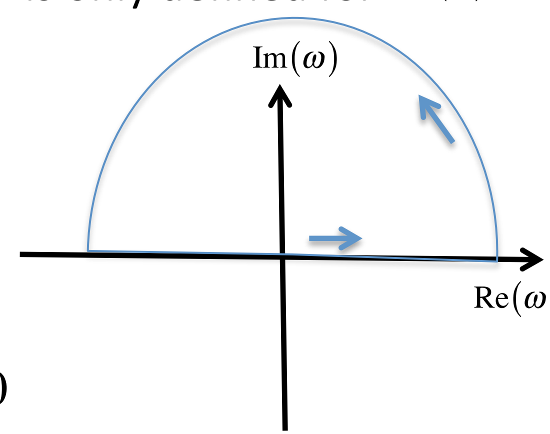
- The inverse transformations are

$$w_{//}(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty} Z_{//}(\omega)e^{-i\omega s/c}\,d\omega$$

$$w_{\perp}(s)=\frac{i}{2\pi}\int_{-\infty}^{\infty} Z_{\perp}(\omega)e^{-i\omega s/c}\,d\omega$$

* In complex ω plane, $Z_{//}(\omega)$ and $Z_{\perp}(\omega)$ should not have singularities in the upper half plane, i.e. $\mathrm{Im}(\omega)\geq 0$, in order to satisfy the causality condition:

$$w_{//}(s<0)=0 \qquad w_{\perp}(s<0)=0$$



Properties of Impedances

- Symmetry properties about positive and negative frequency

$$Z_{//}*(\omega)=\frac{1}{c}\int_0^\infty w_{//}(s)e^{-i\omega s/c}\,ds=Z_{//}(-\omega)\quad\Rightarrow\begin{cases}\mathrm{Re}\left[Z_{//}(\omega)\right]=\mathrm{Re}\left[Z_{//}(-\omega)\right]\\ \mathrm{Im}\left[Z_{//}(\omega)\right]=-\mathrm{Im}\left[Z_{//}(-\omega)\right]\end{cases}$$

$$Z_\perp*(\omega)=-Z_\perp(-\omega)\quad\Rightarrow\begin{cases}\mathrm{Re}\left[Z_\perp(\omega)\right]=-\mathrm{Re}\left[Z_\perp(-\omega)\right]\\ \mathrm{Im}\left[Z_\perp(\omega)\right]=\mathrm{Im}\left[Z_\perp(-\omega)\right]\end{cases}$$

- Relations between real part and imaginary part of impedances

$$w_{//}(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Z_{//}(\omega)e^{-i\omega s/c}d\omega\quad\Rightarrow\quad w_{//}(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\mathrm{Re}\left[Z_{//}(\omega)\right]\cos\left(\frac{\omega s}{c}\right)-\mathrm{Im}\left[Z_{//}(\omega)\right]\sin\left(\frac{\omega s}{c}\right)\right\}d\omega$$

$$w_l(s<0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\mathrm{Re}\left[Z_l(\omega)\right]\cos\left(\frac{\omega s}{c}\right)+\mathrm{Im}\left[Z_l(\omega)\right]\sin\left(\frac{\omega|s|}{c}\right)\right\}d\omega=0\quad\Rightarrow\quad\int_{-\infty}^{\infty}\mathrm{Im}\left[Z_l(\omega)\right]\sin\left(\frac{\omega|s|}{c}\right)d\omega=-\int_{-\infty}^{\infty}\mathrm{Re}\left[Z_l(\omega)\right]\cos\left(\frac{\omega|s|}{c}\right)d\omega$$

$$\Rightarrow\qquad w_l(s>0)=\frac{2}{\pi}\int_0^\infty\mathrm{Re}\left[Z_l(\omega)\right]\cos\left(\frac{\omega s}{c}\right)d\omega\qquad\qquad w_\perp(s>0)=\frac{2}{\pi}\int_0^\infty\mathrm{Re}\left[Z_\perp(\omega)\right]\sin\left(\frac{\omega s}{c}\right)d\omega$$

Kramers-Kronig relations (homework):

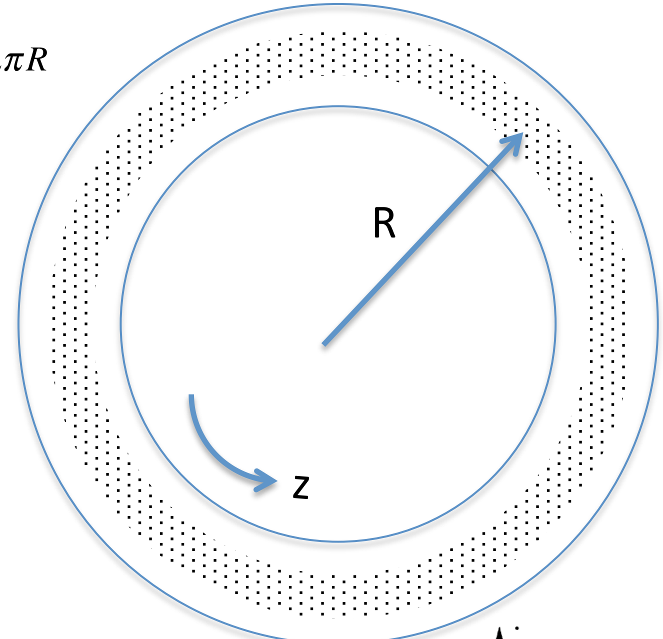
$$Z_{//}(\omega)=-\frac{i}{\pi}P.V.\int_{-\infty}^{\infty}\frac{Z_{//}(\omega')}{\omega'-\omega}d\omega'\quad\Rightarrow\quad\mathrm{Re}\left[Z_{//}(\omega)\right]=\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{\mathrm{Im}\left[Z_{//}(\omega')\right]}{\omega'-\omega}d\omega'\qquad\mathrm{Im}\left[Z_{//}(\omega)\right]=-\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{\mathrm{Re}\left[Z_{//}(\omega')\right]}{\omega'-\omega}d\omega'$$

Longitudinal Microwave Instability

Unperturbed phase space density:

$$\psi_0(z,\Delta E)=\psi_0(\Delta E)=\frac{N}{C_0}f_0(\Delta E) \qquad \rho_0(z)=\rho_0=\frac{N}{C_0}$$

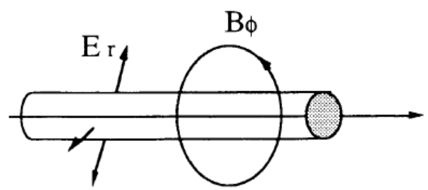
$C_0 = 2\pi R$



DC current does not excite wake

$$V_{//}(z_0)=\int_{z_0}^{\infty}\lambda(z_1)w_{//}(z_1-z_0)dz_1$$

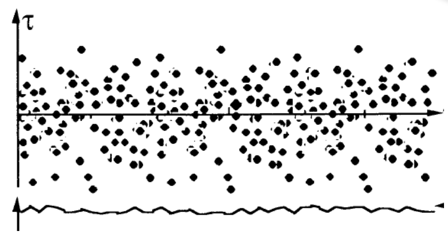
$$=\rho_0\int_{z_0}^{\infty}W_0^{'}(z_1-z_0)dz_1=-\rho_0 W_0(0)=0$$

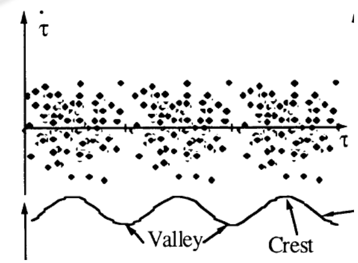


Consider perturbation in phase space density: n-th azimuthal mode

$$\psi_1(s,\Delta E,0)=\hat{\psi}_1(\Delta E)e^{inz/R}$$

Ansatz: $\psi_1(s,\Delta E,t)=\hat{\psi}_1(\Delta E)e^{inz/R-i\Omega t}$





*Note that if a perturbation is static,

$$\psi_1{*}(s,\Delta E,t)=\hat{\psi}_1{*}(\Delta E)e^{in(z-v_0t)/R}=\hat{\psi}_1{*}(\Delta E)e^{inz/R-i\Omega^* t}\Rightarrow\Omega^*=nv_0/R=n2\pi v_0/C=n\omega_0$$

But the system is not likely to be static and we need to solve Vlasov equation self-consistently to know the answer for Ω and hence $\psi_1(s,\Delta E,t)$

$$\frac{\partial}{\partial t}\psi_1(z,\Delta E,t)+\frac{dz}{dt}\cdot\frac{\partial}{\partial z}\psi_1(z,\Delta E,t)+\frac{d\Delta E}{dt}\cdot\frac{\partial}{\partial \Delta E}\psi_0(\Delta E)=0 \qquad \frac{dz}{dt}=v(\Delta E)$$

Longitudinal Microwave Instability

$$c\Delta p_z(z,t) = -eQ_e V_{//}(z,t) = -e^2 v_0 \int_{-\infty}^{t} \rho_1(z,t_1) w_{//}(t-t_1)\,dt_1 = -e^2 v_0 \int_{0}^{\infty} \rho_1(z,t-\tau) w_{//}(\tau)\,d\tau$$

$\rho_1 v_0 dt$ gives particle number in the slice (t,t+dt). $T_0 = \frac{C_0}{v_0}$ is revolution period

$$\frac{d\Delta E(z,t)}{dt} = -\frac{c\Delta p_z(z,t)}{T_0} = -\frac{e^2 v}{T_0}\int_0^{\infty} \rho_1(z,t-\tau) w_{//}(\tau)\,d\tau$$

$$w_{//}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_{//}(\omega) e^{-i\omega\tau}\,d\omega \qquad \rho_1(z,t) = \int_{-\infty}^{\infty} \psi_1(z,\Delta E,t)\,d\Delta E = \hat{\rho}_1 e^{inz/R - i\Omega t} \qquad \hat{\rho}_1 \equiv \int_{-\infty}^{\infty} \hat{\psi}_1(\Delta E)\,d\Delta E$$

$$\frac{d\Delta E(z,t)}{dt} = -\hat{\rho}_1 \frac{e^2 v_0}{2\pi T_0} e^{inz/R - i\Omega t} \int_{-\infty}^{\infty} d\omega Z_{//}(\omega) \int_{-\infty}^{\infty} e^{i(\Omega-\omega)\tau}\,d\tau = -\hat{\rho}_1 \frac{e^2 v_0}{T_0} e^{inz/R - i\Omega t} Z_{//}(\Omega)$$

$$-i\Omega\psi_1(z,\Delta E,t) + v(\Delta E)\cdot\frac{in}{R}\psi_1(z,\Delta E,t) - \hat{\rho}_1 \frac{e^2 v_0}{T_0} e^{inz/R - i\Omega t} Z_{//}(\Omega)\cdot\frac{\partial}{\partial \Delta E}\psi_0(\Delta E) = 0$$

$$\int_{-\infty}^{\infty} d\Delta E \rightarrow \quad \psi_1(z,\Delta E,t) = \frac{ie^2 v_0 Z_{//}(\Omega)}{T_0}\frac{\hat{\rho}_1 e^{inz/R - i\Omega t}}{\Omega - \omega(\Delta E)n}\frac{d\psi_0(\Delta E)}{d\Delta E} \qquad \omega(\Delta E) = \frac{v(\Delta E)}{R}$$

Longitudinal Microwave Instability

Dispersion relation: $$\boxed{1 = \frac{ieI_0 Z_{//}(\Omega)}{T_0} \int_{-\infty}^{\infty} \frac{f_0'(\Delta E)}{\Omega - \omega(\Delta E) n} d\Delta E}$$

$I_0 = eN/T_0$

$$\psi_0(\Delta E) = \frac{N}{C_0} f_0(\Delta E)$$

$$\omega(\Delta E) = \omega_0 + \Delta\omega(\Delta E) = \omega_0 - \eta\omega_0 \frac{\Delta p_z}{p_{0,z}} = \omega_0 - \frac{\eta\omega_0}{\beta^2}\frac{\Delta E}{E_0}$$

Phase slip factor: $\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$

Cold Beam: $f_0(\Delta E) = \delta(\Delta E)$

$$1 = \frac{ieI_0 Z_{//}(\Omega)}{T_0} \frac{\eta n \omega_0}{E_0 \beta^2} \int_{-\infty}^{\infty} \frac{f_0(\Delta E)}{\left(\Omega - n\omega_0 + \frac{\eta n \omega_0}{E_0\beta^2}\Delta E\right)^2} d\Delta E$$

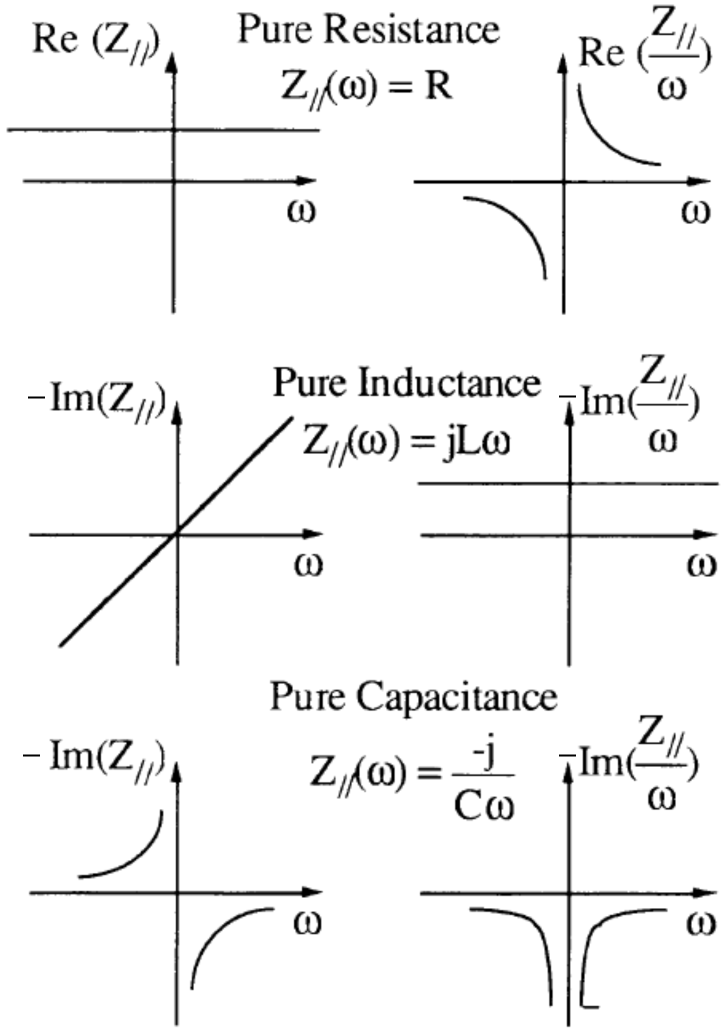
$$\Rightarrow \quad \Omega = n\omega_0 \pm \omega_0 \sqrt{\frac{ieI_0 \eta n Z_{//}(\Omega)}{2\pi E_0 \beta^2}} \approx n\omega_0 \pm \omega_0 \sqrt{\frac{ieI_0 \eta n Z_{//}(n\omega_0)}{2\pi E_0 \beta^2}}$$

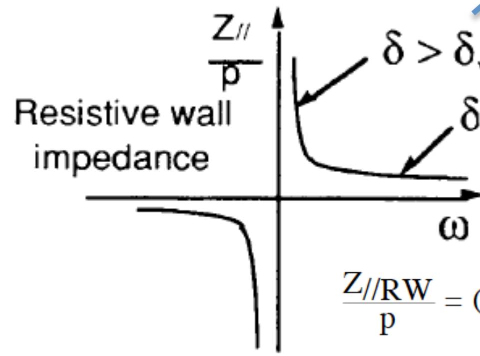
Perturbative appraoch assuming $\frac{|\Omega - n\omega_0|}{n\omega_0} << 1$

Typical Longitudinal Impedance

$$j = -i$$

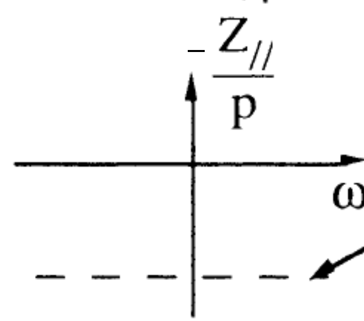
Taken from ‘Coasting beam longitudinal coherent instabilities’ by J.L. Laclare





$$\frac{Z_{//RW}}{p} = \frac{R}{\sigma b \delta_W}\frac{1}{p}$$

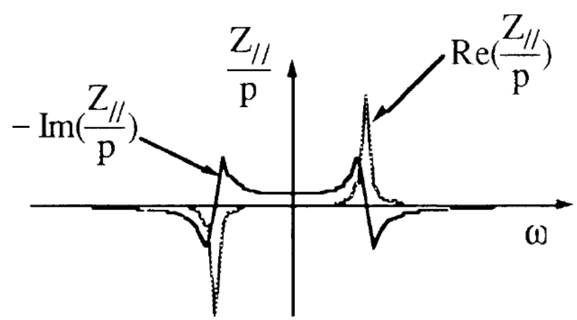
$$\frac{Z_{//RW}}{p} = (1+j)\frac{R}{\sigma b \delta}\frac{\omega_0}{\omega} = (1+j)\frac{Z_0 \beta_0 \delta_0}{2b}\frac{1}{\sqrt{p}}$$



Space charge negative inductance

$$\frac{Z_{//SC}(\omega)}{p} = -j\frac{Z_0\, g}{2\beta_0\gamma_0^2} \text{ where p stands for } \frac{\omega}{\omega_0}$$

Resonator model (cavities)

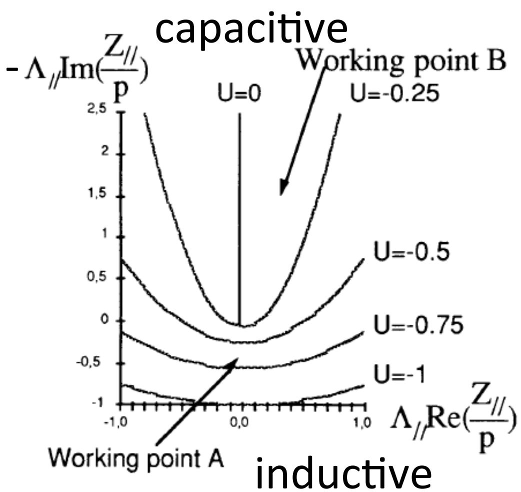


$$Z_{//} = \frac{R_S}{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}$$

Longitudinal Microwave Instabilities

Cold beam continued:

(assuming $\eta > 0$) $\Omega \approx n\omega_0 \pm \omega_0 \sqrt{\dfrac{ieI_0 \eta n Z_{//}(n\omega_0)}{2\pi E_0 \beta^2}}$



Taken from 'Accelerator Physics' by S.Y. Lee

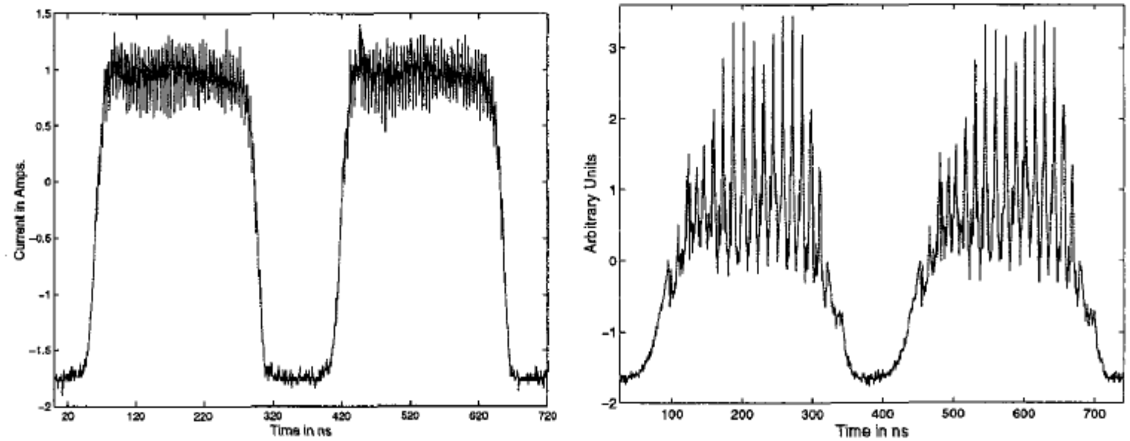


Figure 3.36: The longitudinal beam profiles observed at PSR the bunched coasting beam in the presence of inductive inserts, where three 1-m long ferrite ring cavities were installed in the PSR ring. [Courtesy of R. Macek, LANL]

Taken from S.Y. Lee

Warm Beam:

$$1 = \frac{ieI_0 Z_{//}(n\omega_0)}{T_0} \frac{1}{\sqrt{2\pi}\sigma_E^2} \int_{-\infty}^{\infty} \frac{-\dfrac{\Delta E}{\sigma_E} \exp\left(-\dfrac{\Delta E^2}{2\sigma_E^2}\right)}{\Omega - \omega(\Delta E) n} d\Delta E$$

$$= i\frac{1}{2}\left\{\frac{eI_0\left[Z_{//}(n\omega_0)/n\right]E_0\beta^2}{2\pi\eta\sigma_E^2}\right\} J_G(\tilde{\Omega})$$

$$= i\frac{2\ln(2)}{\pi}\left\{\frac{eI_0\left[Z_{//}(n\omega_0)/n\right]E_0\beta^2}{\eta\sigma_{E,FWHM}^2}\right\} J_G(\tilde{\Omega})$$

$$f_0(\Delta E) = \frac{1}{\sqrt{2\pi}\sigma_E} \exp\left(-\frac{\Delta E^2}{2\sigma_E^2}\right)$$

$$U' - iV' \equiv \frac{eI_0\left[Z_{//}(n\omega_0)/n\right]E_0\beta^2}{\eta\sigma_{E,FWHM}^2}$$

$$U' \sim \mathrm{Re}\left(Z_{//}(n\omega_0)\right) \qquad V' \sim -\mathrm{Im}\left(Z_{//}(n\omega_0)\right)$$

$$\boxed{U' - iV' = \frac{-i\pi}{2\ln(2) J_G\left(\mathrm{Re}\,\tilde{\Omega} + i\,\mathrm{Im}\,\tilde{\Omega}\right)}}$$

$$J_G(\tilde{\Omega}) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{-x\exp\left(-\dfrac{x^2}{2}\right)}{\left[\tilde{\Omega}\dfrac{E_0\beta^2}{\eta n\omega_0\sigma_E} + x\right]} dx$$

$$\tilde{\Omega} = \mathrm{Re}(\tilde{\Omega}) + \mathrm{Im}(\tilde{\Omega}) \equiv \Omega - \omega_0 n$$

Longitudinal Microwave instability

Gaussian with various growth rate, $\mathrm{Im}(\tilde{\Omega})$

Contours with $\mathrm{Im}(\tilde{\Omega})=0$ for various energy distribution

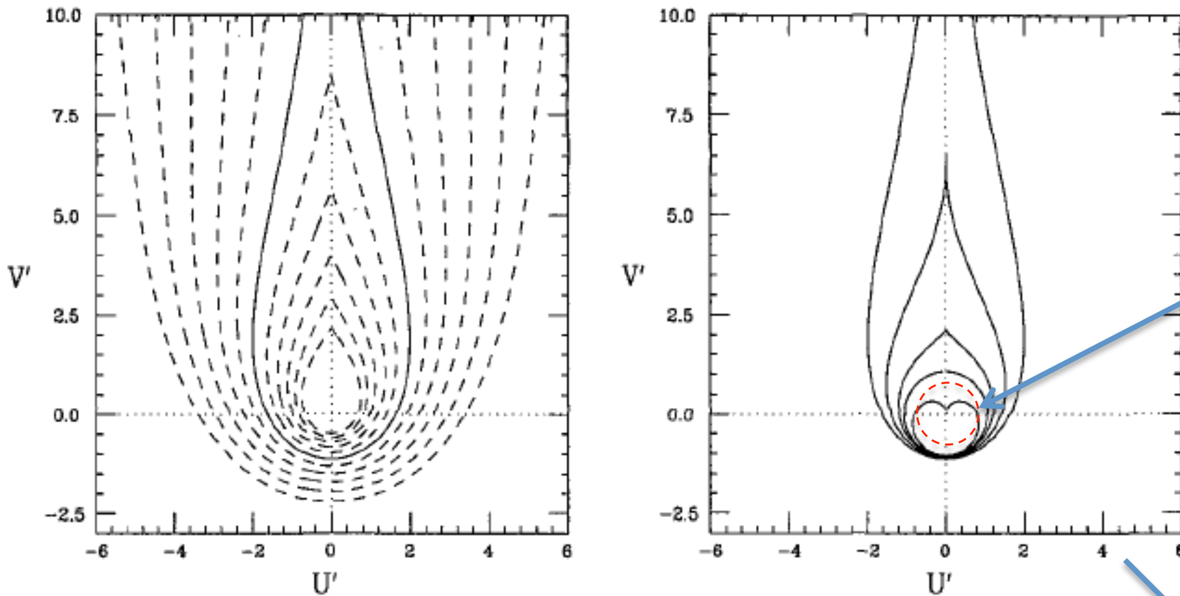


Figure 3.34: Left: The solid line shows the parameters V' vs U' for a Gaussian beam distribution at a zero growth rate. Dashed lines inside the threshold curve are stable. They correspond to $-\mathrm{Im}\,\Omega/(\sqrt{2\ln 2}\,\omega_0\eta\sigma_\delta) = -0.1, -0.2, -0.3, -0.4$, and -0.5. Dashed lines outside the threshold curve have growth rates $-\mathrm{Im}\,\Omega/(\sqrt{2\ln 2}\,\omega_0\eta\sigma_\delta) = 0.1, 0.2, 0.3, 0.4$, and 0.5 respectively. Right: The threshold V' vs U' parameters for various beam distributions.

Simplified estimation for stability condition:

Keil-Schnell criterion

$$\left|Z_{//}(n\omega_0)/n\right| \le \frac{2\pi|\eta|\sigma_E^2}{E_0\beta^2 e I_0}F$$

F depends on distributiion and for Gaussian energy distribution, it is 1.

from inside outward, for the normalized distribution functions $\Psi_0(x) = 3(1-x^2)/4$, $8(1-x^2)^{3/2}/3\pi$, $15(1-x^2)^2/16$, $315(1-x^2)^4/32$, and $(1/\sqrt{2\pi})\exp(-x^2/2)$. All dis-