

PHY 554

Fundamentals of Accelerator Physics

Lecture 15: Beam Dynamics in an Electron Storage Ring Part 2

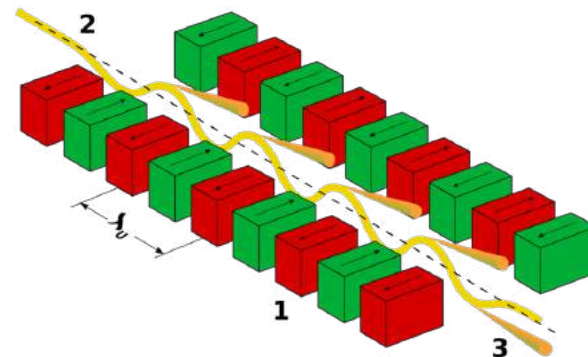
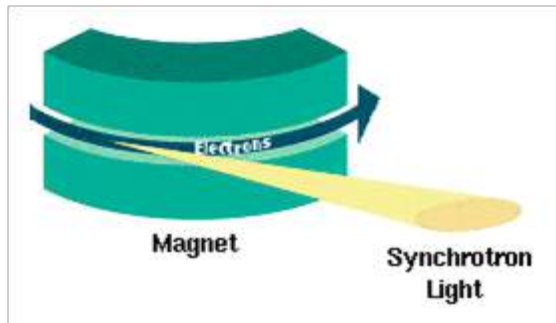
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Why electron/positron storage rings are different?

SYNCHROTRON RADIATION

- Energy emitted to infinity or other boundary condition.
 - Form: Electromagnetic waves
 - Source: acceleration of charged particles
 - Direction: Along the tangent of the beam trajectory



Effects of Synchrotron Radiation

- ❑ In last class we learned that SR could cause damping of betatron and synchrotron oscillations.
- ❑ Does it mean that beam sizes will collapse to zero and we will have zero beam sizes and infinitely dense beam?
- ❑ The answer is no – there are effects which prevent this collapse
- ❑ Today we will study the main effect which does exactly this – quantum fluctuations of synchrotron radiation

Radiation Damping

$$\mathbf{E}_o = \gamma_o mc^2; \gamma \gg 1, \beta \cong 1; r_c = e^2 / mc^2$$

$$\frac{\Delta E_{SR}}{\mathbf{E}_o} = -\frac{2e^2}{3\mathbf{E}_o \beta_o^2} \gamma_o^4 I_2 \cong -\frac{2r_c}{3} \gamma_o^3 \cdot I_2$$

$$\bar{D} = \frac{I_4}{I_2}; I_4 = \oint ds \cdot DK_o (2K_x - K_o^2)$$

$$\xi_y(C) \cong \frac{r_c}{3} \gamma_o^3 \cdot I_2$$

$$I_2 = \oint_C \frac{ds}{\rho^2} \equiv \oint_C K_o^2(s) ds$$

$$y = a_y \sqrt{\beta_y} \cos \psi_y e^{-\xi_y(s)}$$

$$\xi_x(C) = \frac{r_c}{3} \gamma_o^3 \cdot I_2 (1 - \bar{D})$$

$$x = a_x \sqrt{\beta_x} \cos \psi_x e^{-\xi_x(s)}$$

$$\xi_{SR} = -\frac{\Delta E_{SR}}{2\mathbf{E}_o}; \quad \xi_{x,y,s} = \zeta_{x,y,s} \cdot \xi_{SR};$$

$$\zeta_x = 1 - \bar{D} \quad \zeta_y = 1 \quad \zeta_s = 2 + \bar{D}$$

$$\zeta_x + \zeta_y + \zeta_s = 4$$

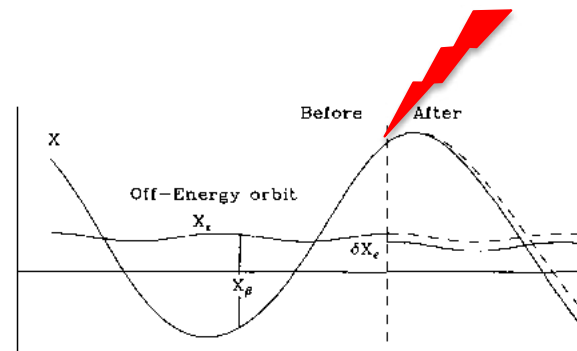
$$\xi_s = \frac{r_c}{3} \gamma_o^3 \cdot I_2 (2 + \bar{D})$$

$$\delta = a_\delta \cos(n\Omega_s + \varphi) \cdot e^{-n\xi_s}$$

Quantum Fluctuations of Synchrotron Radiation

- Synchrotron radiation is not a continuous emission
- Instead, photons are radiated at random moments and energy loss has quantum nature
- The emission obeys Poisson distribution
- It serves as a source of the noise that excites (heat) motion of particles in the electron beam.
- Since energy jumps are random there is not correlations, and we need to find how to handle this source of “random noise”

Sudden change in energy δE_{ph}
not only excite energy
oscillations but also induce
horizontal oscillations about
“new $E - \delta E_{ph}$ orbit”



How to handle random jumps?

- 1 D case (Brownian motion) – no definitive x

$$\dot{x} = \sum \delta x_i \cdot \delta(t - t_i); \langle \delta x_i \rangle = 0 \Rightarrow x(t) = \sum_{t > t_i} \delta x_i = ?$$

- It is not true for x^2

$$\delta x^2 = (x + \delta x_i)^2 - x^2 = 2x\delta x_i + \delta x_i^2 \rightarrow \langle \delta x^2 \rangle = 2\langle x\delta x_i \rangle \downarrow_0 + \langle \delta x_i^2 \rangle > 0$$

$$\langle x^2(t + \Delta t) \rangle = \langle x^2(t) \rangle + \left\langle \sum_{t < t_i < t + \Delta t} \delta x_i^2 \right\rangle = \langle x^2(t) \rangle + \dot{N} \langle \delta x_i^2 \rangle \Delta t$$

- Rate of jumps and diffusion coefficient:

$$\dot{N}(t) = \lim_{\Delta t \rightarrow 0} \frac{\left\langle \sum_{t < t_i < t + \Delta t} \delta x_i^2 \right\rangle}{\Delta t \cdot \langle \delta x_i^2 \rangle}; \quad \frac{d\langle x^2 \rangle}{dt} = \mathbf{D}; \quad \mathbf{D} = \dot{N}(t) \langle \delta x_i^2 \rangle = \langle \dot{N} \delta x_i^2 \rangle$$

- Statistically average $\langle x^2 \rangle$ is well defined!

How to handle random noise?

- What about an oscillator?

$$x = a \cdot \sin \varphi; p = a\omega \cdot \cos \varphi; x \rightarrow x + \delta x_i; p \rightarrow p + \delta p_i$$

$$a^2 = x^2 + \frac{p^2}{\omega^2}; \langle \delta a^2 \rangle = 2 \langle x \delta x_i \rangle \downarrow_0 + \frac{2}{\omega^2} \langle p \delta p_i \rangle \downarrow_0 + \langle \delta x_i^2 \rangle + \frac{\langle \delta p_i^2 \rangle}{\omega^2} > 0$$

- Diffusion coefficient for amplitude squared

$$N'_x(s) = \lim_{\Delta s \rightarrow 0} \frac{\left\langle \sum_{s < s_i < s + \Delta s} \delta x_i^2 \right\rangle}{\Delta s \langle \delta x_i^2 \rangle}; N'_p(s) = \lim_{\Delta s \rightarrow 0} \frac{\left\langle \sum_{s < s_j < s + \Delta s} \delta p_j^2 \right\rangle}{\Delta s \langle \delta p_j^2 \rangle};$$

$$\frac{d \langle a^2 \rangle}{ds} = \mathbf{D}(s); \mathbf{D}(s) = \langle N'_x \delta x_i^2 \rangle + \frac{\langle N'_p \delta p_j^2 \rangle}{\omega^2}$$

- Statistically average a^2 is well defined!

Noise +damping

- 1 D case

$$\dot{x} = -\xi x + \sum \delta x_i \cdot \delta(t - t_i); \langle \delta x_i \rangle = 0; t_1 = t + \Delta t; d\tilde{x} = \sum_{t < t_i < t + \Delta t} \delta x_i$$

$$\Delta x^2 = (x + \delta\tilde{x} - \xi x \cdot \Delta t)^2 - x^2 = 2x(\delta\tilde{x} - \xi x \cdot \Delta t) + (\delta\tilde{x} - \xi x \cdot \Delta t)^2$$

$$\langle \Delta x^2 \rangle = -2\xi \langle x^2 \rangle \Delta t + \mathbf{D} \Delta t \rightarrow \frac{d\langle x^2 \rangle}{dt} = \mathbf{D} - 2\xi \langle x^2 \rangle$$

- Statistically average x^2 is well defined

$$\frac{d\langle x^2 \rangle}{dt} = \mathbf{D} - 2\xi \langle x^2 \rangle = 0 \Rightarrow \langle x^2 \rangle = \frac{\mathbf{D}}{2\xi} = \frac{\langle \dot{N} \delta x_i^2 \rangle}{2\xi}$$

Noise +damping

- Damping of the oscillator?

$$a = a_o e^{-\xi(s)} \Rightarrow \frac{da^2}{ds} = -2\xi'(s)a^2$$

- Combine with diffusion and assume slow evolution

$$\frac{d\langle a^2 \rangle}{ds} = \mathbf{D}(s) - 2\xi'(s)\langle a^2 \rangle \Rightarrow \frac{d\langle a^2 \rangle}{ds} = \langle \mathbf{D}(s) \rangle - 2\langle \xi'(s) \rangle \langle a^2 \rangle$$

- Defines average action

$$\langle a^2 \rangle = \frac{\langle \mathbf{D}(s) \rangle}{2\langle \xi'(s) \rangle}; \quad \mathbf{D}(s) = \langle N'_x \delta x_i^2 \rangle + \frac{\langle N'_x \delta p_j^2 \rangle}{\omega^2}$$

Quantum Fluctuations of Synchrotron Radiation

- If ε is the emitted photon energy, the average amplitude of energy deviation is

$$\frac{d\langle \delta^2 \rangle}{dt} = \frac{\langle \dot{N}_i \varepsilon_i^2 \rangle}{\mathbf{E}_o^2}; \langle \dot{N}_i \varepsilon_i^2 \rangle = \sum_i \dot{N}_i \varepsilon_i^2 \cong \int \varepsilon^2 f(\varepsilon) d\varepsilon$$

$$\left(\frac{\delta}{a_\delta} \right)^2 + \left(\frac{\Delta\phi}{a_\phi} \right)^2 = 1 \quad \frac{a_\delta}{a_\phi} = \sqrt{\frac{eV_{rf} |\cos\phi_s|}{2\pi h |\eta| \beta^2 \mathbf{E}_o}} = \frac{Q_s}{h |\eta|}$$

$$\frac{d\langle a_\delta^2 \rangle}{dt} = \frac{1}{\mathbf{E}_o^2} \int \varepsilon^2 f(\varepsilon) d\varepsilon - 2 \frac{\xi_s}{T_o} \langle a_\delta^2 \rangle; T_o = \frac{C}{v_o} \cong \frac{C}{c}$$

- And we need to evaluate the integral on RHS

How to calculate $\langle u^2 \rangle$

- We need to use spectral properties of synchrotron radiation

$$P_{SR} = \sum_i \bar{n}_i \varepsilon_i = \int_0^\infty \dot{n}(\varepsilon) d\varepsilon \equiv \int_0^\infty U(u) du; \quad u = \frac{\varepsilon}{\varepsilon_c} = \frac{\hbar\omega}{\hbar\omega_c} \equiv \frac{\omega}{\omega_c}; \quad \omega_c = \frac{3}{2} \frac{c}{|\rho|} \gamma^2$$

$$U(u) = \frac{3\sqrt{3}}{4\pi} \frac{e^2 c}{\rho^2} \gamma^4 \cdot u \int_u^\infty K_{5/3}(u') du'$$

similarly, we can calculate diffusion

$$\sum_i \bar{n}_i \varepsilon_i^2 = \int_0^\infty \varepsilon \cdot \dot{n}(\varepsilon) d\varepsilon \equiv \hbar\omega_c \int_0^\infty u U(u) du; \quad \varepsilon = u \cdot \hbar\omega_c$$

$$\sum_i \bar{n}_i \varepsilon_i^2 = \frac{3\sqrt{3}}{4\pi} \frac{e^2 c}{\rho^2} \gamma^4 \cdot \hbar\omega_c \int_0^\infty \left(u^2 \int_u^\infty K_{5/3}(u') du' \right) du = \frac{55}{24\sqrt{3}} \frac{\gamma^7}{|\rho|^3} \hbar c^2 e^2$$

$$\mathbf{D}_\delta = \frac{\langle \delta E^2 \rangle}{E_o^2} = \frac{55}{24\sqrt{3}} \gamma^5 \frac{c}{|\rho|^3} \frac{\hbar}{mc} \frac{e^2}{mc^2} \equiv \frac{55}{24\sqrt{3}} \gamma^5 \frac{c}{|\rho|^3} \hat{\lambda}_c r_c$$

$$\hat{\lambda}_c = \frac{\hbar}{mc}; \quad r_c = \frac{e^2}{mc^2}$$

Equilibrium energy spread

- The damping and excitation will reach a steady state.

$$\langle a_\delta^2 \rangle = \frac{\langle \mathbf{D}_\delta \rangle}{2\xi_s} \cdot \frac{C_o}{c} = \frac{1}{2\xi_s c} \oint \mathbf{D}_\delta ds, \quad \xi_s = \frac{r_c}{3} \gamma_o^3 \cdot I_2 (2 + \bar{D})$$

$$\oint \mathbf{D}_\delta ds / c = \frac{55}{24\sqrt{3}} \gamma_o^5 \tilde{\lambda}_c r_c I_3; \quad I_3 = \oint \frac{ds}{|\rho(s)|^3} \equiv \oint |K_o^3(s)| ds;$$

$$\langle a_\delta^2 \rangle = \frac{55}{16\sqrt{3}} \tilde{\lambda}_c \gamma_o^2 \cdot \frac{I_3}{I_2 (2 + \bar{D})}$$

Since phases of oscillations are random, $\langle \cos^2(\Omega_s n + \varphi) \rangle = \frac{1}{2}$
and the relative rms energy spread is twice larger than $\langle a_\delta^2 \rangle$

$$\delta = a_\delta \cos(\Omega_s n + \varphi); \quad \sigma_\delta^2 \equiv \langle \delta^2 \rangle = \langle a_\delta^2 \rangle \langle \cos^2(\Omega_s n + \varphi) \rangle = \frac{\langle a_\delta^2 \rangle}{2}$$

$$\sigma_\delta^2 = \frac{55}{32\sqrt{3}} \tilde{\lambda}_c \gamma_o^2 \cdot \frac{I_3}{I_2 (2 + \bar{D})}$$

Equilibrium energy spread II

- The equilibrium of the relative energy spread:

$$\sigma_\delta^2 \equiv \left(\frac{\sigma_E}{E_o} \right) = \frac{55}{32\sqrt{3}} \hat{\lambda}_c \gamma_o^2 \cdot \frac{I_3}{I_2(2 + \bar{D})} = \frac{C_q \cdot \gamma_o^2}{(2 + \bar{D})} \frac{\langle |\rho|^{-3} \rangle}{\langle \rho^{-2} \rangle};$$

Radiation Integral I_3

Radiation Integral I_2

$$C_q = \frac{55}{32\sqrt{3}} \hat{\lambda}_c = 3.83 \cdot 10^{-13} m$$

- It does not depend on the RF voltage, but the bunch length does:

$$\frac{\sigma_\delta}{\sigma_\phi} = \frac{a_\delta}{a_\phi} = \sqrt{\frac{eV_{rf} |\cos \phi_s|}{2\pi h |\eta| \beta^2 E_o}} = \frac{Q_s}{h |\eta|} \rightarrow \sigma_\phi = \frac{h |\eta|}{Q_s} \sigma_\delta$$

Excitation of horizontal betatron oscillations

- Radiation of the photon changes electron's energy, but does not change its location

$$\Delta E = -\varepsilon_{ph}; \Delta x = 0; \Delta x' = -x' \cdot \frac{\varepsilon_{ph}}{E_o};$$

- At the same time, the closed orbit for new electron changes, which results in excitation of betatron oscillations

$$\Delta x = \Delta x_{\beta} + D \frac{\Delta E}{E_o} = 0; \Delta x' = \Delta x'_{\beta} + D' \frac{\Delta E}{E_o} = -x' \cdot \frac{\varepsilon_{ph}}{E_o};$$

$$\Delta x_{\beta} = -D \frac{u_{ph}}{E_o}; \Delta x'_{\beta} = -D' \frac{\varepsilon_{ph}}{E_o} - x' \cdot \frac{\varepsilon_{ph}}{E_o}$$

- The second term on the right-hand side with $\Delta x'$ is a part of radiation damping, and is already considered

Excitation of horizontal betatron oscillations

- Let's calculate the induced amplitude of betatron oscillations

$$x = a_x \sqrt{\beta_x} \cdot \cos \psi; x' = -\frac{a_x}{\sqrt{\beta_x}} \cdot (\sin \psi + \alpha_x \cos \psi)$$

$$\Delta(a_x \sqrt{\beta_x} \cdot \cos \psi) = -D \frac{\epsilon_{ph}}{E_o}; \Delta\left(\frac{a_x}{\sqrt{\beta_x}} (\alpha_x \cos \psi + \sin \psi)\right) = D' \frac{\epsilon_{ph}}{E_o}$$

$$\Delta a_x \cdot \cos \psi - \sin \psi \cdot a_x \Delta \psi = -\frac{D}{\sqrt{\beta_x}} \frac{\epsilon_{ph}}{E_o}$$

$$\Delta a_x (\alpha_x \cos \psi + \sin \psi) + (\cos \psi - \alpha_x \sin \psi) \cdot a_x \Delta \psi = \sqrt{\beta_x} D' \frac{\epsilon_{ph}}{E_o}$$

$$\begin{bmatrix} \cos \psi & -\sin \psi \\ \alpha_x \cos \psi + \sin \psi & \cos \psi - \alpha_x \sin \psi \end{bmatrix} \cdot \begin{bmatrix} \Delta a_x \\ a_x \Delta \psi \end{bmatrix} = \frac{\epsilon_{ph}}{E_o} \begin{bmatrix} -\frac{D}{\sqrt{\beta_x}} \\ \sqrt{\beta_x} D' \end{bmatrix}$$

$$\begin{bmatrix} \Delta a_x \\ a_x \Delta \psi \end{bmatrix} = \frac{\epsilon_{ph}}{E_o} \begin{bmatrix} \cos \psi - \alpha_x \sin \psi & \sin \psi \\ -\alpha_x \cos \psi - \sin \psi & \cos \psi \end{bmatrix} \begin{bmatrix} -\frac{D}{\sqrt{\beta_x}} \\ \sqrt{\beta_x} D' \end{bmatrix}$$

- giving us

$$\Delta a_x = -\frac{\epsilon_{ph}}{E_o} \left(\frac{D}{\sqrt{\beta_x}} (\cos \psi - \alpha_x \sin \psi) - \sqrt{\beta_x} D' \sin \psi \right); \langle \cos \psi \sin \psi \rangle = 0; \langle \cos^2 \psi \rangle = \langle \sin^2 \psi \rangle = \frac{1}{2}$$

$$\langle \Delta a_x^2 \rangle = \frac{1}{2 E_o^2} \left\langle \epsilon_{ph}^2 \left(\frac{D^2}{\beta_x} (1 + \alpha_x^2) + 2 D' D \alpha_x + \beta_x D'^2 \right) \right\rangle = \frac{\langle \epsilon_{ph}^2 H \rangle}{2 E_o^2};$$

$$H(s) = \frac{D^2 (1 + \alpha_x^2) + 2 D' D \beta_x \alpha_x + (\beta_x D')^2}{\beta_x} = \gamma_x \cdot D^2 + 2 \alpha_x \cdot D' D + \beta_x D'^2$$

Horizontal betatron oscillations

- Equation for the amplitude square is

$$\frac{d\langle a_x^2 \rangle}{dt} = \mathbf{D}_x - 2\xi'_x \langle a_x^2 \rangle; \mathbf{D}_x = \frac{H\langle \varepsilon_{ph}^2 \rangle}{2E_o^2} = \frac{55c}{24\sqrt{3}} \gamma^5 \tilde{\lambda}_c r_c \frac{H(s)}{|\rho|^3};$$

$$\frac{d\langle a_x^2 \rangle}{dn} = \oint \mathbf{D}_x ds - 2\xi_x(C) \langle a_x^2 \rangle; \xi_x(C) = \frac{r_c}{3} \gamma_o^3 \cdot I_2(1 - \bar{D}); \oint \mathbf{D}_x ds = \frac{55c}{24\sqrt{3}} \gamma^5 \tilde{\lambda}_c r_c I_5$$

$$I_5 = \oint \frac{H(s)}{|\rho(s)|^3} ds \equiv \oint ds |K_o(s)|^3 \left(\gamma_x(s) \cdot D(s)^2 + 2\alpha_x(s) \cdot D'(s) D(s) + \beta_x(s) D'(s)^2 \right)$$

- Or in simplified form

$$\frac{d\langle a_x^2 \rangle}{dn} = \frac{55c}{24\sqrt{3}} \gamma^5 \tilde{\lambda}_c r_c \cdot I_5 - \frac{2r_c}{3} \gamma_o^3 \cdot I_2(1 - \bar{D})$$

Transverse equilibrium

- Horizontal emittance

$$\frac{d\langle a_x^2 \rangle}{dn} = \frac{55c}{24\sqrt{3}} \gamma_o^5 \tilde{\lambda}_c r_c \cdot I_5 - \frac{2r_c}{3} \gamma_o^3 \cdot I_2 (1 - \bar{D}) = 0;$$

$$\epsilon_x = \left\langle \frac{a_x^2}{2} \right\rangle = \frac{55c}{32\sqrt{3}} \gamma_o^2 \tilde{\lambda}_c \cdot \frac{I_5}{I_2 (1 - \bar{D})} = C_q \cdot \gamma_o^2 \cdot \frac{I_5}{I_2 (1 - \bar{D})};$$

$$C_q \frac{55}{32\sqrt{3}} \tilde{\lambda}_c = 3.83 \cdot 10^{-13} m$$

- and beam size

$$\sigma_x^2 = \left\langle \left(x_\beta + D\delta \right)^2 \right\rangle = \left\langle x_\beta^2 \right\rangle + 2D \left\langle x_\beta \delta \right\rangle_0 + D^2 \left\langle \delta^2 \right\rangle$$

$$\left\langle x_\beta^2 \right\rangle = \beta_x \left\langle a_x^2 \right\rangle \left\langle \cos^2 \psi \right\rangle = \beta_x \epsilon_x \Rightarrow \sigma_x^2 = \beta_x \epsilon_x + D^2 \sigma_\delta^2$$

Vertical emittance

- In storage ring with ideal plane orbit, there is no direct coupling between energy jumps and vertical oscillations. The same treatment as for horizontal direction will result in zero vertical emittance
- There is minuscule diffusion in vertical direction as result of finite angular spread of the radiated photons:

$$\delta y' \propto \frac{1}{\gamma_o} \frac{\varepsilon_{ph}}{E_o} \Rightarrow \mathbf{D}_y \propto \frac{\mathbf{D}_x}{\gamma_o^2}$$

- Estimated vertical equilibrium would not depend on the beam energy and would be unrealistically small

$$\varepsilon_y \propto C_q \frac{\langle \beta_z \rangle}{\rho}$$

- IN reality, vertical emittance is defined by coupling with horizontal motion. It is result of imperfections in the beam's orbit and skew-quadrupole errors

Transverse Coupling (definition)

- Coupling of emittances κ is defined as

$$\varepsilon_{nat} = C_q \cdot \gamma_o^2 \cdot \frac{I_5}{I_2(1 - \bar{D})}$$

$$\varepsilon_x = \frac{1}{1 + \kappa} \varepsilon_{nat} \quad \varepsilon_y = \frac{\kappa}{1 + \kappa} \varepsilon_{nat}$$

Summary: Radiation Integral

Index	Integrals	Properties
1	$I_1 = \oint D/\rho ds$	$\alpha_c = I_1 \times C$
2	$I_2 = \oint 1/\rho^2 ds$	$U_{SR} = \frac{C_\gamma E^4}{2\pi} I_2$
3	$I_3 = \oint 1/ \rho ^3 ds$	$\left(\frac{\sigma_E}{E}\right)^2 = C_q \gamma^2 \frac{I_3}{2I_2 + I_4}$
4	$I_4 = \oint (D/\rho) (1/\rho^2 + 2K(s)) ds$	$\bar{D} = I_4/I_2$
5	$I_5 = \oint H(s) / \rho ^3 ds$	$\epsilon_x = C_q \gamma^2 \frac{I_5}{I_2 - I_4}$

Typical “good” numbers

- Revolution time: \sim micro-second (300 m)
- Longitudinal oscillation: sub-millisecond
- Damping time: few thousand turns
 - Several millisecond
- Relative energy spread $\sim 10^{-3}$
- Rms transverse emittance \sim nm-rad
- Rms vertical emittance \sim few pm-rad

Additional sources of emittance growth

- Emittance and energy spread can also grow from other effects, for example:
 - Scattering on residual gas
 - Collisions of particles inside the beam (called intra-beam scattering or IBS)
 - Beam instabilities (will be discussed later in this course)
 - High frequency noise in RF and magnetic fields

Beam life-time

- Quantum lifetime
 - Although the equilibrium emittance is small, there is chance that, for one single electron, continuous random emission drive the electron out of aperture
 - Longitudinal or Transverse.
- Touschek lifetime
 - Coulomb scattering in the bunch may transfer transverse momentum to longitudinal plane and cause beam loss.