PHY 564
Advanced Accelerator Physics
Lecture 23

Free Electron Lasers II: FELs in High Gain Regime

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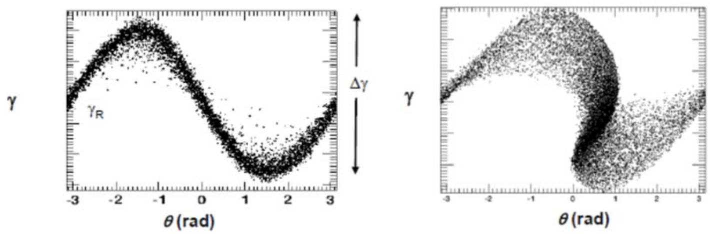
Outline

- Introduction
  - Concept of the FEL instability
- 1-D FEL theory in high gain regime
  - Linearized Vlasov equation for electrons in a FEL
  - Wave equation for radiation field
  - Combined Vlasov-Maxwell equation system
    - Integra-differential equation
    - Solutions for cold electron beam
    - Solutions for warm electron beam

High Gain Regime: Concept

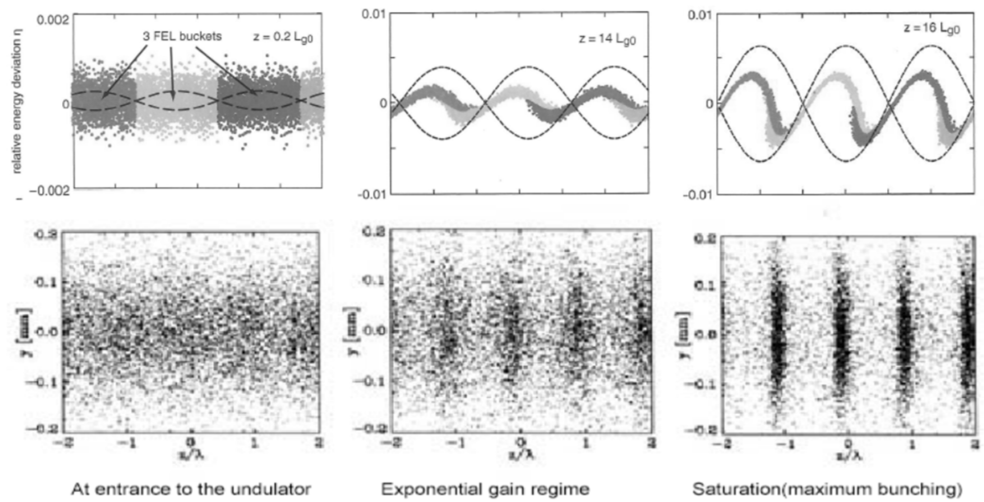
1. Energy kick from radiation field + dispersion/drift -> electron density bunching;

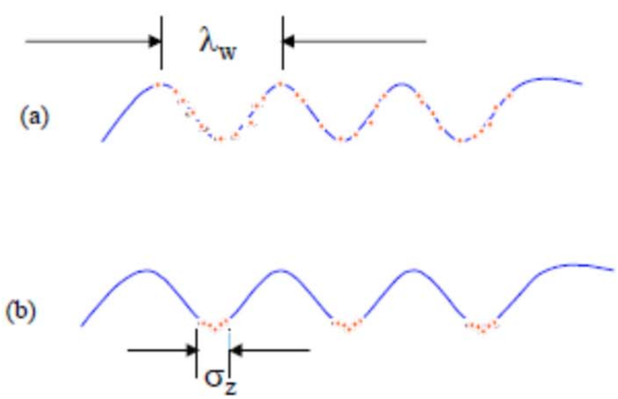


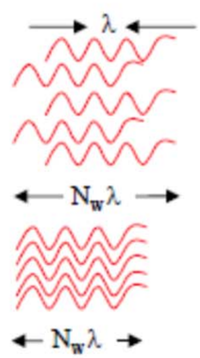
*The plots are for illustration only. The right plot actually shows somewhere close to saturation.

2. Electron density bunching makes more electrons radiates coherently -> higher radiation field;

3. Higher radiation fields leads to more density bunching through 1 and hence closes the positive feedback loop -> FEL instability.







$$|E| \propto \sqrt{N_e}$$

$$I_{incoherent} \propto N_e$$

$$|E| \propto N_e$$

$$I_{coherent} \propto N_e^2$$

The positive feedback loop between radiation field and electron density bunching is the underlying mechanism of high gain FEL regime.

High Gain Regime: 1-D FEL Theory

- Ignoring the space charge effects, the Hamiltonian for electrons in a FEL can be written as (see additional material):

$$H(\psi, P, z) = CP + \frac{\omega}{2c\gamma_z^2 E_0} P^2 - \left(U(z)e^{i\psi} + U^*(z)e^{-i\psi}\right)$$

$$U = -\frac{e\theta_s \tilde{E}(z)}{2i} \qquad E_x + iE_y = \tilde{E}(z)\exp[i\omega(z/c - t)]$$

Slow varying phase

$$\Rightarrow \begin{cases} \frac{dP}{dz} = -\frac{\partial H}{\partial \psi} = 2\frac{\partial}{\partial \psi}\mathrm{Re}\left[Ue^{i\psi}\right] = -\mathrm{Re}\left[e\theta_s \tilde{E}(z)e^{i\psi}\right] = -e\theta_s \left|\tilde{E}(z)\right| \cos(\psi + \varphi(z)) \\ \frac{d\psi}{dz} = \frac{\partial H}{\partial P} = C + \frac{\omega}{c\gamma_z^2 \mathcal{E}_0} P \end{cases}$$

Linearization of Vlasov Equation

Vlasov equation: $$\frac{\partial f}{\partial z}+\frac{\partial H}{\partial P}\frac{\partial f}{\partial \psi}-\frac{\partial H}{\partial \psi}\frac{\partial f}{\partial P}=0$$

$$f(\psi,P,z)=f_0(P)+\tilde{f}_1(P,z)e^{i\psi}+\tilde{f}_1^*(P,z)e^{-i\psi} \qquad \psi=k_u z+k(z-ct)$$

Linearized Vlasov equation: $$\frac{\partial \tilde{f}_1}{\partial z}+i\left[C+\frac{\omega}{c\gamma_z^2\mathcal{E}_0}P\right]\tilde{f}_1+iU\frac{\partial f_0}{\partial P}=0$$

$$\frac{\partial}{\partial z}\left\{\tilde{f}_1\exp\left[i\left(C+\frac{\omega}{c\gamma_z^2\mathcal{E}_0}P\right)z\right]\right\}+iU\exp\left[i\left(C+\frac{\omega}{c\gamma_z^2\mathcal{E}_0}P\right)z\right]\frac{\partial f_0}{\partial P}=0$$

Assuming that there is no initial modulation in the electrons, i.e. $\tilde{f}_1(0)=0$

$$\tilde{f}_1(P,z)=-in_0\frac{\partial F_0(P)}{\partial P}\int_0^z dz_1 U\exp\left[i\left(C+\frac{\omega}{c\gamma_z^2\mathcal{E}_0}P\right)(z_1-z)\right]dz_1 \qquad f_0(P)=n_0F(P)$$

Integrate over energy deviation: $-ec\int_{-\infty}^{\infty}\tilde{f}_1(P,z)dP=\tilde{j}_1(z) \qquad j_z=-j_0+j_{z,1}=-j_0+\tilde{j}_1e^{i\psi}+\tilde{j}_1^*e^{-i\psi}$

$$j_0=en_0c$$

$$\tilde{j}_1(z)=ij_0\int_0^z dz_1U(z_1)\int_{-\infty}^{\infty}\frac{\partial F_0(P)}{\partial P}\exp\left[i\left(C+\frac{\omega}{c\gamma_z^2\mathcal{E}_0}P\right)(z_1-z)\right]dP$$

Wave Equation

$$\psi = k_w z + k(z - ct)$$

1-D theory and hence $\partial/\partial x = 0$ and $\partial/\partial y = 0$

Wave equation for transverse vector potential: $$\frac{\partial^2 \vec{A}_\perp}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \vec{A}_\perp}{\partial t^2} = -\mu_0 \vec{j}_\perp \qquad (1)$$

Transverse current perturbation: $$j_x + i j_y = \frac{1}{v_z}\left(v_x + i v_y\right) j_{z,1} = \theta_s e^{-ik_w z}\left(\tilde{j}_1 e^{i\psi} + + \tilde{j}_1^{\,*} e^{-i\psi}\right) \qquad (2)$$

We seek the solution for vector potential of the form:

$$A_{x,y}(z,t) = \tilde{A}_{x,y}(z) e^{i\omega(z/c - t)} + \tilde{A}_{x,y}^{\,*}(z) e^{-i\omega(z/c-t)} \qquad (3)$$

Inserting eq. (2) and (3) into eq. (1) yields

$$e^{i\omega(z/c-t)}\left\{\frac{2i\omega}{c}\frac{\partial}{\partial z}\begin{pmatrix}\tilde{A}_x\\ \tilde{A}_y\end{pmatrix} + \frac{\partial^2}{\partial z^2}\begin{pmatrix}\tilde{A}_x\\ \tilde{A}_y\end{pmatrix}\right\} + C.C. = -\mu_0 \theta_s \begin{pmatrix}\cos(k_w z)\\ -\sin(k_w z)\end{pmatrix}\left(\tilde{j}_1 e^{i\psi} + C.C.\right)$$

$$\left\{\frac{2i\omega}{c}\frac{\partial}{\partial z}\begin{pmatrix}\tilde{A}_x\\ \tilde{A}_y\end{pmatrix} + \frac{\partial^2}{\partial z^2}\begin{pmatrix}\tilde{A}_x\\ \tilde{A}_y\end{pmatrix}\right\} = -\frac{\mu_0\theta_s}{2}\begin{pmatrix}e^{ik_w z} + e^{-ik_w z}\\ ie^{ik_w z} - ie^{-ik_w z}\end{pmatrix}\tilde{j}_1 e^{ik_w z}$$

1. Ignoring fast oscillating term $\sim e^{2ik_w z}$

2. Ignoring second derivative by assuming that the variation of $\tilde{A}_x'$ is negligible over the optical wave length.

Wave Equation

After neglecting the fast oscillation terms, we get the following relation between the current perturbation and the vector potential of the radiation field:

$$\frac{\partial}{\partial z}\tilde{A}_x = -\frac{c\mu_0\theta_s}{4i\omega}\tilde{j}_1 \qquad \frac{\partial}{\partial z}\tilde{A}_y = \frac{\mu_0 c\theta_s}{4\omega}\tilde{j}_1$$

In order to relate the vector potential to the electric field, we use the Maxwell equation:

$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \nabla\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0 \Rightarrow \left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = \vec{\nabla}\varphi \Rightarrow E_{x,y} = -\frac{\partial A_{x,y}}{\partial t}$$

$$\Rightarrow \tilde{E}e^{i\omega(z/c-t)} = E_x + iE_y = -\frac{\partial}{\partial t}\left[\left(\tilde{A}_x + i\tilde{A}_y\right)e^{i\omega(z/c-t)}\right]$$

$$\Rightarrow \tilde{E} = i\omega\left(\tilde{A}_x + i\tilde{A}_y\right)$$

Finally, the relation between the radiation field and the current modulation is obtained:

$$\frac{d}{dz}\tilde{E} = i\omega\left(\frac{\partial}{\partial z}\tilde{A}_x + i\frac{\partial}{\partial z}\tilde{A}_y\right) = -\frac{c\mu_0\theta_s}{2}\tilde{j}_1$$

Integra-differential Equation

Let's put together what we achieved so far…

$$\tilde{j}_1(z) = ij_0 \int_0^z dz_1 U(z_1) \int_{-\infty}^{\infty} \frac{\partial F_0(P)}{\partial P} \exp\left[i\left(C + \frac{\omega}{c\gamma_z^2 \mathcal{E}_0} P \right)(z_1 - z) \right] dP$$

$$\frac{d}{dz}\widetilde{E}(z) = -\frac{c\mu_0\theta_s}{2}\tilde{j}_1(z) \qquad U \equiv -\frac{e\theta_s \widetilde{E}(z)}{2i}$$

After inserting the latter two equations back into the first equation, we arrive at

$$\frac{d}{d\hat{z}}\widetilde{E}(\hat{z}) = \int_0^{\hat{z}} d\hat{z}_1 \widetilde{E}(\hat{z}_1) \int_{-\infty}^{\infty} \frac{dF_0(\hat{P})}{d\hat{P}} \exp\left[i(\hat{C} + \hat{P})(\hat{z}_1 - \hat{z}) \right] d\hat{P}$$

where the following normalized variables are used to make the equation more compact:

Gain parameter: $\Gamma = \left[\frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma I_A} \right]^{1/3}$ Pierce Parameter: $\rho = \gamma_z^2 \Gamma c / \omega$

$$\hat{C} = C/\Gamma \qquad \hat{z} = z\Gamma \qquad \hat{P} = \frac{\mathcal{E} - \mathcal{E}_0}{\mathcal{E}_0 \rho} \qquad I_A \equiv 4\pi\varepsilon_0 \frac{m_e c^3}{e} \approx 17KA$$

Solution for Cold Beam

After integration by parts: $$\frac{d}{d\hat{z}}\widetilde{E}(\hat{z}) = -i\int_0^{\hat{z}} d\hat{z}_1 \widetilde{E}(\hat{z}_1)(\hat{z}_1 - \hat{z})\int_{-\infty}^{\infty} F_0(\hat{P})\exp\left[i(\hat{C}+\hat{P})(\hat{z}_1 - \hat{z})\right]d\hat{P}$$

For cold beam: $$F_0(\hat{P}) = \delta(\hat{P})$$

$$e^{i\hat{C}\hat{z}}\frac{d}{d\hat{z}}\widetilde{E}(\hat{z}) = -i\int_0^{\hat{z}} \widetilde{E}(\hat{z}_1)(\hat{z}_1 - \hat{z})e^{i\hat{C}\hat{z}_1}d\hat{z}_1$$

Taking derivative: $$\frac{d}{d\hat{z}}\left[e^{i\hat{C}\hat{z}}\frac{d}{d\hat{z}}\widetilde{E}(\hat{z})\right] = i\int_0^{\hat{z}} \widetilde{E}(\hat{z}_1)e^{i\hat{C}\hat{z}_1}d\hat{z}_1$$

Taking another derivative: $$\frac{d^2}{d\hat{z}^2}\left[e^{i\hat{C}\hat{z}}\frac{d}{d\hat{z}}\widetilde{E}(\hat{z})\right] = i\widetilde{E}(\hat{z})e^{i\hat{C}\hat{z}}$$

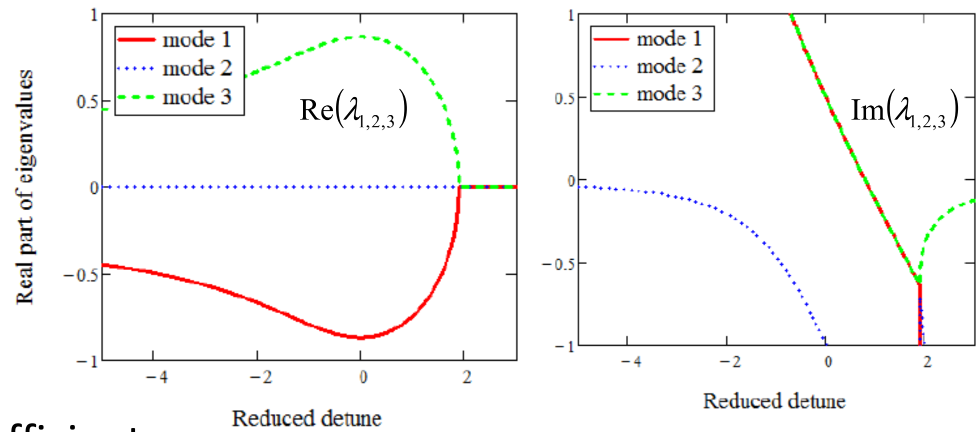
We obtain a third order homogenous ODE: $$\frac{d^3}{d\hat{z}^3}\widetilde{E}(\hat{z}) + 2i\hat{C}\frac{d^2}{d\hat{z}^2}\widetilde{E}(\hat{z}) - \hat{C}^2\frac{d}{d\hat{z}}\widetilde{E}(\hat{z}) = i\widetilde{E}(\hat{z})$$

Solution for Cold Beam

The general solution of the ODE reads:

$$\widetilde{E}(\hat{z}) = \sum_{k=1}^{3} B_k e^{\lambda_k \hat{z}}$$

$$\lambda^3 + 2i\hat{C}\lambda^2 - \hat{C}^2\lambda = i$$



Applying initial condition to get the coefficients

$$\begin{pmatrix} \widetilde{E}(0) \\ \widetilde{E}'(0) \\ \widetilde{E}''(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \Rightarrow \qquad \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{E}(0) \\ \widetilde{E}'(0) \\ \widetilde{E}''(0) \end{pmatrix}$$

For $\widetilde{E}(0) = E_{ext}$ and $\widetilde{E}'(0) = \widetilde{E}''(0) = 0$, the solution can be explicitly written as

$$\widetilde{E}(\hat{z}) = E_{ext}\left[\frac{\lambda_2\lambda_3 e^{\lambda_1\hat{z}}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2\hat{z}}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3\hat{z}}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right]$$

Low Gain Limit of High Gain Solution

Can we reproduce the previously obtained low gain solution by taking the proper limit of the high gain solution?

$$\tau \equiv \frac{2\pi j_0 \theta_s^2 \omega}{c\gamma_z^2\gamma}\frac{l_w^3}{I_A} = 2\Gamma^3 l_w^3$$

$$g_l = \frac{(E_{ext}+\Delta E)^2 - E_{ext}^2}{E_{ext}^2} \approx \frac{2\Delta E}{E_{ext}} = \tau \cdot f\left(\hat{C}_l\right) = 2\Gamma^3 l_w^3 f_l\left(\hat{C}_l\right)$$

$$f_l\left(\hat{C}_l\right) = \frac{2}{\hat{C}_l^{\,3}}\left(1-\cos\hat{C}_l - \frac{\hat{C}_l}{2}\sin\hat{C}_l\right)$$

$$\hat{C}_l = Cl_w$$

$$g_h\left(\hat{C}_l\right) = \frac{\widetilde{E}^2 - E_{ext}^2}{E_{ext}^2} = \left|\frac{\lambda_2\lambda_3 e^{\lambda_1 \hat{l}_w}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2 \hat{l}_w}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3 \hat{l}_w}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right|^2 - 1$$

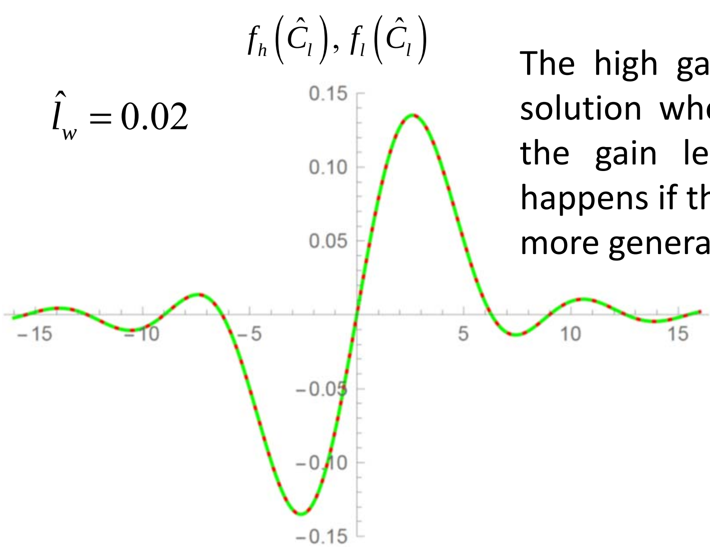
$$= 2\Gamma^3 l_w^3 f_h\left(\hat{C}_l\right) \qquad \hat{l}_w = l_w\Gamma$$

$$f_h\left(\hat{C}_l\right) = \frac{1}{2\hat{l}_w^3}\left\{\left|\frac{\lambda_2\lambda_3 e^{\lambda_1 \hat{l}_w}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2 \hat{l}_w}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3 \hat{l}_w}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right|^2 - 1\right\}$$

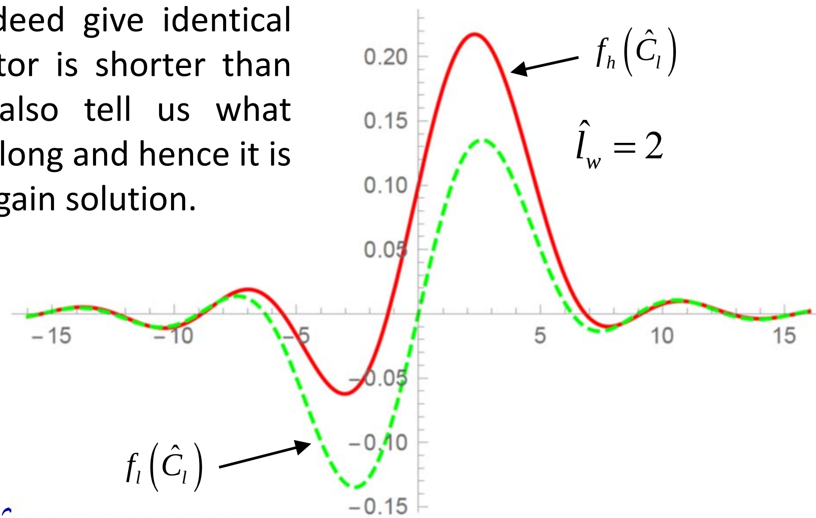
The normalization factor for high gain is different from that of low gain:

$$\hat{C}_h = C/\Gamma = Cl_w/\hat{l}_w = \hat{C}_l/\hat{l}_w$$

$$\lambda^3 + 2i\frac{\hat{C}_l}{\hat{l}_w}\lambda^2 - \left(\frac{\hat{C}_l}{\hat{l}_w}\right)^2\lambda = i$$



The high gain solution indeed give identical solution when the undulator is shorter than the gain length. But it also tell us what happens if the undulator is long and hence it is more general than the low gain solution.



High Gain FEL with Warm Beam

- For warm electron beam with general energy distribution, the method of solving the integro-differential equation directly in the time domain is usually difficult.

$$\frac{d}{d\hat{z}}\widetilde{E}(\hat{z}) = \int_0^{\hat{z}} d\hat{z}_1 \widetilde{E}(\hat{z}_1) \int_{-\infty}^{\infty} \frac{dF_0(\hat{P})}{d\hat{P}} \exp\left[i\left(\hat{C}+\hat{P}\right)(\hat{z}_1 - \hat{z})\right] d\hat{P}$$

- For a general initial value problem, Laplace transformation is frequently proved to be helpful (Remember that we actually used similar technique in solving the longitudinal microwave instability problem.). In the following slides, we will try to apply the Laplace transformation technique to solve above equation.

Laplace Transformation

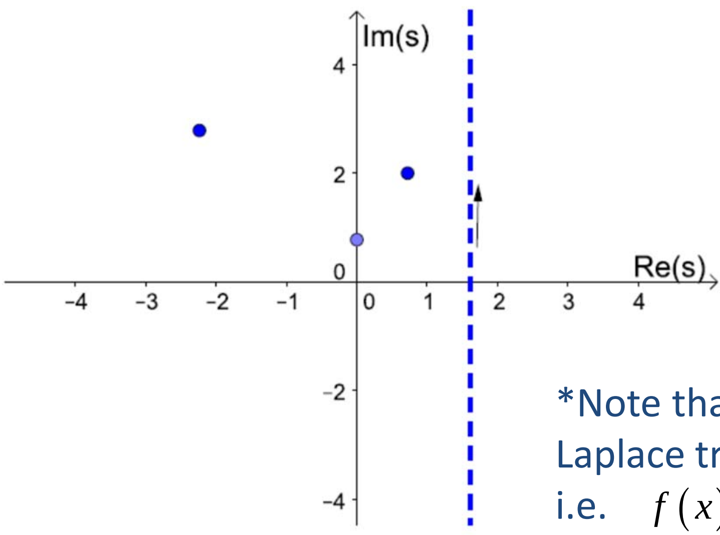
The Laplace transform of the function f(x), denoted by F(s), is defined by the integral

$$F(s) = \int_0^\infty e^{-sx} f(x)\,dx \qquad \text{for} \quad \mathrm{Re}(s) > 0$$

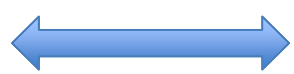
The inversion of the Laplace transform is accomplished for analytic function F(s) by means of the inversion integral*

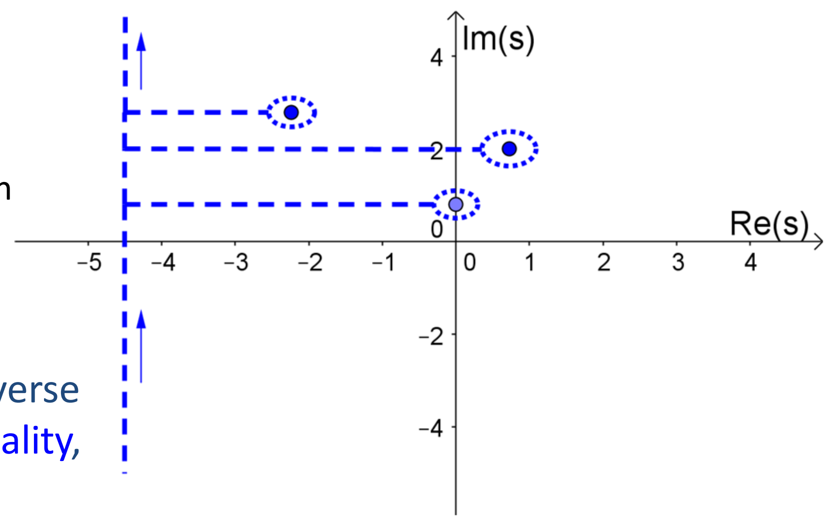
$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{sx} F(s)\,ds \qquad \text{for} \quad \mathrm{Re}(s) > 0$$

where γ is a real constant that exceeds the real part of all the singularities of F(s).



Analytic Continuation





*Note that the definition of inverse Laplace transform implies causality, i.e. $f(x) = 0 \quad \text{for} \quad x < 0$

Solution of the Initial Value Problem by Laplace Transform

Let’s get back to the integro-differential equation:

$$\frac{d}{d\hat{z}}\tilde{E}(\hat{z}) = \int_0^{\hat{z}} d\hat{z}_1 \tilde{E}(\hat{z}_1) \int_{-\infty}^{\infty} \frac{dF_0(\hat{P})}{d\hat{P}} \exp\left[i(\hat{C}+\hat{P})(\hat{z}_1-\hat{z})\right] d\hat{P} \qquad (1)$$

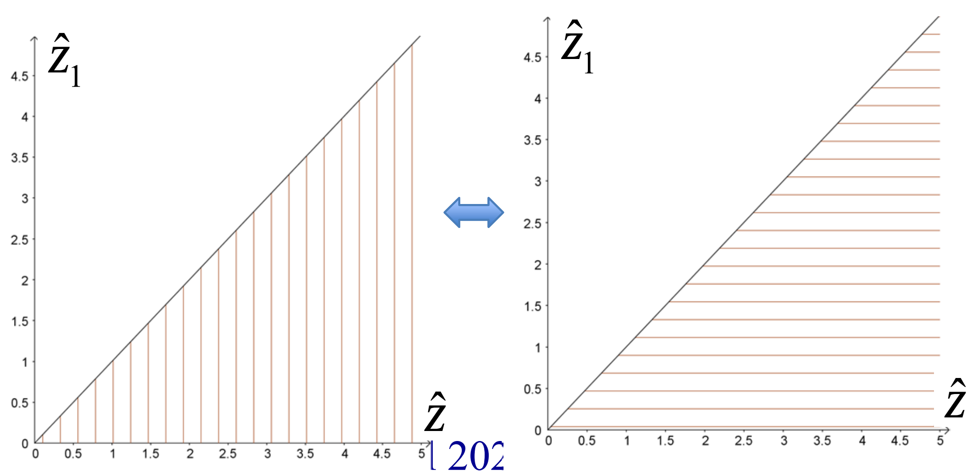
Multiplying both sides by $\exp(-\lambda\hat{z})$ and integrate over $\hat{z}$ from 0 to ∞ lead to

$$\int_0^{\infty} \exp(-\lambda\hat{z}) \frac{d}{d\hat{z}}\tilde{E}(\hat{z}) d\hat{z} = \exp(-\lambda\hat{z})\tilde{E}(\hat{z})\Big|_{\hat{z}=0}^{\hat{z}=\infty} + \lambda\int_0^{\infty}\tilde{E}(\hat{z})\exp(-\lambda\hat{z}) d\hat{z} = \lambda\tilde{E}(\lambda) - \tilde{E}_{ext} \qquad (2)$$

$$\tilde{E}_{ext} \equiv \tilde{E}(\hat{z}=0)$$

$$\int_0^{\infty} \exp(-\lambda\hat{z}) \int_0^{\hat{z}} d\hat{z}_1 \tilde{E}(\hat{z}_1) \exp\left[i(\hat{C}+\hat{P})(\hat{z}_1-\hat{z})\right] d\hat{z} = \int_0^{\infty} d\hat{z} \int_0^{\hat{z}} d\hat{z}_1 \tilde{E}(\hat{z}_1) \exp\left[i(\hat{C}+\hat{P})\hat{z}_1\right] \exp\left[-(i\hat{C}+i\hat{P}+\lambda)\hat{z}\right]$$

$$= \int_0^{\infty} d\hat{z}_1 \tilde{E}(\hat{z}_1) \exp\left[i(\hat{C}+\hat{P})\hat{z}_1\right] \int_{\hat{z}_1}^{\infty} \exp\left[-(i\hat{C}+i\hat{P}+\lambda)\hat{z}\right] d\hat{z}$$



Solution in Laplace Domain

$$\int_0^\infty \exp(-\lambda\hat{z})\int_0^{\hat{z}} d\hat{z}_1\widetilde{E}(\hat{z}_1)\exp\left[i\left(\hat{C}+\hat{P}\right)(\hat{z}_1-\hat{z})\right]d\hat{z} = \int_0^\infty d\hat{z}_1\widetilde{E}(\hat{z}_1)\exp\left[i\left(\hat{C}+\hat{P}\right)\hat{z}_1\right]\int_{\hat{z}_1}^\infty \exp\left[-\left(i\hat{C}+i\hat{P}+\lambda\right)\hat{z}\right]d\hat{z}$$

$$= \int_0^\infty d\hat{z}_1 \frac{\widetilde{E}(\hat{z}_1)\exp\left[i\left(\hat{C}+\hat{P}\right)\hat{z}_1\right]}{-\left(i\hat{C}+i\hat{P}+\lambda\right)}\left[0-\exp\left[-\left(i\hat{C}+i\hat{P}+\lambda\right)\hat{z}_1\right]\right]$$

$$= \frac{1}{\lambda+i\left(\hat{C}+\hat{P}\right)}\int_0^\infty \widetilde{E}(\hat{z}_1)\exp(-\lambda\hat{z}_1)d\hat{z}_1$$

$$= \frac{\widetilde{E}(\lambda)}{\lambda+i\left(\hat{C}+\hat{P}\right)}$$

Eq. (3)

Inserting eq. (2) and eq. (3) back into eq. (1) yields

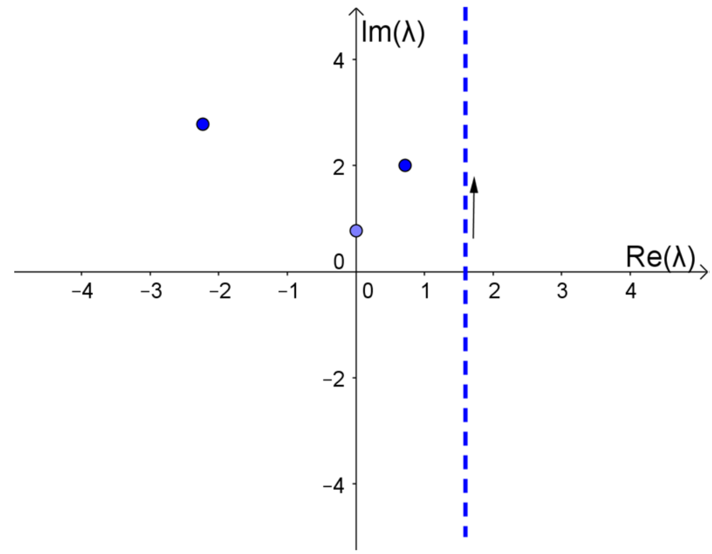
$$\lambda\widetilde{E}(\lambda)-\widetilde{E}_{ext} = \widetilde{E}(\lambda)\int_{-\infty}^{\infty}\frac{F_0'\left(\hat{P}\right)}{\lambda+i\left(\hat{C}+\hat{P}\right)}d\hat{P} \qquad F_0'\left(\hat{P}\right)\equiv\frac{d}{d\hat{P}}F_0\left(\hat{P}\right)$$

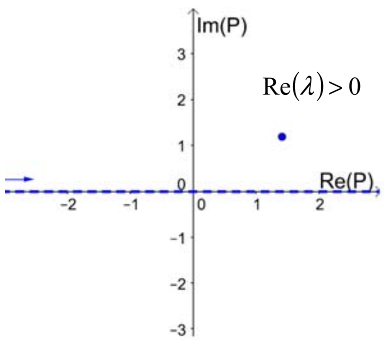
$$\boxed{\widetilde{E}(\lambda)=\frac{\widetilde{E}_{ext}}{\lambda-\hat{D}(\lambda)}} \qquad \hat{D}(\lambda)\equiv\int_{-\infty}^{\infty}\frac{F_0'\left(\hat{P}\right)}{\lambda+i\left(\hat{C}+\hat{P}\right)}d\hat{P}$$

* Notice that $\hat{D}(\lambda)$ is only defined for $\mathrm{Re}(\lambda)>0$.

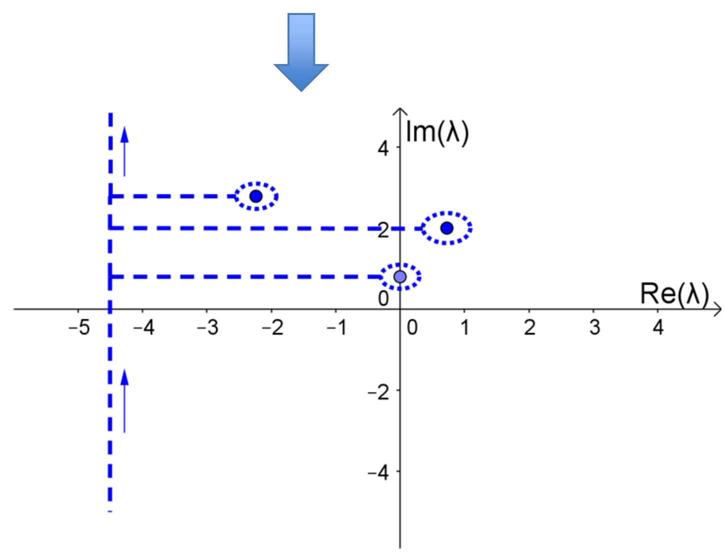
Define $\hat{D}(\lambda)$ for $\mathrm{Re}(\lambda) \le 0$ by Analytic Continuition

Inverse Laplace transform: $$\widetilde{E}(\hat{z}) = \frac{\widetilde{E}_{ext}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda \hat{z}}}{\lambda - \hat{D}(\lambda)} d\lambda$$

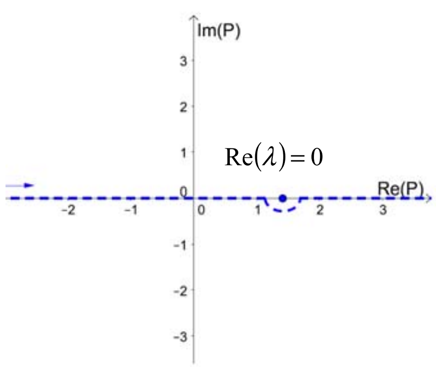


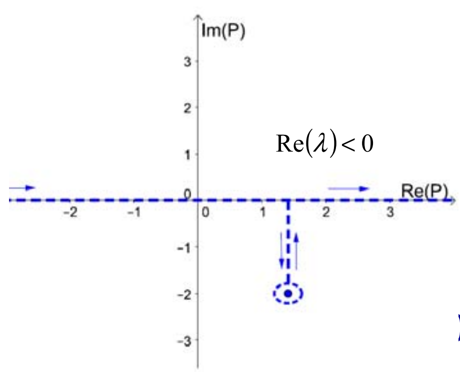


$$\hat{D}(\lambda) \equiv \int_{-\infty}^{\infty} \frac{F_0'(\hat{P})}{\lambda + i(\hat{C} + \hat{P})} d\hat{P}$$
$$= -i \int_{-\infty}^{\infty} \frac{F_0'(\hat{P})}{\hat{P} - (i\lambda - \hat{C})} d\hat{P}$$



In order to use the residue theorem, we need to define the integrand of above integration for $\mathrm{Re}(\lambda) \le 0$ through analytic continuation:





Solution in z (time) Domain

After analytic continuation, the definition of $\hat{D}(\lambda)$ in the whole complex λ plane reads:

$$\hat{D}(\lambda)=\begin{cases}\int_{-\infty}^{\infty}\frac{F'(\hat{P})}{\lambda+i(\hat{P}+\hat{C})}d\hat{P} & \text{for } \mathrm{Re}(\lambda)>0\\ P.V.\int_{-\infty}^{\infty}\frac{F'(\hat{P})}{\lambda+i(\hat{P}+\hat{C})}d\hat{P}+\pi F'(i\lambda-\hat{C}) & \text{for } \mathrm{Re}(\lambda)=0\\ \int_{-\infty}^{\infty}\frac{F'(\hat{P})}{\lambda+i(\hat{P}+\hat{C})}d\hat{P}+2\pi F'(i\lambda-\hat{C}) & \text{for } \mathrm{Re}(\lambda)<0\end{cases}$$

Using Cauchy's residue theorem, the radiation field in the z (time) domain is given by

L'Hospital's Rule

$$\widetilde{E}(\hat{z})=\frac{\widetilde{E}_{ext}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda\hat{z}}}{\lambda-\hat{D}(\lambda)}d\lambda=\widetilde{E}_{ext}\sum_j \exp(\lambda_j\hat{z})\lim_{\lambda\to\lambda_j}\frac{(\lambda-\lambda_j)}{(\lambda-\hat{D}(\lambda))}=\widetilde{E}_{ext}\sum_j\frac{\exp(\lambda_j\hat{z})}{1-\hat{D}'(\lambda_j)}$$

λ_j are roots of the following dispersion relation: $\lambda-\hat{D}(\lambda)=0$

*The asymptotic solution at $\hat{z}>>1$ is determined by the term with greatest $\mathrm{Re}(\lambda_j)$.

Example: Lorentzian Energy Distribution

Consider energy distribution of the form: $F_0(\hat{P}) = \frac{1}{\pi\hat{q}} \frac{1}{1+\left(\frac{\hat{P}}{\hat{q}}\right)^2}$

$$\frac{d}{d\hat{P}} F_0(\hat{P}) = -\frac{\hat{q}}{\pi} \frac{2\hat{P}}{\left(\hat{q}^2+\hat{P}^2\right)^2}$$

$$\hat{D}(\lambda) = -i\int_G \frac{F_0'(\hat{P})}{\hat{P}-(i\lambda-\hat{C})} d\hat{P}$$

$$= i\frac{2\hat{q}}{\pi}\int_G \frac{\hat{P}}{\left[\hat{P}-(i\lambda-\hat{C})\right]\left(\hat{P}-i\hat{q}\right)^2\left(\hat{P}+i\hat{q}\right)^2} d\hat{P}$$

$$= 4\hat{q}\frac{d}{d\hat{P}}\left\{\frac{\hat{P}}{\left[\hat{P}-(i\lambda-\hat{C})\right]\left(\hat{P}-i\hat{q}\right)^2}\right\}_{\hat{P}=-i\hat{q}}$$

$$= 4\hat{q}\left\{\frac{1}{\left[\hat{P}-(i\lambda-\hat{C})\right]\left(\hat{P}-i\hat{q}\right)^2}\left[1-\frac{\hat{P}}{\hat{P}-(i\lambda-\hat{C})}-\frac{2\hat{P}}{\hat{P}-i\hat{q}}\right]\right\}_{\hat{P}=-i\hat{q}}$$

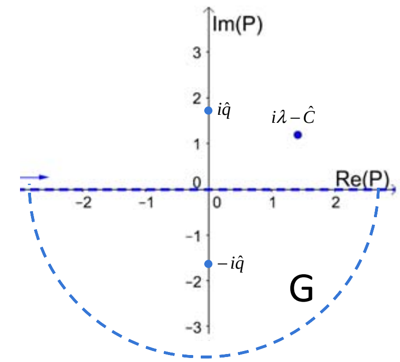
$$= \frac{i}{\left(\hat{q}+\lambda+i\hat{C}\right)^2}$$

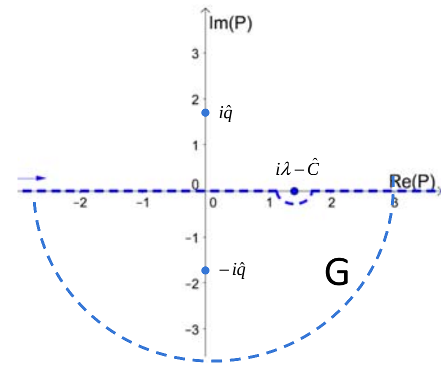
*Note: the contour is closed from the lower half plane and hence there is only one pole at $\hat{P} = -i\hat{q}$

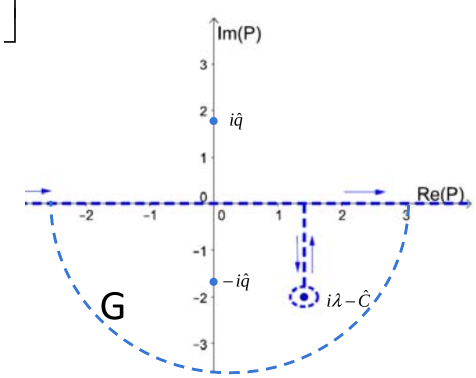
* Note: the contour G is clockwise and hence there is a minus sign.

*Residue at mth order pole:

$$\mathrm{Res}(f;z_0) = \lim_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$







Example: Lorentzian Energy Distribution

The eigenvalues are determined by the dispersion relation:

$$\lambda - \hat{D}(\lambda) = 0 \Rightarrow$$

$$\boxed{\lambda\left(\lambda + \hat{q} + i\hat{C}\right)^2 = i}$$

* Note: in the limit of $\hat{q} = 0$, the dispersion relation reduces to the dispersion relation of a cold beam: $\lambda^3 + 2i\hat{C}\lambda^2 - \hat{C}^2\lambda = i$

For the roots of the dispersion relation, the following relation holds:

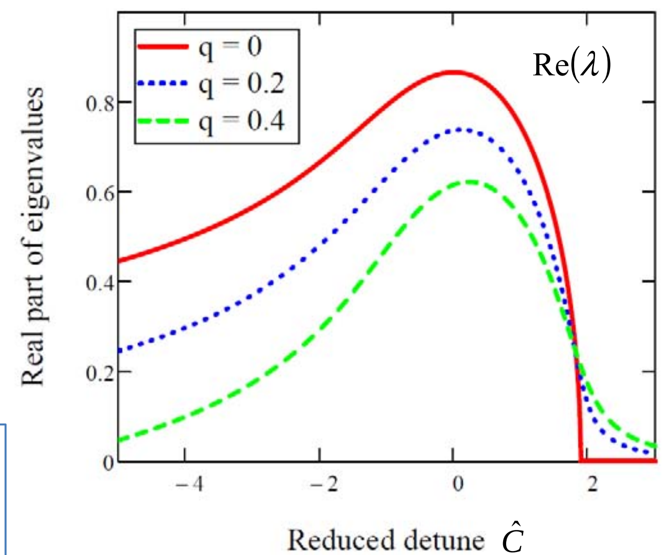
$$\lambda_j\left(\lambda_j + \hat{q} + i\hat{C}\right)^2 = i \Rightarrow \left(\lambda_j + \hat{q} + i\hat{C}\right)^3 = \frac{-1}{\lambda_j^2\left(\lambda_j + \hat{q} + i\hat{C}\right)}$$

and hence $\hat{D}'\left(\lambda_j\right) = \dfrac{-2i}{\left(\lambda_j + \hat{q} + i\hat{C}\right)^3} = 2i\lambda_j^2\left(\lambda_j + \hat{q} + i\hat{C}\right)$

Using above relation, the radiation field in time domain is

$$\boxed{\widetilde{E}(\hat{z}) = \widetilde{E}_{ext}\sum_j \frac{\exp\left(\lambda_j\hat{z}\right)}{1 - \hat{D}'\left(\lambda_j\right)} = \widetilde{E}_{ext}\sum_j \frac{\exp\left(\lambda_j\hat{z}\right)}{1 - 2i\lambda_j^2\left(\lambda_j + \hat{q} + i\hat{C}\right)}}$$

Growth rate for various energy spread parameter, $\hat{q} = 0, 0.2, 0.4$



References:

[1] ‘The Physics of Free Electron Lasers’ by E.L. Saldin, E.A. Schneidmiller and M.V. Yurkov;

[2] L. D. Landau, J. Phys. USSR 10, 25 (1946)

What we learned today

- The positive feedback loop between radiation field and electron density bunching is the underlying mechanism of high gain FEL regime.
- Starting from 1-D linearized Vlasov equation and wave equation, we derived an Integra-differential equation for the evolution of radiation field in a high gain FEL with helical undulator.
- For cold electron beam, we obtained the solution of radiation field and compared it with the low gain solution
- For warm electron beam, Laplace transformation is used to obtain the dispersion relation. As an example, we then solved the dispersion relation for Lorentzian energy distribution.