Lecture 1: Special Relativity

1.1 Einstein principle of relativity.

There is nothing more un-natural than "non-relativistic" electrodynamics. And there are very few thing in our world as natural as relativistic electrodynamics. We can consider non-relativistic classical or quantum mechanics for objects which can rest or move slowly. But how we can describe electromagnetic wave without using speed of the light? which is the universal, as far as we know, physical constant:

$$\mathbf{c} = \mathbf{2.99792458(1.2) \cdot 10^{10}\ cm/sec}; \qquad (1\text{-}1)$$

The "**c**" does not depend on the system of reference . The standard non-relativistic Galileo's relativity principle claims

1. Free particle propagates with constant velocity (the law of inertia) $\vec{v} = conts$;

2. Time does not depend on the choice of inertial frame moving with velocity $\vec{V}$ with respect to initial frame of reference:

$$t = t'; \ \vec{r} = \vec{r}\,' + \vec{V}t \qquad (1\text{-}2)$$

and velocity transformation is

$$\vec{v} = \vec{v}\,' + \vec{V}. \qquad (1\text{-}3)$$

Many modern experimental facts disagree with Galileo's principle and confirm that:

The speed of the light does not depend of the reference frame.

Galileo assumed that we are leaving in Euclidean world. What is wrong in Galileo's principle is the assumption that time and distance between two points in 3-D space are absolute, i.e. independent from the reference frame.

In 1905 Einstein modified principle of relativity to satisfy new experimental data (read **J11.1-J11.2** for details).

The **Einstein principle of relativity** comprises of two postulates:

1. **POSTULATE OF RELATIVITY** (the same as Galileo):

The laws of nature and results of all experiments are independent of translational motion of the system (reference frame) as whole. Precisely: there are a triply infinite set of equivalent Euclidean (3D) reference frames moving with constant velocities in rectilinear paths relative to one other in which all physical phenomena occur in an identical manner.

2. **POSTULATE OF THE CONSTANCY OF THE SPEED OF THE LIGHT** (Einstein):

The speed of the light (maximum velocity of propagation of interaction) is independent on the motion of its source. In other words: there is maximum velocity of propagation of any physical object (a particle, a wave, etc.), which interact with our world.

[1] All through the course I will refer to three textbook as following:
Jn.m - means Jackson, Chapter **n**, Section **m**;

Ln.m - means Landau, Chapter **n**, § **m**;
An.m - means Arfken, Chapter **n**, Section **m**.

Galileo principle and formulae for velocity transformation (1-3) do not satisfy second Einstein postulate. Therefore, Newton (or classical) mechanics based on the Galileo principles must be modified to satisfy experimental results. The most of famous experimental result contradicting to Galileo principle was Michelson-Morley experiment (1887). They tried to measure "ether drift" (the ether is imaginary substance in which electromagnetic waves are propagating; similar to the air for acoustic waves). They tried to measure difference between speed of the light in the direction of the Earth rotation and the opposite direction. According to the Galeleo law (1-3), there must be difference of $\pm v_{rot}$. The result showed no difference.

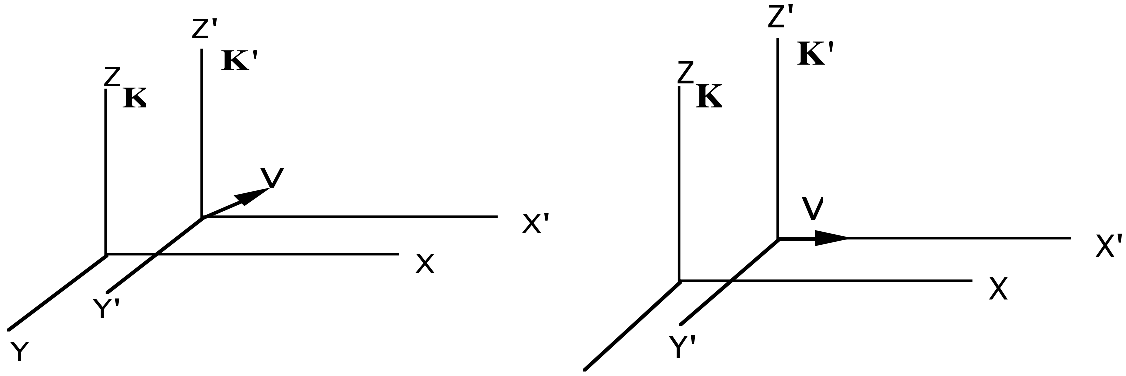

Fig. 1. Two Inertial Reference Frames: system K' moves with velocity $\vec{V}$ with respect to system K. By choice of coordinate system (rotation in 3D space) we can make $\vec{V}$ parallel to the X axis.

1.2 Events, 4-vectors, 4D-Intervals.

Let's introduce an important object in relativistic theory - an *EVENT*. An event is described by the location (in 3D coordinate system) **where** it occurred and by time **when** it occurred. As far as we know, it is full description of any event. We do not have any firm prove about the existence of other coordinates, so far...

Therefore, an event is defined by four coordinates (4-vector) in 4-dimensional time-space:

$$x^i = (x^0, x^1, x^2, x^3) \equiv (x^0, \vec{r}); \qquad \begin{matrix} x_o = ct; x^1 = x;\ x^2 = y;\ x^3 = z \\ (x^4 = ix_o \quad - Minkowski\ \ metric) \end{matrix}. \qquad (1\text{-}4)$$

Let's look at two event A and B: A is the event when we sent a signal propagating with maximum possible speed c, B is the event when signal arrived in different point of space. Both events can be described in any reference system:

K-system: Event A: the signal was sent from location $\vec{r}_A = \hat{e}_x x_A + \hat{e}_y y_A + \hat{e}_z z_A$ at time t'_A : $X^i_A = (x^0_A, \vec{r}_A)$;

Event B: the signal was observed in location $\vec{r}_B = \hat{e}_x x_B + \hat{e}_y y_B + \hat{e}_z z_B$ at time t_B: $X^i_B = (x^0_B, \vec{r}_B)$.

K'-system: Event A: the signal was sent from location $\vec{r}'_A = \hat{e}_x x'_A + \hat{e}_y y'_A + \hat{e}_z z'_A$ at time t'_B : $X'^i_A = (x'^0_A, \vec{r}'_A)$;

Event B: the signal was observed in location $\vec{r}'_B = \hat{e}_x x'_B + \hat{e}_y y'_B + \hat{e}_z z'_B$ at time t'_B: $X'^i_B = (x'^0_B, \vec{r}'_B)$.

Signal propagates with the speed of the light in both systems. Therefore:

$$c^2(t_B-t_A)^2-(\vec{r}_B-\vec{r}_A)^2=c^2(t_B-t_A)^2-(x_B-x_A)^2-(y_B-y_A)^2-(z_B-z_A)^2=0; \qquad (1\text{-}5)$$

$$c^2(t'_B-t'_A)^2-(\vec{r}'_B-\vec{r}'_A)^2=c^2(t'_B-t'_A)^2-(x'_B-x'_A)^2-(y'_B-y'_A)^2-(z'_B-z'_A)^2=0. \qquad (1\text{-}5')$$

The quantity for any arbitrary events A and B, defined as:

$$s_{AB}=\sqrt{c^2(t_B-t_A)^2-(x_B-x_A)^2-(y_B-y_A)^2-(z_B-z_A)^2}; \qquad (1\text{-}6)$$

is of special importance in special relativity. It is called ***the interval between two events***. We have found that if interval is equal zero in one system it is equal to zero in all inertial system of references (eqs. (1-5) and (1-5')). Let's look at to events, which are infinitely close to each another: $\vec{r}_B=\vec{r}_A+d\vec{r}$; $t_B=t_A+dt$; and interval ds between them:

$$ds^2=c^2dt^2-dx^2-dy^2-dz^2. \qquad (1\text{-}7)$$

If ds^2=0, then it is equal zero in any other system ds'^2=0. In addition, ds and ds' are infinitesimals of the same order. Therefore, ds^2, ds'^2 must be proportional to each other:

$$ds^2=a\,ds'^2. \qquad (1\text{-}8)$$

The coefficient a can not depend on time or position not to violate homogeneity of the space and time. Similarly, it can not depend on direction of relative velocity not to contradict the isotropy of the space. Therefore, it can depend only on absolute value of relative velocity of the systems $a=a\left(\left|\vec{V}\right|\right)$. Let's consider three reference systems: K,K',K'':

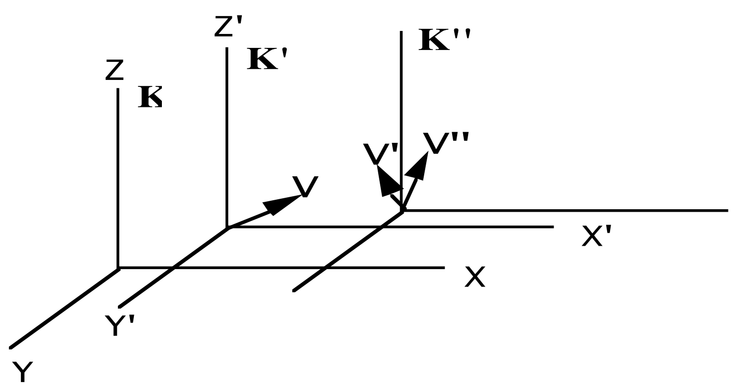

Fig. 2 Three inertial reference systems K.K'.K''. K' moves with velocity $\vec{V}$ with respect to K, K'' moves with velocity $\vec{V}'$ with respect to K' and with velocity $\vec{V}''$ with respect to K. $\vec{V}''$ depends on both values and direction of $\vec{V},\vec{V}'$.

Using relation (1-8) we have for K-system:

$$ds^2=a\left(\left|\vec{V}\right|\right)ds'^2;\;\; ds^2=a\left(\left|\vec{V}''\right|\right)ds''^2;$$

and for K'-system:

$$ds'^2=a\left(\left|\vec{V}'\right|\right)ds''^2;$$

yields the ratio:

$$a\left(\left|\vec{V}''\right|\right)=a\left(\left|\vec{V}\right|\right)a\left(\left|\vec{V}'\right|\right).$$

Left side depends on value of $\vec{V}''$ which depends on both values and direction of $\vec{V},\vec{V}'$, while right side depends only on absolute values of $\vec{V},\vec{V}'$. Therefore, we should conclude that a does not depend on velocity at all: $a = const$. The above relation reduces to $a=a^2$, i.e. $a=1$ (we drop trivial $a=0$). This great ratio gives us equality of infinitesimal intervals:

$$ds^2 = ds'^2; \tag{1-9}$$

and as result invariance of any finite intervals:

$$s_{AB} = \int_A^B ds = \int_A^B ds' = s'_{AB}. \tag{1-10}$$

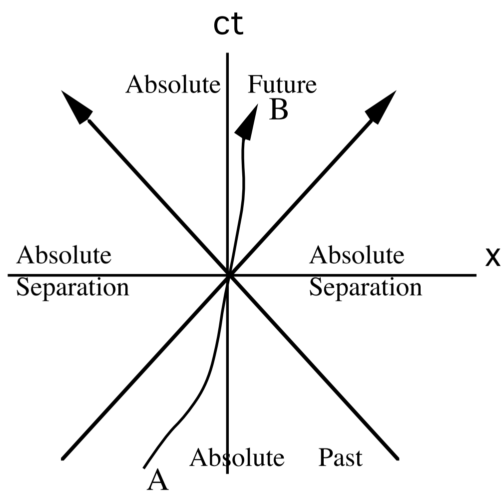

Fig. 3 World line (A-B) of the system and the light cone.

There are three distinctive values of $s^2{}_{AB}$: positive, negative and zero. The sign and the value of $s^2{}_{AB}$ does not depend on system of reference:

$$\left\{\begin{array}{r} s^2{}_{AB} < 0,\ spacelike\ separation \\ s^2{}_{AB} > 0,\ timelike\ separation \\ s^2{}_{AB} = 0, lightlike\ separation \end{array}\right\}$$

<u>Spacelike interval</u>: there is a system K' where two events occur at the same time, but in different points of space

$$s^2{}_{AB} = c^2(t_B - t_A)^2 - (\vec{r}_B - \vec{r}_A)^2 < 0; \Rightarrow s^2{}_{AB} = -(\vec{r}'_B - \vec{r}'_A)^2 < 0;$$

<u>Timelike interval</u>: there is a system K' where two events occur at the same place, but in different points of time

$$s^2{}_{AB} = c^2(t_B - t_A)^2 - (\vec{r}_B - \vec{r}_A)^2 > 0; \Rightarrow s^2{}_{AB} = c^2(t'_B - t'_A)^2 < 0;$$

<u>Lightlike interval</u>: two events can be connected by light signal $s^2{}_{AB} = 0$.

If we put event O in the origin, then $\vec{r}^{\,2} = c^2t^2$ will define the light cone. All events inside the light cone (closer to t axis) can or could be connected with event O in future or in the past. Events outside this cone are absolutely remote with respect to this event: any exchange of information between these events and the event O is impossible. Fig. 3 illustrates this puncture for 1D space with light cone equation of $x = \pm ct$.

<u>1.3</u> Lorentz transformations.

Transformation related to the change of reference system **<u>must</u>** preserve the value of interval $s^2{}_{AB}$ between two arbitrary events: $s^2{}_{AB} = c^2(t_B - t_A)^2 - (\vec{r}_B - \vec{r}_A)^2$. An example of such transformation is rotation in 3D space which does not change time and preserves $(\vec{r}_B - \vec{r}_A)^2$. We should look for some type of rotation in 4D space which preserves the interval.

There are six independent rotation in 4D space: for example in planes xy, yz, zx, xt, yt, zt. Three of them are 3 independent rotation in 3D space. The rest are special - they rotate **THE TIME**. Let's consider xt "rotation", which does not change values of y and z. To preserve interval we should use hyperbolic functions instead of trigonometric:

$$\begin{aligned} x &= x'\cosh\psi + ct'\sinh\psi;\ \ y = y';\\ ct &= ct'\cosh\psi + x'\sinh\psi;\ z = z'; \end{aligned} \tag{1-11}$$

$$\begin{aligned} s^2 &= (ct'\cosh\psi + x'\sinh\psi)^2 - (x'\cosh\psi + ct'\sinh\psi)^2 - y'^2 - z'^2 = \\ &(ct')^2\left(\cosh^2\psi - \sinh^2\psi\right) - x'^2\left(\cosh^2\psi - \sinh^2\psi\right) - y'^2 - z'^2 = s'^2 \end{aligned}.$$

Let's relate the angle of "rotation" and the movement of K' origin $x' = 0$ (i.e. its velocity):

$$x = ct'\sinh\psi; ct = ct'\cosh\psi;\ \Rightarrow \frac{V}{c} = \frac{x}{ct} = \tanh\psi;$$

and yields final expression for Lorentz transformation:

$$\sinh\psi = \frac{V}{c}\Big/\sqrt{1-\frac{V^2}{c^2}} = \beta\gamma;\ \cosh\psi = 1\Big/\sqrt{1-\frac{V^2}{c^2}} = \gamma$$

with conventional dimensionless parameters $0 \le \beta < 1;\ 1 \le \gamma < \infty$:

$$\beta = \frac{V}{c}; \vec{\beta} = \frac{\vec{V}}{c};\ \gamma = 1\Big/\sqrt{1-\frac{V^2}{c^2}} = 1\Big/\sqrt{1-\beta^2}. \tag{1-12}$$

Therefore, the Lorentz transformation in compact form is:

$$x = \gamma(x' + \beta ct');\ \ ct = \gamma(ct' + \beta x'); y = y'; z = z'; \tag{1-13}$$

gives us all necessary relation to proceed further. The inverse Lorentz transformation is following from (1-13):

$$x' = \gamma(x - \beta ct);\ \ ct' = \gamma(ct - \beta x); y' = y; z' = z; \tag{1-14}$$

which gives us identity relations if combined with (1-13):

$$\begin{aligned} x &= \gamma(x' + \beta ct') = \gamma(\gamma(x - \beta ct) + \beta\gamma(ct - \beta x)) = \gamma^2(1-\beta^2)x = x;\\ ct &= \gamma(ct' + \beta x') = \gamma(c\gamma(ct - \beta x) + \beta\gamma(x - \beta ct)) = \gamma^2(1-\beta^2)ct = ct; \end{aligned} \tag{1-15}$$

using identity ratio:

$$\gamma^2(1-\beta^2) = \frac{1-\beta^2}{1-\beta^2} = 1. \tag{1-16}$$

More general approach to the same derivation (we leave aside y and z which do not transform). In matrix form interval is:

$$s^2 = X^T SX; \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \tag{1-17}$$

and arbitrary Lorentz transformation in (x,t) is:

$$X = L \cdot X'; \quad L = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \tag{1-18}$$

with condition to preserve 4-interval (we chose +):

$$L^T SL = S \;\Rightarrow \det L = \pm 1; \quad \text{"+"} \;\; ad - bc = 1; \tag{1-19}$$

$$X' = L^{-1} \cdot X; L^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Applying standard conditions : coordinates move with ±V:

$$x' = 0; \quad x = \beta ct; \;\; c = \beta a; \;\; \beta = V/c; \;\; x = 0; \quad x' = -\beta ct'; \;\; c = -\beta d; \Rightarrow a = d;$$

we got

$$L = \begin{bmatrix} a & b \\ \beta a & a \end{bmatrix}.$$

Constant speed of light gives the symmetry of (x,ct):

$$x = ct; x' = ct'; \begin{bmatrix} ct' \\ ct' \end{bmatrix} = L \begin{bmatrix} ct \\ ct \end{bmatrix}; \Rightarrow a + b = a + \beta a; \Rightarrow b = \beta a$$

Finally, detL=1 resolves the rest of puzzle:

$$L = a \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix}; \det L = 1 \Rightarrow a = \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{1-20}$$

<u>2.1</u> Proper Time, Proper Length and Proper Volume.

Proper time is defined in moving system K', i.e. in the rest frame of an object. (a clock). Let's consider a clock located in the origin of K'. Therefore, $d\vec{r}\,' = 0$ and we can write proper time for moving object:

$$ds^2 = c^2 dt^2 - d\vec{r}^{\,2} = c^2 dt'^2$$

$$dt' = dt\sqrt{1 - \frac{d\vec{r}^{\,2}}{c^2 dt^2}} = dt\sqrt{1 - \frac{v^2}{c^2}} = dt\sqrt{1-\beta^2}\,; \qquad (2\text{-}1)$$

$$t'_B - t'_A = \int_A^B dt\sqrt{1 - \frac{v^2}{c^2}} = \int_A^B \frac{dt}{\gamma}.$$

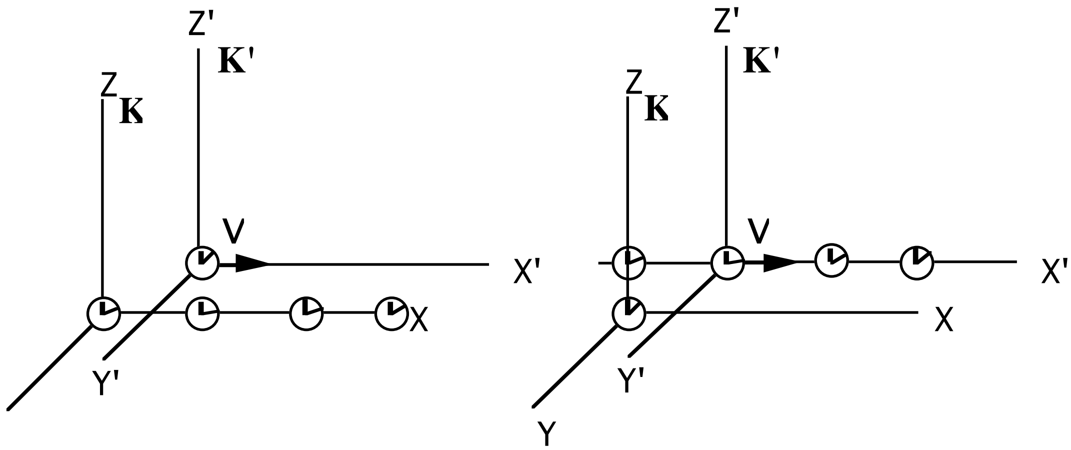

Fig. 4 To find the proper time at origin of K', we compare one clock in K' with set of clocks in K (left); to find proper time at origin of K', we compare one clock in K with set of clocks in K (right). This process is asymmetric and a clock compared with a set of clocks always lags behind.

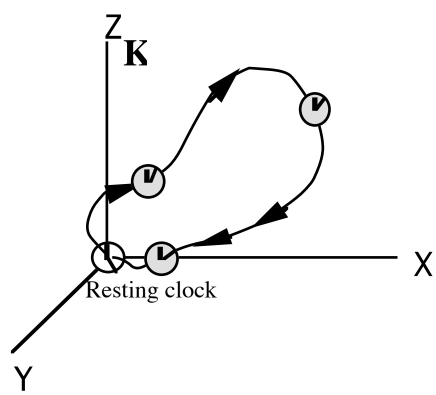

Fig. 5 The only correct way to compare clocks: use two clocks, start them at the same point of space, leave one at the rest and bring second at the same point to compare elapsed time. The clock at rest will show more time then moving clock. Why? It is impossible to return clock using rectilinear motion; i.e. moving clock must be accelerated. Therefore, the system related to traveling clock is not inertial and is not identical to inertial system where first clock rests.

Thus, a moving clock will show less time elapsed then a resting one. On other hand, we can look for the motion of K system from point of view of K'. Now we should locate a clock at the origin of K, and $d\vec{r} = 0$. Similar to eq. (2-1) we have:

$$dt = dt'\sqrt{1-\frac{d\vec{r}'^2}{c^2 dt'^2}} = dt\sqrt{1-\frac{v^2}{c^2}}; t_B - t_A = \int_A^B dt'\sqrt{1-\frac{v^2}{c^2}} = \int_A^B \frac{dt'}{\gamma}. \tag{2-2}$$

It looks as a contradiction: time in K' system is both faster and slower then in K system. What is not correct is to compare different clocks in the resting system with fixed one in the moving system. The solution of "paradox" is illustrated by Figures 4 and 5.

"Time paradox" is directly related to the *Lorentz contraction*. Suppose that there is a rod at rest in K system measured $l = x_B - x_A$ where x_B, x_A are coordinates of two end of the rod. We should determine length of the same rod in K' system: $x_A = \gamma(x'_A + \beta ct'); x_B = \gamma(x'_B + \beta ct');$ <u>at the same moment of time</u> t':

$$l' = x'_B - x'_A = (x_B - x_A)/\gamma = l/\gamma. \tag{2-3}$$

Therefore, observed from a moving system the resting rod contracts by factor γ. The same will be correct if we look from K system on the rot resting in K' system <u>at the same moment of time</u> t using $x'_A = \gamma(x_A - \beta ct); x'_B = \gamma(x_B - \beta ct);$

$$l = l'/\gamma. \tag{2-4}$$

Again, there is no contradiction. We are looking for the length of the rod by observing its ends at the same moment of time, but in different systems. The source of "asymmetry": time and space coordinates depend of the system of observation.

As we derived, coordinates transverse to the relative velocity of the system do not change $y' = y; z' = z$. Therefore, the volume of the body will decrease proportionally to the contraction of coordinate parallel to the relative velocity of the system (x). This volume is called <u>proper volume</u>:

$$V = V_0/\gamma. \tag{2-5}$$

To finish discussion, let's consider a synchronization procedure of the clocks. The natural way to set clocks located at different positions x in K system is to send periodical light signal from the origin and set them at time $t = x/c$ when light reach them. The traveling clock, fixed at origin of K', sees the distances in K system contracting by factor γ, and therefore the clock "thinks" that elapse time is $t' = x/\gamma c$.

What is most important that 4-dimentional volume

$$d\Omega = cdtdV \equiv dx^0 dx^1 dx^2 dx^3$$

is invariant of Lorentz transformations (we will discuss it at next lecture). It is direct consequence of the unit determinant of Lorentz transformation matrix:

$$d\Omega = \det[L] d\Omega';$$

$$L = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \ \det[L] = \gamma^2(1-\beta^2) = 1$$

2.2 Transformation of velocities.

Lorentz transformation of coordinates and time give us all necessary information to calculate velocity of the particles is arbitrary inertial system:

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{e}_x \frac{dx}{dt} + \hat{e}_y \frac{dy}{dt} + \hat{e}_z \frac{dz}{dt}; \vec{v}' = \frac{d\vec{r}'}{dt'} = \hat{e}_x \frac{dx'}{dt'} + \hat{e}_y \frac{dy'}{dt'} + \hat{e}_z \frac{dz'}{dt'};.$$

Let's rewrite (1-13) in form of differentials:

$$cdt = \gamma(cdt' + \beta dx'); dx = \gamma(dx' + \beta cdt'); dy = dy'; dz = dz'; \quad (2\text{-}6)$$

and divide coordinate differential by time differential:

$$v_x = \frac{dx}{dt} = \frac{c\gamma(dx' + \beta cdt')}{\gamma(cdt' + \beta dx')} = \frac{\frac{dx'}{dt'} + \beta c}{1 + \beta / c \cdot \frac{dx'}{dt'}} = \frac{v'_x + V}{1 + v'_x V / c^2};$$

$$v_y = \frac{dy}{dt} = \frac{cdy'}{\gamma(cdt' + \beta dx')} = \frac{\frac{dy'}{dt'}}{\gamma(1 + \frac{\beta dx'}{cdt'})} = \frac{v'_y \sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{v'_x V}{c^2}}; \quad (2\text{-}7)$$

$$v_z = \frac{dz}{dt} = \frac{cdz'}{\gamma(cdt' + \beta dx')} = \frac{dz' / dt'}{\gamma(1 + \beta / c \cdot dx' / dt')} = \frac{v'_z \sqrt{1 - \frac{V^2}{c^2}}}{1 + v'_x V / c^2}.$$

The transformation of velocities is more complex then transformation of space-time coordinates. It should not be of any surprise; e.g. the 3-D velocity is not a 4D object and it combines time and coordinates in "unnatural way for 4D world".

Let check that non-relativistic transformation of velocities is correct: for $|v| << c, |V| << c$ or $c \to \infty$. Truly, we got $v_x \cong v'_x + V; v_y \cong v'_y; v_z \cong v'_z$. Let see that speed of the light transfer itself in the speed of the light:

$$v' = c; \Rightarrow v'^2_x + v'^2_y + v'^2_z = c^2; \; v_y^2 + v_z^2 = c^2 - v'^2_x$$

$$\vec{v}^2 = v_x^2 + v_y^2 + v_z^2 = \frac{(v'_x + V)^2 + (v'^2_y + v'^2_z)(1 - V^2/c^2)}{(1 + v'_x V / c^2)^2} = \frac{(v'_x + V)^2 + (c^2 - v'^2_x)(1 - V^2/c^2)}{(1 + v'_x V / c^2)^2} = \quad (2\text{-}8)$$

$$\frac{v'^2_x + 2v'_x V + V^2 + c^2 - V^2 - v'^2_x + v'^2_x V^2 / c^2}{(1 + v'_x V / c^2)^2} = \frac{c^2 + 2v'_x V + v'^2_x V^2 / c^2}{(1 + v'_x V / c^2)^2} = c^2$$

Therefore, Lorentz transformation preserves speed of the light, as was required.

Before ending this section, let's look on generalization of Lorentz transformation when velocity of K' system is not parallel to the X-axis.

$$ct' = \gamma\left(ct - \vec{\beta}\vec{r}\right); \quad \vec{r}' = \vec{r} + \vec{\beta}\frac{\gamma - 1}{\beta^2}(\vec{\beta}\vec{r}) - \gamma\vec{\beta}ct; \quad (2\text{-}9)$$

where $\vec{\beta} = \vec{V} / c$. This transformation defers from (1-14) by rotation in 3D space of X-axis into direction of $\vec{V}$. It derivation is rather simple:

$$\beta x = \vec{\beta}\vec{r} = \gamma(\vec{\beta}\vec{r}\,' + \beta^2 ct');$$
$$\vec{r}_{\perp} = \hat{y}\cdot y + \hat{z}\cdot z = \vec{\beta}\times[\vec{r}\times\vec{\beta}]/\beta^2 = \vec{r} - \vec{\beta}(\vec{\beta}\vec{r})/\beta^2;\ \vec{r}_{\perp} = \vec{r}_{\perp}';$$
$$\vec{r} = \vec{r}_{\perp} + \vec{r}_{//} = \vec{r} - \vec{\beta}(\vec{\beta}\vec{r})/\beta^2 + \gamma\vec{\beta}(\vec{\beta}\vec{r}\,' + \beta^2 ct')/\beta^2$$

Other useful form of (2-9) is (using $\vec{\beta} = \hat{n}\beta$):

$$\vec{r} = \vec{r}_{\perp} + \vec{r}_{//} = \hat{n}\times[\vec{r}\,'\times\hat{n}] + \gamma\hat{n}(\hat{n}\vec{r}\,' + \beta ct');$$
$$ct = \gamma(ct' + \beta\hat{n}\vec{r}\,');$$