

PHY 564

Advanced Accelerator Physics

Lecture 6

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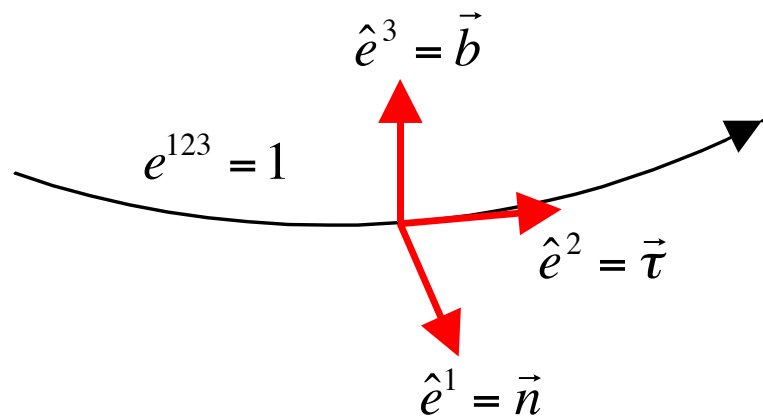
Recap of previous lecture

Accelerator Hamiltonian: continued

Figures 2 and 3 illustrate the Frenet-Serret coordinate system and define 3 orthogonal unit vectors: Normal $\hat{e}^1 = \vec{n}(s)$, tangent $\hat{e}^2 = \vec{\tau}(s)$, and normal and bi-normal $\hat{e}^3 = \vec{b}(s) = [\vec{n} \times \vec{\tau}]$:

$$(\vec{n} \cdot \vec{\tau}) = (\vec{b} \cdot \vec{n}) = (\vec{b} \cdot \vec{\tau}) = 0.$$

Coordinates



$$\vec{\tau} = \frac{d\vec{r}_o(s)}{ds} = \vec{r}'_o$$

$$\vec{n} = - \frac{\vec{r}_o''}{|\vec{r}_o''|}$$

$$\vec{b} = [\vec{n} \times \vec{\tau}]$$

Fig. 3. Unit vectors in the Frenet-Serret coordinate system and their definitions

The reference trajectory must be smooth, with finite second derivatives, etc....etc... The position of any particle located in close proximity to the reference trajectory can uniquely expressed as

$$\vec{r} = \vec{r}_o(s) + x \cdot \vec{n}(s) + y \cdot \vec{b}(s). \quad (100)$$

i.e., it is fully described by 3 contra-variant coordinates:

$$q^1 = x; q^2 = s, q^3 = y. \quad (100-1)$$

The vectors $\{\vec{n}, \vec{\tau}, \vec{b}\}$ satisfy Frenet-Serret formulae:

$$\frac{d\vec{\tau}}{ds} = K(s) \cdot \vec{n}; \quad \frac{d\vec{n}}{ds} = K(s) \cdot \vec{\tau} - \kappa(s) \cdot \vec{b}; \quad \frac{d\vec{b}}{ds} = \kappa(s) \cdot \vec{n}; \quad (101)$$

where

$$K(s) = 1/\rho(s) \quad (101-1)$$

is the curvature of the trajectory, and $\kappa(s)$ is its torsion. If the torsion is equal to zero, the trajectory remains in one plane, as designed for majority of accelerators. Curvature of trajectory is more common – each dipole magnet makes trajectory to curve.

Coordinates

A co-variant basis vector is readily derived from the orthogonal conditions:

$$\vec{a}_i \vec{a}^j = \delta_i^j; \quad \vec{a}^1 = \vec{n} - \frac{\kappa y}{1 + Kx} \vec{\tau}; \quad \vec{a}^2 = \frac{\vec{\tau}}{1 + Kx} + \kappa(\vec{n}y - \vec{b}x); \quad \vec{a}^3 = \vec{b} + \frac{\kappa x}{1 + Kx} \vec{\tau}; \quad (104)$$

The components of the co- and contra-variant metric tensors are defined as follows:

$$g_{ik} = \vec{a}_i \cdot \vec{a}_k = \begin{bmatrix} 1 & \kappa y & 0 \\ \kappa y & (1 + Kx)^2 + \kappa^2(x^2 + y^2) & -\kappa x \\ 0 & -\kappa x & 1 \end{bmatrix}$$

$$g^{ik} = \vec{a}^i \cdot \vec{a}^k = \frac{1}{(1 + Kx)^2} \cdot \begin{bmatrix} (1 + Kx)^2 + \kappa^2 y^2 & -\kappa y & -\kappa^2 xy \\ -\kappa y & 1 & \kappa x \\ -\kappa^2 xy & \kappa x & (1 + Kx)^2 + \kappa^2 x^2 \end{bmatrix} \quad (105)$$

$$g_o = \det[g_{ik}] = (1 + Kx)^2$$

Curvilinear Coordinates

Any vector can be expanded about both co- and contra-variant bases, as well can $\{\vec{n}, \vec{\tau}, \vec{b}\}$:

$$\vec{R} \equiv R_x \vec{n} + R_s \vec{\tau} + R_y \vec{b} \equiv \sum_k R^k \vec{a}_k \equiv \sum_k R_k \vec{a}^k$$

$$R_k = \vec{R} \cdot \vec{a}_k; R_1 = R_x; R_2 = (1 + Kx)R_s + \kappa(R_x y - R_y x); R_3 = R_y;$$

$$R^k = \vec{R} \cdot \vec{a}^k; R^1 = R_x - \frac{\kappa y}{1 + Kx} R_s; R^2 = \frac{R_s}{1 + Kx} + \kappa(R_x y - R_y x); R^3 = R_y + \frac{\kappa x}{1 + Kx} R_s;$$

(106)

All this is trivial, and finally differential operators will look like:

$$\vec{\nabla} \varphi = \vec{a}^k \frac{\partial \varphi}{\partial q^k}; \quad \text{div} \vec{A} = (\vec{\nabla} \cdot \vec{A}) = \frac{1}{\sqrt{g_o}} \frac{\partial}{\partial q^k} (\sqrt{g_o} A^k);$$

$$\text{curl} \vec{A} = [\vec{\nabla} \times \vec{A}] = \frac{e^{ikl}}{\sqrt{g_o}} \frac{\partial A_l}{\partial q^k} \vec{a}_i; \quad \Delta \varphi = \vec{\nabla}^2 \varphi = \frac{1}{\sqrt{g_o}} \frac{\partial}{\partial q^i} \left(\sqrt{g_o} g^{ik} \frac{\partial \varphi}{\partial q^k} \right).$$

(107)

Curvilinear Coordinates

From least action we get the Most General Form of the Accelerator Hamiltonian

$$h^* = -(1 + Kx) \sqrt{\frac{(H - e\varphi)^2}{c^2} - m^2 c^2 - \left(P_1 - \frac{e}{c} A_1\right)^2 - \left(P_3 - \frac{e}{c} A_3\right)^2} + \frac{e}{c} A_2 + \kappa x \left(P_3 - \frac{e}{c} A_3\right) - \kappa y \left(P_1 - \frac{e}{c} A_1\right)$$

$$x' = \frac{dx}{ds} = \frac{\partial h^*}{\partial P_1}; \quad \frac{dP_1}{ds} = -\frac{\partial h^*}{\partial x}; \quad y' = \frac{dy}{ds} = \frac{\partial h^*}{\partial P_3}; \quad \frac{dP_3}{ds} = -\frac{\partial h^*}{\partial y}$$

$$t' = \frac{dt}{ds} = \frac{\partial h^*}{\partial P_t} \equiv -\frac{\partial h^*}{\partial H}; \quad \frac{dP_t}{ds} = -\frac{\partial h^*}{\partial t} \rightarrow \frac{dH}{ds} = \frac{\partial h^*}{\partial t}$$

with s as independent variable.

And $t(s)$ is arrival time of a particle to the azimuth s .

Lecture 6. Choosing a specific gauge

This allows us to express 4-potential as explicit function of electric and magnetic fields – it is useful when you explain what you need to build to engineers...

The equations (114) and (115) are the general form of the single-particle Hamiltonian equation in an accelerator. It undoubtedly is nonlinear (the square root signifies relativistic mechanics), and cannot be solved analytically in general. Only few specific cases allow such solutions.

The only additional option we have is to choose a gauge for the 4-potential. One good choice (my preference) is to make the vector potential equal to zero at the reference trajectory. Two other auxiliary conditions will allow us to express the components of the 4-vector potential in a form of the Taylor series:

$$a) \vec{A}(s,0,0,t) = 0; \quad b) \left. \frac{\partial^n A_1}{\partial x^n} \right|_{s,0,0,t} = \left. \frac{\partial^n A_3}{\partial y^n} \right|_{s,0,0,t} = 0; \quad c) \left. \frac{\partial A_1}{\partial x} \right|_{s,0,0,t} + \left. \frac{\partial A_3}{\partial y} \right|_{s,0,0,t} = 0 \quad (116)$$

that can be achieved by gauge transformation

$$\vec{A} = \vec{\tilde{A}} - \vec{\nabla}f; \varphi = \varphi + \frac{1}{c} \frac{\partial f}{\partial t}; f = f_a = f_b + f_c$$

$$f_a = \int_0^s \tilde{A}_2(s_1, 0, 0, t) ds_1 + \tilde{A}_1(s_1, 0, 0, t) \cdot x + \tilde{A}_3(s_1, 0, 0, t) \cdot y$$

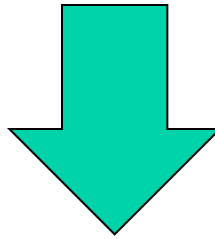
$$f_b = \sum_{n=1} \left(\partial_x^n \tilde{A}_1 \Big|_{s,0,0,t} \frac{x^{n+1}}{(n+1)!} + \partial_y^n \tilde{A}_3 \Big|_{s,0,0,t} \frac{y^{n+1}}{(n+1)!} \right) \quad (117)$$

$$f_c = \frac{1}{2} \sum_{n,k=0} \partial_x^n \partial_y^k (\partial_y \tilde{A}_1 + \partial_x \tilde{A}_3) \frac{x^{n+1}}{(n+1)!} \frac{y^{k+1}}{(n+1)!}$$



$$a) \vec{A}(s, 0, 0, t) = 0; \quad b) \partial_x^n A_1 \Big|_{s,0,0,t} = \partial_y^n A_3 \Big|_{s,0,0,t} = 0; \quad c) \frac{\partial A_1}{\partial x} \Big|_{s,0,0,t} + \frac{\partial A_3}{\partial y} \Big|_{s,0,0,t} = 0$$

$$a) \vec{A}(s,0,0,t) = 0; \quad b) \partial_x^n A_1 \Big|_{s,0,0,t} = \partial_y^n A_3 \Big|_{s,0,0,t} = 0; \quad c) \frac{\partial A_1}{\partial x} \Big|_{s,0,0,t} + \frac{\partial A_3}{\partial y} \Big|_{s,0,0,t} = 0$$



Conditions (116) have following important consequences:

$$a) \partial_s^k \partial_t^l \vec{A}(s,0,0,t) \equiv 0; \quad b) A_1(s,x,0,t) \equiv 0;$$

$$A_3(s,0,y,t) \equiv 0; \quad c) \partial_s^k \partial_t^l \partial_x^m \partial_y^n \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_3}{\partial y} \right) \equiv 0 \quad (116++)$$

After a one-page-long exercise, using the first pair of Maxwell equations and conditions (116), one can express the 4-potential in this gauge though the components of the magnetic- and electric- fields, in other words, make an unique vector potential:

$$\begin{aligned}
 A_1 &= \frac{1}{2} \sum_{n,k=0}^{\infty} \partial_x^k \partial_y^n B_s|_{ro} \frac{x^k y^{n+1}}{k! (n+1)!}; \quad A_3 = -\frac{1}{2} \sum_{n,k=0}^{\infty} \partial_x^k \partial_y^n B_s|_{ro} \frac{x^{k+1} y^n}{(k+1)! n!} \\
 A_2 &= \sum_{n=1}^{\infty} \left\{ \partial_x^{n-1} \left((1+Kx) B_y + \kappa x B_s \right)_{ro} \frac{x^n}{n!} - \partial_y^{n-1} \left((1+Kx) B_x - \kappa y B_s \right)_{ro} \frac{y^n}{n!} \right\} + \\
 &+ \frac{1}{2} \sum_{n,k=1}^{\infty} \left\{ \partial_x^{n-1} \partial_y^k \left((1+Kx) B_y + \kappa x B_s \right)_{ro} \frac{x^n y^k}{n! k!} - \partial_x^n \partial_y^{k-1} \left((1+Kx) B_x - \kappa y B_s \right)_{ro} \frac{x^n y^k}{n! k!} \right\}; \\
 \varphi &= \varphi_o(s, t) - \sum_{n=1}^{\infty} \partial_x^{n-1} E_x|_{ro} \frac{x^n}{n!} - \sum_{n=1}^{\infty} \partial_y^{n-1} E_y|_{ro} \frac{y^n}{n!} - \frac{1}{2} \sum_{n,k=1}^{\infty} \left(\partial_x^{n-1} \partial_y^k E_x|_{ro} + \partial_x^n \partial_y^{k-1} E_y|_{ro} \right) \frac{x^n y^k}{n! k!};
 \end{aligned} \tag{118}$$

where $f|_{ro}$; $(f)_{ro}$ denotes that the value of the function f is taken at the reference orbit $r_o(s)$: i.e., at $x = 0$; $y = 0$, but in an arbitrary moment of time t .

We reserve the notions $f|_{ref}$; $(f)_{ref}$ for values taken at the reference trajectory $\vec{r} = \vec{r}_o(s)$ at the reference time $t = t_o(s)$. It is noteworthy that the value of our new Hamiltonian for the reference particle is the full particle's momentum with the minus sign:

$$h^*|_{ref} = -p_o(s) \quad (119)$$

We should note that $\varphi_o(s,t)$ is determined with the accuracy of an arbitrary constant, which can be eliminated by requesting $\varphi_o(s_o, t_o(s_o)) = 0$ at some point along the reference trajectory. The coefficients in (118) can be expanded further using a trivial time series

$$f(t) = f(t_o(s)) + \sum_{n=1}^{\infty} \frac{d^n f}{dt^n} \Big|_{t=t_o(s)} \frac{(t - t_o(s))^n}{n!}.$$

One important feature of this expansion that no conditions in the EM field are assumed; thus, it can be in free-space field (typical for single-particle dynamics) or a field with sources (for example, charges and currents of beam are examples). Hence, the expansion is applicable to any arbitrary accelerator problem.

An equilibrium particle and a reference trajectory.

A particle that follows the reference trajectory is called an equilibrium (or reference) one:

$$\vec{r} = \vec{r}_o(s); \quad t = t_o(s); \quad H = H_o(s) = E_o(s) + \varphi_o(s, t_o(s)), \quad (120)$$

with $x \equiv 0$; $y \equiv 0$; $p_x \equiv 0$; $p_y \equiv 0$. This is where condition (L2.20a) $\vec{A}|_{ref} = 0$ is useful, i.e., for

$$x|_{ref} = 0; \quad y|_{ref} = 0; \quad P_1|_{ref} = p_x|_{ref} + \frac{e}{c} A_1|_{ref} \equiv 0; \quad P_3|_{ref} = p_y|_{ref} + \frac{e}{c} A_3|_{ref} \equiv 0. \quad (121)$$

The differential form of (121)

$$\begin{aligned} \left. \frac{dx}{ds} \right|_{ref} = \left. \frac{\partial h^*}{\partial P_1} \right|_{ref} = 0; \quad \left. \frac{dy}{ds} \right|_{ref} = \left. \frac{\partial h^*}{\partial P_3} \right|_{ref} = 0; \\ \left. \frac{dP_1}{ds} \right|_{ref} = - \left. \frac{\partial h^*}{\partial x} \right|_{ref} = 0; \quad \left. \frac{dP_3}{ds} \right|_{ref} = - \left. \frac{\partial h^*}{\partial y} \right|_{ref} = 0; \end{aligned} \quad (122)$$

should be combined with the expression for the Hamiltonian (L2.19). The two first equations in (122) give us the already known conditions, viz., that of the zero transverse component of momentum. The following two equations are not as trivial; they set the two conditions at the reference orbit.

Completing a trivial differentiation on \mathbf{x} (where most of the terms are turned into zero at the reference orbit, except $\partial_x \varphi$ and $\partial_x A_2$) we have

$$-\left. \frac{\partial h^*}{\partial x} \right|_{ref} = K \sqrt{G} \Big|_{ref} - (1 + Kx) \Big|_{ref} \frac{\left(\left[\frac{eE}{c^2} \frac{\partial \varphi}{\partial x} \right] + p_x \frac{e}{c} \frac{\partial A_1}{\partial x} + p_y \frac{e}{c} \frac{\partial A_3}{\partial x} \right) \Big|_{ref}}{\sqrt{G} \Big|_{ref}} + \left[\frac{e}{c} \frac{\partial A_2}{\partial x} \right] \Big|_{ref} + \kappa (p_y) \Big|_{ref} + \kappa \left(\frac{e}{c} \frac{\partial A_1}{\partial x} y - \frac{e}{c} \frac{\partial A_3}{\partial x} x \right) \Big|_{ref} = 0$$

$$E \equiv (H - e\varphi); \quad G = \frac{E^2}{c^2} - m^2 c^2 - p_x^2 - p_y^2; \quad \sqrt{G} \Big|_{ref} = p_o \dots$$

Note: The term(s) that do not vanish at the limit are identified by the square brackets [...]

and using the above expansions, we derive the well-know equation for the curvature of the trajectory:

$$K(s) \equiv \frac{1}{\rho(s)} = -\frac{e}{p_o c} \left(B_y \Big|_{ref} - \frac{E_o}{p_o c} E_x \Big|_{ref} \right). \quad (123)$$

Differentiation on y is similar

$$\begin{aligned}
 -\frac{\partial h^*}{\partial y} \Big|_{ref} &= -(1 + Kx)_{ref} \frac{\left(\left[\frac{eE}{c^2} \frac{\partial \phi}{\partial y} \right] + p_x \frac{e}{c} \frac{\partial A_1}{\partial y} + p_y \frac{e}{c} \frac{\partial A_3}{\partial y} \right)_{ref}}{\sqrt{G} \Big|_{ref}} \\
 &+ \left[\frac{e}{c} \frac{\partial A_2}{\partial y} \right]_{ref} - (p_x)_{ref} + \kappa \left(\frac{e}{c} \frac{\partial A_1}{\partial y} y - \frac{e}{c} \frac{\partial A_3}{\partial y} x \right)_{ref} = 0
 \end{aligned}$$

and yields

$$B_x \Big|_{ref} = \frac{E_o}{\rho_o c} E_y \Big|_{ref}, \tag{124}$$

The conditions in (120) for the arrival time of the reference particle and values of its Hamiltonian are also informative, but not surprising. First, the condition on the arrival time

$$\frac{dt_o(s)}{ds} = -\frac{\partial h^*}{\partial H} \Big|_{ref} = \frac{H - e\varphi}{c^2 \sqrt{G}} \Big|_{ref} = \frac{H_o - e\varphi_o}{p_o c^2} \equiv \frac{E_o}{p_o c^2} = \frac{1}{v_o(s)} \quad (125)$$

gives an understandable definition of velocity along trajectory: $\mathbf{v} = d\mathbf{s}/dt$, and the velocity of the reference particle $\mathbf{V}_o = p_o c^2 / E_o$.

The condition on energy (3D Hamiltonian) gives

$$\begin{aligned} \left. \frac{\partial h^*}{\partial t} \right|_{ref} &= (1 + Kx)_{ref} \frac{\left(\left[\frac{eE}{c^2} \frac{\partial \varphi}{\partial t} \right] + p_x \frac{e}{c} \frac{\partial A_1}{\partial t} + p_y \frac{e}{c} \frac{\partial A_3}{\partial t} \right)_{ref}}{\sqrt{G} \Big|_{ref}} \\ &+ \left(\frac{e}{c} \frac{\partial A_2}{\partial t} \right)_{ref} + \kappa \left(\frac{e}{c} \frac{\partial A_1}{\partial x} y - \frac{e}{c} \frac{\partial A_3}{\partial x} x \right)_{ref} = \frac{eE_o}{p_o c^2} \left. \frac{\partial \varphi}{\partial t} \right|_{ref} \quad (126)s \\ \frac{dH_o(s)}{ds} &= \left. \frac{\partial h^*}{\partial t} \right|_{ref} = \frac{eE_o}{p_o c^2} \left. \frac{\partial \varphi}{\partial t} \right|_{ref} ; \end{aligned}$$

which can be transferred using $H = E + e\varphi$ and $d\varphi_o(s, t_o(s)) = \frac{\partial \varphi_o}{\partial s} ds + \frac{\partial \varphi_o}{\partial t} \frac{ds}{v_o(s)}$

into the energy gain of the reference particle along its trajectory:

$$\frac{dE_o(s)}{ds} = \frac{d(H_o(s) - \varphi_o(s, t_o(s)))}{ds} = -e \left. \frac{\partial \varphi}{\partial s} \right|_{ref} \equiv eE_2(s, t_o(s)). \quad (127)$$

As discussed before, accelerator designers face the problem of ensuring that the reference particle faithfully follows the reference trajectory. Our goal is to use the above conditions to the maximum, and, as we see below, to eliminate zero-order terms from the equations of motion. By selecting the reference trajectory as basis for our coordinate system, we set the transverse coordinates and momenta at zero at the reference orbit. Hence, two canonical pairs have a good and solid origin.

The third pair $(-t, H)$ is odd; it is not zero for the reference particle. Furthermore, it has different units. Hence, we can move step forward with a more natural Canonical pair $\{q_\tau = -ct, p_\tau = H/c\}$ - whose generating function is obvious: $\Phi(q = -t, \tilde{P} = p_\tau) = -ct \cdot p_\tau$. In this case, the analogy is complete: $q_\tau = -ct$ has the dimension of distance and is just $-\mathbf{x}_o$ in 4D space, while $p_\tau = H/c$ has the dimension of momentum and is just \mathbf{P}_o in 4D space.

We also should select variables that are zero at the reference orbit. The following pair is one of better choices:

$$\{\tau = -c(t - t_o(s)), \delta = (H - E_o(s) - e\varphi_o(s,t))/c\}, \quad (128)$$

which are zero for the reference particle. Generation function is easily to come with:

$$\Phi(q, \tilde{P}, s) = \tilde{P}_1 x + \tilde{P}_3 y - (E_o(s) + c\delta)(t - t_o(s)) - e \int^t \varphi_o(s, t_1) dt_1, \quad (129)$$

and it produces what is desired:

$$P_1 = \frac{\partial \Phi}{\partial x} = \tilde{P}_1; \quad P_3 = \frac{\partial \Phi}{\partial y} = \tilde{P}_3; \quad H = \frac{\partial \Phi}{\partial(-t)} = E_o + c\delta + e\varphi_o(s,t);$$

$$\tilde{q}_1 = \frac{\partial \Phi}{\partial \tilde{P}_1} = x; \quad \tilde{q}_3 = \frac{\partial \Phi}{\partial \tilde{P}_3} = y; \quad \tilde{q}_\delta = \frac{\partial \Phi}{\partial \delta} = -c(t - t_o(s)) = \tau$$

$$\tilde{h} = h + \frac{\partial \Phi}{\partial s} = h + \frac{E_o(s) + c\delta}{v_o(s)} + E'_o(s)\tau/c - e \int^t \varphi'_o(s, t_1) dt_1$$

The change to the Hamiltonian comprises meaningful terms as well as just a trivial function of s , $g(s)$:

$$\frac{\partial \Phi}{\partial s} = \frac{c}{v_o(s)} \delta - e\varphi_{//}(s, \tau) + g(s);$$

$$g(s) = E_o(s) / v_o(s) - e \int^{t_o(s)} \varphi'_o(s, t_1) dt_1$$

$$\begin{aligned} \varphi_{//}(s, \tau) =_{def} \frac{\partial}{\partial s} \int_0^{-\tau/c} (\varphi_o(s, t_o(s) + \zeta) - \varphi_o(s, t_o(s))) d\zeta \equiv \\ - \int_0^{-\tau/c} (E_2(s, t_o(s) + \zeta) - E_2|_{ref}) d\zeta \end{aligned} \quad (131)$$

where we used eq. (127) as $E'_o(s) = -e \frac{\partial \varphi}{\partial s} \Big|_{ref}$. Additive $g(s)$ simply can be dropped from the Hamiltonian - it does not change equations of motion.

Now the only remaining task is to express the new Hamiltonian function with an updated canonical pair (130) and (115):

$$\begin{aligned} \tilde{h} = & -(1+Kx)\sqrt{p_o^2 + \frac{2E_o}{c}\left(\delta - \frac{e}{c}\varphi_{\perp}\right) + \left(\delta - \frac{e}{c}\varphi_{\perp}\right)^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2} + \\ & + \frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right) + \frac{c}{v_o}\delta - \frac{e}{c}\varphi_{\parallel}(s,\tau) \end{aligned} \quad (132)$$

where we used following trivial expansion and definition:

$$\frac{(E_o + c\delta + e\varphi_o(s,t) - e\varphi)^2}{c^2} - m^2c^2 = p_o^2 + \frac{2E_o}{c}\left(\delta - \frac{e}{c}\varphi_{\perp}\right) + \left(\delta - \frac{e}{c}\varphi_{\perp}\right)^2;$$

$$\varphi_{\perp def} = \varphi(s,x,y,t) - \varphi_o(s,t) \equiv \varphi(s,x,y,t) - \varphi(s,0,0,t)$$

Scaling variables.

Frequently, it is useful to scale one of canonical variables. Typical scaling in accelerator physics involves dividing the canonical momenta P_1, P_3, δ by the momentum of the reference particle:

$$\pi_1 = \frac{P_1}{P_o}; \quad \pi_3 = \frac{P_3}{P_o}; \quad \pi_o = \frac{\delta}{P_o}.$$

These variables are dimensionless and also are close to $x', y', \delta E / p_o c$ for small deviations. Such scaling only is allowed in Hamiltonian mechanics when the scaling parameter is constant, i.e., is not function of s . Thus, scaling by the particle's momentum remains within the framework of Hamiltonian mechanics only if the reference particle's momentum is constant, that is, when the longitudinal electric field is zero along the reference particle's trajectory (i.e. at moment $t=t_o(s)$). One similarly can scale the coordinates by a constant.

$$\xi_1 = \frac{x}{L}; \quad \xi_3 = \frac{y}{L}; \quad \xi_o = \frac{\tau}{L}.$$

Scaling by a constant is easy; divide the Hamiltonian by the constant and rename the variables. Hence, transforming (134) with constant, called p_o , will make Hamiltonian (132) into

$$\begin{aligned} \tilde{h} = & -(1+Kx) \sqrt{1 + \frac{2E_o}{p_o c} \left(\delta - \frac{e}{p_o c} \varphi_{\perp} \right) + \left(\delta - \frac{e}{p_o c} \varphi_{\perp} \right)^2 - \left(\pi_1 - \frac{e}{p_o c} A_1 \right)^2 - \left(\pi_3 - \frac{e}{p_o c} A_3 \right)^2} + \\ & + \frac{e}{p_o c} A_2 + \kappa x \left(\pi_3 - \frac{e}{p_o c} A_3 \right) - \kappa y \left(\pi_1 - \frac{e}{p_o c} A_1 \right) + \frac{c}{v_o} \delta - \frac{e}{p_o c} \varphi_{\parallel}(s, \tau) \end{aligned}$$

(132 @ constant energy)

Usage of this Hamiltonian is very popular for storage rings or transport channels, wherein the energy of the particles remains constant in time. It should not be employed for particles undergoing an acceleration.

Expanding the Hamiltonian.

Expanding the Hamiltonian (132) is a nominal tool in accelerator physics that allows the separation of the effects of various orders and sometimes the use of perturbation-theory approaches. Having completed the process of creating canonical variables, which are zero for the reference particle, the next step is to assume (which is true for operational accelerators) that the relative deviations of momenta are small

$$\left| \frac{P_1}{p_o} \right| \ll 1; \quad \left| \frac{P_3}{p_o} \right| \ll 1; \quad \left| \frac{\delta}{p_o} \right| \ll 1;$$

and that the EM fields are sufficiently smooth around the reference trajectory to allow expansion in terms of x ; y ; τ . We will consider all six variables to be of the same order (of infinitesimally, α). We call the order of expansion to be the maximum total power in a product that is any combination of $x, y, \tau, P_1, P_2, \delta$. Unless there is a good reason not to do so, we truncate the series using this rule.

The general expansion of Hamiltonian (132) can be accomplished via the already derived expansion for 4-potential and the well-known expansion of the square root:

$$\begin{aligned}
 \tilde{h} = & -(1 + Kx)p_o \sqrt{1 + \frac{2E_o}{p_o c} \left(\frac{\delta}{p_o} - \frac{e}{p_o c} \varphi_{\perp} \right) + \left(\frac{\delta}{p_o} - \frac{e}{p_o c} \varphi_{\perp} \right)^2} \\
 & - \left(\frac{P_1}{p_o} - \frac{e}{p_o c} A_1 \right)^2 - \left(\frac{P_3}{p_o} - \frac{e}{p_o c} A_3 \right)^2 ; \quad (135) \\
 & + \frac{e}{c} A_2 + \kappa x \left(P_3 - \frac{e}{c} A_3 \right) - \kappa y \left(P_1 - \frac{e}{c} A_1 \right) + \frac{c}{v_o} \delta - \frac{e}{c} \varphi_{//}(s, \tau) \\
 & \sqrt{1 + g} = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{g^n}{2^n} \frac{(2n-3)!!}{n!} = 1 + \frac{y}{2} + O(y^2).
 \end{aligned}$$

Let us discuss a few general issues first. If we expand (134) into an finite series

$$\begin{aligned} \tilde{h} = & -p_o(s) + \sum_{v=1}^N \sum_{\substack{i+j+k+ \\ l+m+n=v}} C_{ijklmn} x^i y^j \tau^k P_1^l P_3^m \delta^n + O(\alpha^{N+1}) = \\ & -p_o + \sum_{v=1}^N \sum_{\substack{\sum p_i=v \\ i=1,6}} C_{v\{p_i\}} \prod_{i=1}^6 x_i^{p_i} + O(\alpha^{N+1}) \end{aligned} \quad ; \quad (136)$$

where we introduce our phase-space vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \equiv \begin{bmatrix} x \\ P_1 \\ y \\ P_2 \\ \tau \\ \delta \end{bmatrix} ; \quad (137)$$

which we will use in its 2n-dimensional phase-space appearance.

$$X^T = [q^1 \quad P_1 \quad \dots \quad \dots \quad q^n \quad P_n] = [x_1 \quad x_2 \quad \dots \quad \dots \quad x_{2n-1} \quad x_{2n}], \quad (138)$$

where T stands for “transposed”. Using this notion, Hamiltonian equations can be written as one

$$\frac{dX}{ds} = S \cdot \frac{\partial H}{\partial X} \quad \Leftrightarrow \quad \frac{dx_i}{ds} = S_{ij} \cdot \frac{\partial H}{\partial x_j} \equiv \sum_{j=1}^{2n} S_{ij} \cdot \frac{\partial H}{\partial x_j} \quad (139)$$

wherein we introduce matrix S – a generator of the symplectic group (see further). The matrix, S , is asymmetric, with $S_{2m-1,2m} = 1 = -S_{2m,2m-1}$, $m = 1, \dots, n$, and other elements are zero.

In matrix form S has n diagonal blocks with a 2×2 matrix σ , and the rest is the field of zeros:

$$S = \begin{bmatrix} \sigma & 0 & \dots & 0 \\ 0 & \sigma & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \sigma \end{bmatrix}_{2n \times 2n} ; \quad \sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (140)$$

This is one of the very important objects in Hamiltonian mechanics.

Using expansion (136), we would dispose of $p_o(s)$ as valueless, and look initially at the first-order terms in the expansion:

$$\tilde{h} = \sum_{i=1,6} C_{1i} x_i + O(\alpha^2); \quad (136-0)$$

That should give us the zero- order terms in the equation of motion:

$$\frac{dx_k}{ds} = \sum_{j=1}^{2n} S_{kj} \cdot C_{1j} + O(\alpha)$$

By design of our variables (section 2.2) $\left. \frac{dx_k}{ds} \right|_{ref} = 0$, i.e. all zero order terms are equal to zero. Hence, the expansion of our Hamiltonian does not have first-order terms:

$$h = \sum_{v=2}^N \sum_{\substack{\sum_{i=1,6} p_i = v}} C_{v\{p_i\}} \prod_{i=1}^6 x_i^{p_i} + O(\alpha^{N+1}); \quad (141)$$

We finish this section with the explicit form of the first non-trivial term in the expansion of (135):

$$\begin{aligned} \tilde{h} = & \frac{P_1^2 + P_3^2}{2p_o} + F \frac{x^2}{2} + Nxy + G \frac{y^2}{2} + L(xP_3 - yP_1) + \\ & \frac{\delta^2}{2p_o} \cdot \frac{m^2 c^2}{p_o^2} + U \frac{\tau^2}{2} + g_x x \delta + g_y y \delta + F_x x \tau + F_y y \tau \end{aligned} \quad ; \quad (144)$$

with

$$\begin{aligned} \frac{F}{p_o} = & \left[-K \cdot \frac{e}{p_o c} B_y - \frac{e}{p_o c} \frac{\partial B_y}{\partial x} + \left(\frac{e B_s}{2 p_o c} \right)^2 \right] - \frac{e}{p_o v_o} \frac{\partial E_x}{\partial x} - 2K \frac{e E_x}{p_o v_o} + \left(\frac{m e E_x}{p_o^2} \right)^2 ; \\ \frac{G}{p_o} = & \left[\frac{e}{p_o c} \frac{\partial B_x}{\partial y} + \left(\frac{e B_s}{2 p_o c} \right)^2 \right] - \frac{e}{p_o v_o} \frac{\partial E_y}{\partial y} + \left(\frac{m e E_z}{p_o^2} \right)^2 ; \\ \frac{2N}{p_o} = & \left[\frac{e}{p_o c} \frac{\partial B_x}{\partial x} - \frac{e}{p_o c} \frac{\partial B_y}{\partial y} \right] - K \cdot \frac{e}{p_o c} B_x - \frac{e}{p_o v_o} \left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} \right) - 2K \frac{e E_y}{p_o v_o} + \left(\frac{m e E_z}{p_o^2} \right) \left(\frac{m e E_x}{p_o^2} \right) \end{aligned} \quad (145)$$

$$L = \kappa + \frac{e}{2 p_o c} B_s ; \quad \frac{U}{p_o} = \frac{e}{p c^2} \frac{\partial E_s}{\partial t} ; \quad g_x = \frac{(m c)^2 \cdot e E_x}{p_o^3} - K \frac{c}{v_o} ; \quad g_y = \frac{(m c)^2 \cdot e E_y}{p_o^3} ;$$

$$F_x = \frac{e}{c} \frac{\partial B_y}{\partial c t} + \frac{e}{v_o} \frac{\partial E_x}{\partial c t} ; \quad F_y = -\frac{e}{c} \frac{\partial B_x}{\partial c t} + \frac{e}{v_o} \frac{\partial E_y}{\partial c t} .$$

We finish this section with the explicit form of the first non-trivial term in the expansion of (135):

$$\begin{aligned} \tilde{h} = & \frac{P_1^2 + P_3^2}{2p_o} + F \frac{x^2}{2} + Nxy + G \frac{y^2}{2} + L(xP_3 - yP_1) + \\ & \frac{\delta^2}{2p_o} \cdot \frac{m^2 c^2}{p_o^2} + U \frac{\tau^2}{2} + g_x x \delta + g_y y \delta + F_x x \tau + F_y y \tau \end{aligned} \quad ; \quad (143)$$

with

$$\begin{aligned} \frac{F}{p_o} = & \left[-K \cdot \frac{e}{p_o c} B_y - \frac{e}{p_o c} \frac{\partial B_y}{\partial x} + \left(\frac{e B_s}{2 p_o c} \right)^2 \right] - \frac{e}{p_o v_o} \frac{\partial E_x}{\partial x} - 2K \frac{e E_x}{p_o v_o} + \left(\frac{m e E_x}{p_o^2} \right)^2; \\ \frac{G}{p_o} = & \left[\frac{e}{p_o c} \frac{\partial B_x}{\partial y} + \left(\frac{e B_s}{2 p_o c} \right)^2 \right] - \frac{e}{p_o v_o} \frac{\partial E_y}{\partial y} + \left(\frac{m e E_z}{p_o^2} \right)^2; \\ \frac{2N}{p_o} = & \left[\frac{e}{p_o c} \frac{\partial B_x}{\partial x} - \frac{e}{p_o c} \frac{\partial B_y}{\partial y} \right] - K \cdot \frac{e}{p_o c} B_x - \frac{e}{p_o v_o} \left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} \right) - 2K \frac{e E_y}{p_o v_o} + \left(\frac{m e E_z}{p_o^2} \right) \left(\frac{m e E_x}{p_o^2} \right) \end{aligned} \quad ; \quad (144)$$

$$L = \kappa + \frac{e}{2 p_o c} B_s; \quad \frac{U}{p_o} = \frac{e}{p c^2} \frac{\partial E_s}{\partial t}; \quad g_x = \frac{(m c)^2 \cdot e E_x}{p_o^3} - K \frac{c}{v_o}; \quad g_y = \frac{(m c)^2 \cdot e E_y}{p_o^3};$$

$$F_x = \frac{e}{c} \frac{\partial B_y}{\partial ct} + \frac{e}{v_o} \frac{\partial E_x}{\partial ct}; \quad F_y = -\frac{e}{c} \frac{\partial B_x}{\partial ct} + \frac{e}{v_o} \frac{\partial E_y}{\partial ct}.$$

If momentum p_o is constant, we can use (134) and rewrite Hamiltonian of the linearized motion (143) as

$$\begin{aligned} \tilde{h}_n = & \frac{\pi_1^2 + \pi_3^2}{2} + f \frac{x^2}{2} + n \cdot xy + g \frac{y^2}{2} + L(x\pi_3 - y\pi_1) + \\ & \frac{\pi_o^2}{2} \cdot \frac{m^2 c^2}{p_o^2} + u \frac{\tau^2}{2} + g_x x \pi_o + g_y y \pi_o + f_x x \tau + f_y y \tau \end{aligned} ; \quad (144-n)$$

with

$$f = \frac{F}{p_o}; \quad n = \frac{N}{p_o}; \quad g = \frac{G}{p_o}; \quad u = \frac{U}{p_o}; \quad f_x = \frac{F_x}{p_o}; \quad f_y = \frac{F_y}{p_o}; \quad (145-n)$$

Note that

$$x' = \frac{\partial h_n}{\partial \pi_1} = \pi_1 - Ly; \quad y' = \frac{\partial h_n}{\partial \pi_3} = \pi_3 + Lx; \quad ; \quad (146)$$

i.e. as soon as $L=0$, we can use traditional x' and y' as reduced momenta.

For flat reference orbit - $\kappa = 0$, in the absence of transverse coupling ($L=0$, $N=0$) and transverse electric fields, DC magnetic fields it becomes a much simpler and much more familiar form:

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F \frac{x^2}{2} + G \frac{y^2}{2} + \frac{\delta^2}{2p_o} \cdot \frac{m^2 c^2}{p_o^2} + U \frac{\tau^2}{2} + g_x x \delta$$

or ;

(146)

$$\tilde{h}_n = \frac{x'^2 + y'^2}{2p_o} + f \frac{x^2}{2} + g \frac{y^2}{2} + \frac{\pi_o^2}{2} \cdot \frac{m^2 c^2}{p_o^2} + u \frac{\tau^2}{2} + g_x x \delta$$

with

$$f = \left[-K \cdot \frac{e}{p_o c} B_y - \frac{e}{p_o c} \frac{\partial B_y}{\partial x} \right]; \quad g = \frac{e}{p_o c} \frac{\partial B_x}{\partial y}; \quad u = \frac{e}{p c^2} \frac{\partial E_s}{\partial t}; \quad g_x = -K \frac{c}{v_o}. \quad (147)$$

Finally, see an Appendix where Mathematica tool allowing expansion of the accelerator to an arbitrary order is presented.