Free Electron Lasers

Outline

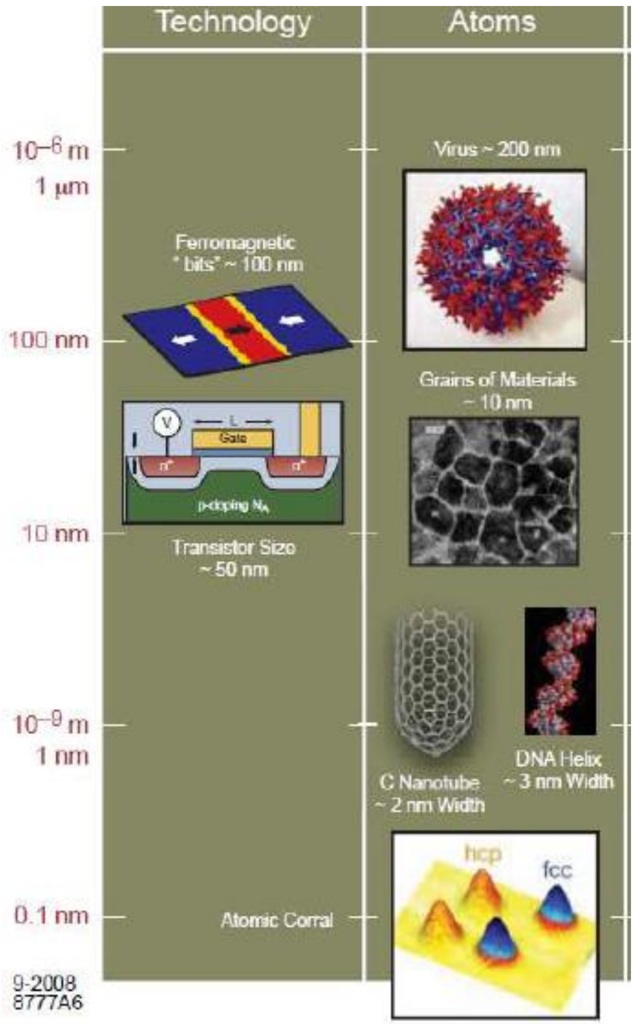
- Introduction
  - What is free electron laser (FEL)
  - Applications and some FEL facilities
  - Basic setup
  - Different types of FEL
- How FEL works
  - Electrons’ trajectory and resonant condition
  - Analysis of FEL process at small gain regime (Oscillator)
  - Analysis of FEL process at high gain regime (Amplifier)

Introduction I: What is free electron lasers

- A free-electron laser (FEL), is a type of laser whose lasing medium consists of very-high-speed electrons moving freely through a magnetic structure, hence the term free electron.
- The free-electron laser was invented by John Madey in 1971 at Stanford University.
- Advantages:
  - ✓ Wide frequency range
  - ✓ Tunable frequency
  - ✓ May work without a mirror (SASE)
- Disadvantages: large, expensive

Introduction II: Applications and FEL facilities

- Medical, Biology (small wavelength and short pulse are required for imaging proteins), Military (~Mwatts)…

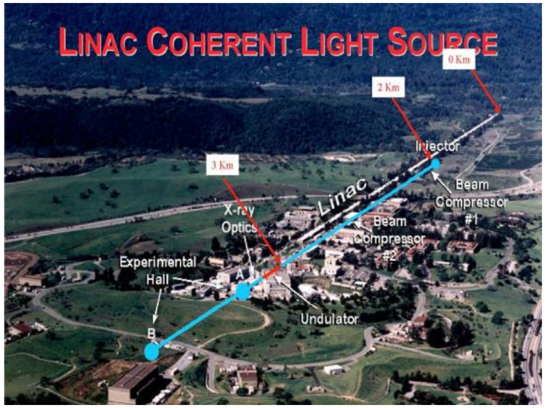


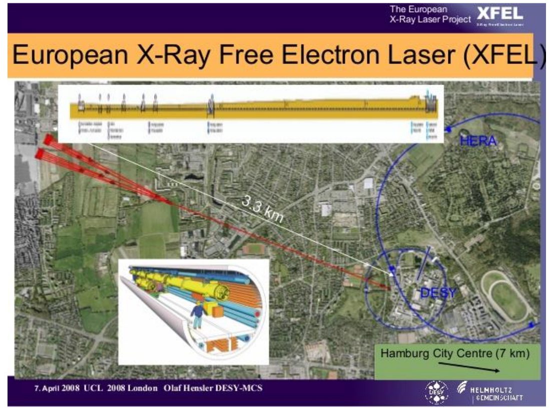
- Operational FEL light sources worldwide (~20):

Free Electron Laser User Facilities Worldwide

LOCATION	FACILITY NAME	WAVELENGTHS	Acc. Type
RIKEN, JAPAN	SACLA	0.63 – 3 Å	NC Linac
SLAC, USA	LCLS	1.2 – 15 Å	NC Linac
DESY, Germany	FLASH	4.1 – 45 nm	SC Linac
ELETTRA, Italy	FERMI	4 – 100 nm	NC Linac
Osaka U., Japan	iFEL	230 nm – 100 μm	NC Linac
Radboud U. Netherlands	FELIX	25 – 420 μm	NC Linac
LURE-Orsay, France	CLIO	3 – 150 μm	NC Linac
Jefferson Lab., USA	Jlab FEL	363 – 438 nm 3.2 – 4.8 μm	SC Linac
SUT, Japan	FEL-SUT	5 – 16 μm	NC Linac
FZ Rossendorf, Germany	FELBE	4 – 250 μm	SC Linac
UCSB, USA	ITST	30 μm – 2.5 mm	electrostatic
PSI, Switzerland	Swiss FEL	1 – 70 Å	NC Linac
DESY, Germany	Euro-XFEL	0.5 – 47 Å	SC Linac
Shanghai, China	SXFEL	1.2 – 10 nm	NC Linac
Dalian, China	DCLS	50 – 150 nm	NC Linac

Introduction III: FEL facilities





PAL XFEL, South Korea

FEL User Facilities for Scientific Research

The Shanghai Soft X-ray Free-Electron Laser Facility.

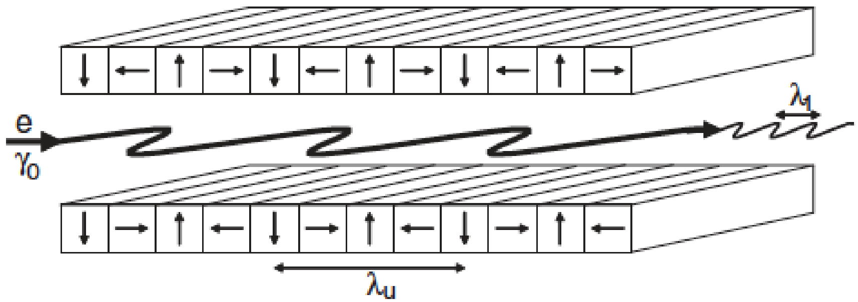
Shanghai SXFEL, China

Introduction IV: Basic Setup

Planar undulator

$$B_y(x,y,z) = B_0 \sin(k_u z)$$

for $x, y << \text{gap size}$



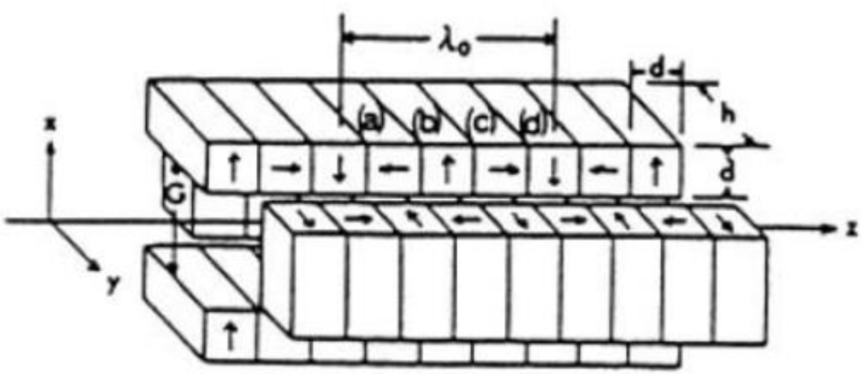
Helical wiggler for CeC PoP

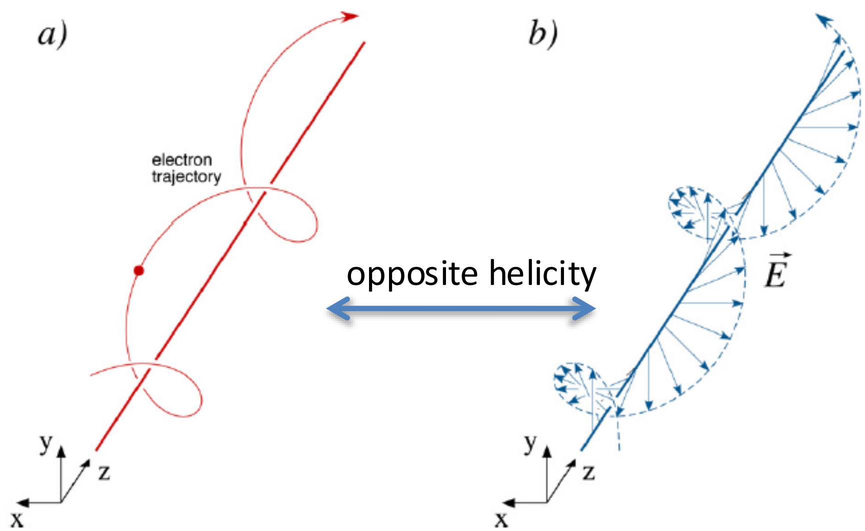
Helical undulator

$$B_x(x,y,z) = B_0 \cos(k_u z)$$

$$B_y(x,y,z) = B_0 \sin(k_u z)$$

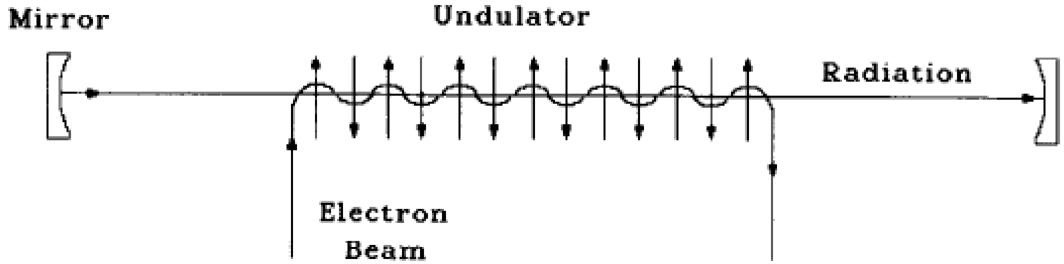
for $x, y << \text{gap size}$



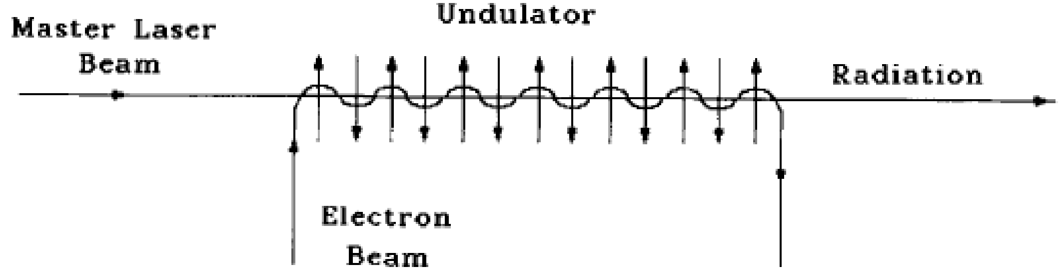


Introduction V: different types of FEL

FEL Oscillator
(Low gain regime)

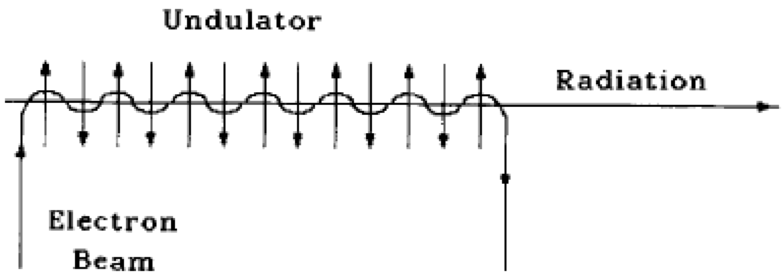


FEL Amplifier
(High gain regime)



SASE FEL
(High gain regime)

Self-Amplified Spontaneous Emittion (SASE)



Unperturbed Electron motion in helical wiggler (in the absence of radiation field)

$$\vec{B}_w(x,y,z) = B_w\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right]$$

$$\vec{F}(x,y,z) = -e\vec{v}\times\vec{B} = -ev_z\hat{z}\times\vec{B} = -ev_z B_w\left[\cos(k_u z)\hat{y} + \sin(k_u z)\hat{x}\right]$$

$$\frac{d(m\gamma v_x)}{dt} = m\gamma\frac{dv_x}{dt} = -ev_z B_w\sin(k_u z) \qquad \frac{d(m\gamma v_y)}{dt} = m\gamma\frac{dv_y}{dt} = -ev_z B_w\cos(k_u z)$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \qquad v = \sqrt{v_x^2+v_y^2+v_z^2} \qquad \tilde{v} \equiv v_x + iv_y$$

$$m\gamma\frac{d\tilde{v}}{dt} = -iev_z B_w\left(\cos(k_u z) - i\sin(k_u z)\right) = -iev_z B_w e^{-ik_u z}$$

$$m\gamma\frac{d\tilde{v}}{dt} = m\gamma\frac{dz}{dt}\frac{d\tilde{v}}{dz} = -iev_z B_w e^{-ik_u z} \Rightarrow m\gamma\frac{d\tilde{v}}{dz} = -ieB_w e^{-ik_u z}$$

Undulator parameter, also called a_w

$$K \equiv \frac{eB_w\lambda_w}{2\pi mc}$$

Electron rotation angle in undulator:

$$\theta_s = K/\gamma$$

$$\frac{\tilde{v}(z)}{c} = \frac{-ieB_w}{mc\gamma}\int e^{-ik_u z_1}dz_1 = \frac{eB_w}{mc\gamma k_u}e^{-ik_u z} = \frac{K}{\gamma}e^{-ik_u z}$$

*Assume the initial velocity of the electron make the integral constant vanishing.

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right] \qquad v_z = const. \qquad \vec{x}(z) = \int_0^z \vec{v}(t_1)dt_1 + \vec{x}(z=0)$$

Energy change of electrons due to radiation field

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right]$$

Consider a circularly polarized electromagnetic wave (plane wave is an assumption for 1D analysis, which is usually valid for near axis analysis) propogating along z direction

$$\vec{E}_\perp(z,t) = E\left[\cos(kz - \omega t)\hat{x} + \sin(kz - \omega t)\hat{y}\right] \qquad E_z = 0$$

$$= E\left[\cos(k(z - ct))\hat{x} + \sin(k(z - ct))\hat{y}\right] \qquad w = kc$$

Energy change of an electron is given by $\frac{d\mathcal{E}}{dt} = \vec{F}\cdot\vec{v} = -e\vec{v}_\perp \cdot \vec{E}_\perp$

To the leading order, electrons move with constant velocity and hence $z = v_z(t - t_0)$

$$\frac{d\mathrm{E}}{dt} = \frac{dz}{dt}\frac{d\mathrm{E}}{dz} = v_z\frac{d\mathrm{E}}{dz} \Rightarrow \frac{d\mathrm{E}}{dz} = \frac{1}{v_z}\frac{d\mathrm{E}}{dt}$$

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s\frac{c}{v_z}\cos(\psi) \approx -eE\theta_s\cos(\psi)$$

Pondermotive phase:

$$y = k_u z + k(z - ct)$$

Resonant Radiation Wavelength

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \cos\left[\left(k_w + k - k\frac{c}{v_z}\right)z + \psi_0\right]$$

We define the resonant radiation wavelength such that

$$k_w + k_0 - k_0\frac{c}{v_z} = 0 \Rightarrow \lambda_0 = \lambda_w\left(\frac{c}{v_z} - 1\right) \approx \frac{\lambda_w}{2\gamma_z^2}$$

$$\gamma_z^{-2} \equiv 1 - v_z^2/c^2 = 1 - \left(v_z^2 + v_\perp^2\right)/c^2 + v_\perp^2/c^2 = \gamma^{-2} + \theta_s^2 = \gamma^{-2}\left(1 + K^2\right)$$

FEL resonant frequency:

$$\boxed{\lambda_0 \approx \frac{\lambda_w\left(1 + K^2\right)}{2\gamma^2}}$$

$$K \equiv \frac{eB_w\lambda_w}{2\pi mc}$$

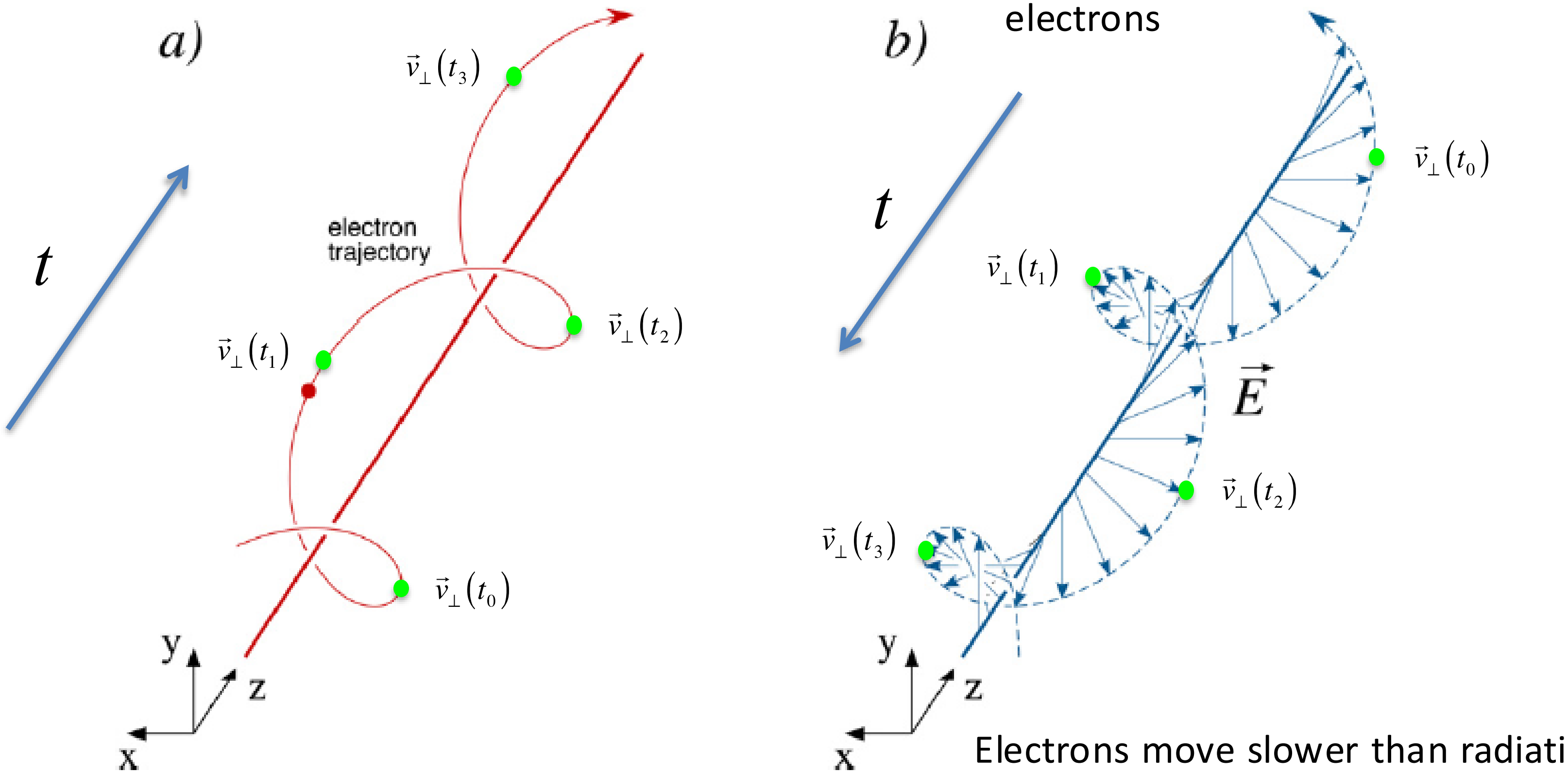
At resonant frequency, the rotation of the electron and the radiation field is synchronized in the x-y plane and hence the energy exchange between them is most efficient.

Helicity of radiation at synchronization

The synchronization requires opposite helicity of radiation with respect to the electrons' trajectories.

Electrons' trajectories

Radiation field observed by electrons



$$t_0 < t_1 < t_2 < t_3$$

Electrons move slower than radiation and hence see the radiation wave slipping ahead. As a result, the rotation direction of the radiation field seen by an electron is the same as its own rotation direction.

Longitudinal equation of motion

In the presence of the radiation field, the longitudinal equation of motion of an electron read

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \cos(\psi) \qquad y = k_w z + k(z - ct)$$

$\mathcal{E}_0$ is the average energy of the beam.

$$\frac{d}{dz}\psi = k_w + k - \frac{\omega}{v_z(\mathcal{E})}$$

$$\approx k_w + k - \omega\left[\frac{1}{v_z(\mathcal{E}_0)} + (\mathcal{E} - \mathcal{E}_0)\frac{d}{d\mathcal{E}}\frac{1}{v_z}\right]$$

$$\approx k_w + k - \frac{\omega}{v_z(\mathcal{E}_0)} + \frac{\omega}{\gamma_z^2 c}\frac{(\mathcal{E} - \mathcal{E}_0)}{\mathcal{E}_0}$$

$$\frac{d}{d\mathcal{E}}\frac{1}{v_z} = \frac{1}{mc^3}\frac{d}{d\gamma}\frac{1}{\beta_z} = \frac{1}{mc^3}\frac{d\gamma_z}{d\gamma}\frac{d}{d\gamma_z}\frac{1}{\beta_z}$$

$$g_z^2 = \frac{g^2}{(1 + K^2)} \qquad \frac{dg_z}{dg} = \frac{g}{g_z(1 + K^2)}$$

$$\frac{d}{dg_z}\frac{1}{b_z} = -\frac{1}{2b_z^3}\frac{d}{dg_z}\left(1 - \frac{1}{g_z^2}\right) = -\frac{1}{b_z^3 g_z^3}$$

$$\Rightarrow \begin{cases} \dfrac{dP}{dz} = -eE\theta_s \cos(\psi) \\ \dfrac{d}{dz}\psi \approx C + \dfrac{\omega}{\gamma_z^2 c \mathcal{E}_0} P \end{cases}$$

Energy deviation: $P \equiv \mathcal{E} - \mathcal{E}_0$

Detuning parameter: $C \equiv k_w + k - \dfrac{\omega}{v_z(\mathcal{E}_0)}$

Low Gain Regime: Pendulum Equation

$$\left.\begin{aligned} \frac{dP}{dz} &= -eE\theta_s \cos(\psi) \\ \frac{d}{dz}\psi &= C + \frac{\omega}{\gamma_z^2 c\mathcal{E}_0} P \end{aligned}\right\} \Rightarrow \qquad \frac{d^2}{dz^2}\psi + \frac{eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0}\cos(\psi) = 0$$

We assume that the change of the amplitude of the radiation field, E, is negligible and treat it as a constant over the whole interaction.

$$\frac{d^2}{d\hat{z}^2} y + \hat{u}\cos(y) = 0 \qquad \hat{u} = \frac{l_w^2 eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0} \qquad \hat{z} = \frac{z}{l_w}$$

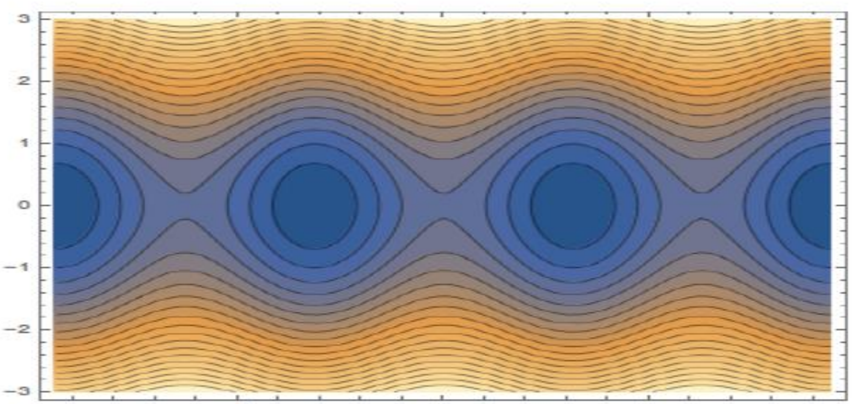
Pendulum equation:

$$\boxed{\frac{d^2}{d\hat{z}^2}\left(y + \frac{p}{2}\right) + \hat{u}\sin\left(y + \frac{p}{2}\right) = 0}$$

Low Gain Regime: Similarity to Synchrotron Oscillation

FEL

y is the angle between the transverse velocity vector and the radiation field vector and hence there is no energy kick for $y = p/2$

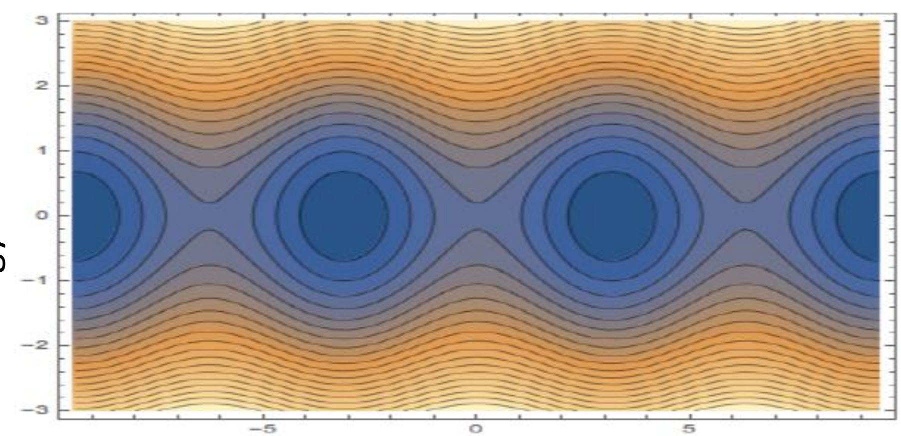


$$\frac{d^2}{d\hat{z}^2}\left(y + \frac{p}{2}\right) + \hat{u}\sin\left(y + \frac{p}{2}\right) = 0$$

$$\hat{u} = \frac{l_w^2 eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0} \qquad y = k_u z + k(z - ct)$$

Synchrotron Oscillation

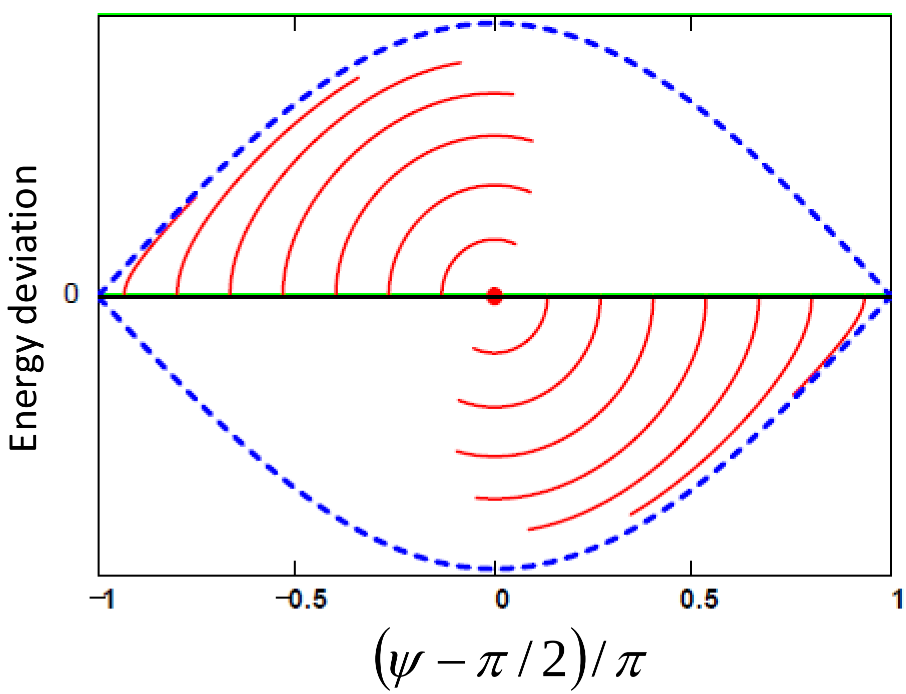
$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau; \quad \frac{d\pi_\tau}{ds} = \frac{1}{C}\frac{eV_{RF}}{p_o c}\sin\left(k_o h_{rf}\tau\right);$$

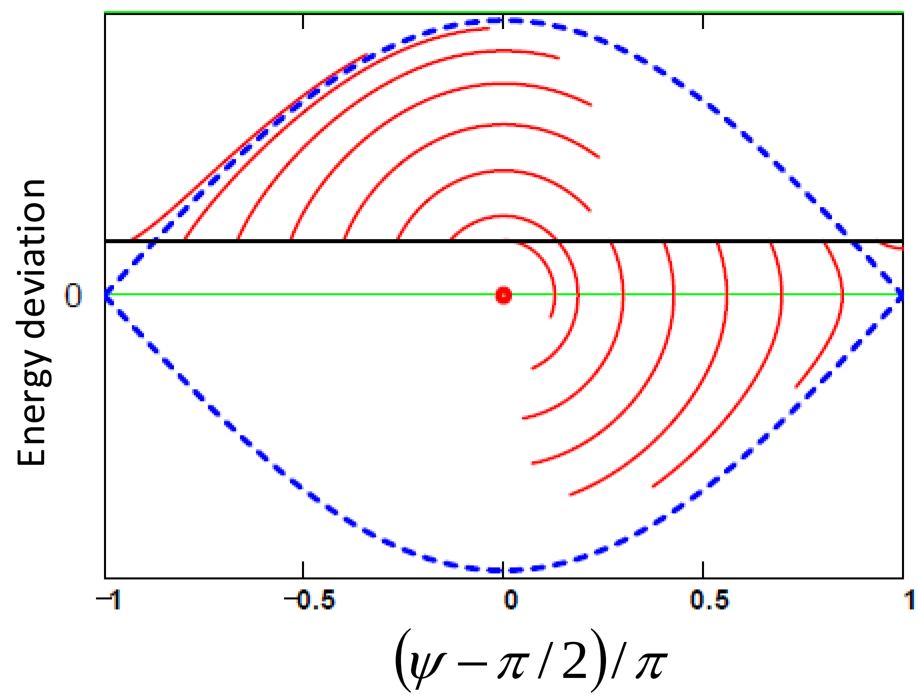


$$\frac{d^2 f_{rf}}{ds^2} = u_{rf}\sin f_{rf}$$

$$u_{rf} = h\frac{1}{C}\frac{eV_{RF}k_0 h_{rf}}{p_0 c} \qquad f_{rf} = k_0 h_{rf} t$$

Low Gain Regime: Qualitative Observation





The average energy of the electrons is right at resonant energy:

$$\lambda_0 \approx \frac{\lambda_w\left(1+K^2\right)}{2\gamma^2} \Rightarrow \gamma = \gamma_0 = \sqrt{\frac{\lambda_w\left(1+K^2\right)}{2\lambda_0}}$$

*Plots are taken from talk slides by Peter Schmuser.

The average energy of the electrons is slightly above the resonant energy:

$$\gamma = \gamma_0 + \Delta\gamma$$

With positive detuning, there is net energy loss by electrons.

Low Gain Regime: Derivation of FEL Gain

Change in radiation power density (energy gain per seconds per unit area):

$$\Delta\Pi_r = c\varepsilon_0\left(E_{ext} + \Delta E\right)^2 - c\varepsilon_0 E_{ext}^2 \approx 2c\varepsilon_0 E_{ext}\Delta E$$

Average change rate in electrons' energy per unit beam area:

$$\Delta\Pi_e = \frac{j_0\langle P\rangle}{e}$$

*The average, <…>, is over all electrons in the beam.

Energy deviation at entrance

Pondermotive phase at entrance

$$\langle P(z)\rangle = \int_{-\infty}^{\infty} dP_0 \int_0^{2\pi} d\psi_0\, f(P_0,\psi_0) P(P_0,\psi_0,z)$$

Assuming radiation has the same cross section area as the electron beam, we obtain the change in electric field amplitude:

$$\Delta\Pi_r + \Delta\Pi_e = 0 \Rightarrow \boxed{\Delta E = -\frac{j_0\langle P\rangle}{2c\varepsilon_0 E_{ext} e}}$$

$$\left.\begin{aligned} \frac{dP}{dz} &= -eE\theta_s\cos(\psi) \\ \frac{d}{dz}\psi &= C + \frac{\omega}{\gamma_z^2 c\mathcal{E}_0}P \end{aligned}\right\} \Rightarrow \langle P\rangle = -eE\theta_s\left\langle \int_0^1 \cos[\psi(\hat{z})]d\hat{z}\right\rangle$$

Low Gain Regime: Derivation of FEL Gain

$$\frac{d^2}{d\hat{z}^2}\psi + \hat{u}\cos\psi = 0$$

$$\psi(\hat{z}) = \psi(0) + \psi'(0)\hat{z} - \hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\psi(\hat{z}_2)\, d\hat{z}_2 \qquad (1)$$

Assuming that all electrons have the same energy and uniformly distributed in the Pondermotive phase at the entrance of FEL: $P_0 = 0$ and $f(\psi_0) = \frac{1}{2\pi}$.

The zeroth order solution for phase evolution is given by ignoring the effects from FEL interaction:

$$\left.\begin{aligned} \frac{dP}{dz} &= -eE\theta_s \cos(\psi) \\ \frac{d}{dz}\psi &= C + \frac{\omega}{\gamma_z^2 c\mathcal{E}_0} P \end{aligned}\right\} \Rightarrow \frac{d}{d\hat{z}}\psi = \hat{C} \Rightarrow \begin{cases} \psi(\hat{z}) = \psi_0 + \hat{C}\hat{z} \\ \psi'(0) = \hat{C} \end{cases} \qquad \hat{C} \equiv C l_w$$

Inserting the zeroth order solution back into eq. (1) yields the 1st order solution:

$$\psi(\hat{z}) = \psi_0 + \hat{C}\hat{z} + \Delta\psi(\psi_0, \hat{z}) \qquad \Delta\psi(\psi_0, \hat{z}) \equiv -\hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\left[\psi_0 + \hat{C}\hat{z}_2\right] d\hat{z}_2$$

Low Energy Regime: Derivation of FEL Gain

$$\Delta\psi(\psi_0,\hat{z}) \equiv -\hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\left[\psi_0 + \hat{C}\hat{z}_2\right] d\hat{z}_2$$

$$= -\frac{\hat{u}}{\hat{C}^2}\left\{\int_0^{\hat{C}\hat{z}} \sin(\psi_0 + x_1)dx_1 - \hat{C}\hat{z}\sin\psi_0\right\} = \frac{\hat{u}}{\hat{C}^2}\left[\cos\left(\psi_0 + \hat{C}\hat{z}\right) - \cos\psi_0 + \hat{C}\hat{z}\sin\psi_0\right]$$

$$\langle P\rangle = -eEl_w\theta_s\left\langle\int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z} + \Delta\psi(\psi_0,\hat{z})\right]d\hat{z}\right\rangle$$

← Average energy loss of electrons

$$= eE\theta_s l_w\left\langle\int_0^1 \sin\left[\psi_0 + \hat{C}\hat{z}\right]\sin(\Delta\psi(\psi_0,\hat{z}))d\hat{z}\right\rangle - eE\theta_s l_w\left\langle\int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z}\right]\cos(\Delta\psi(\psi_0,\hat{z}))d\hat{z}\right\rangle$$

$$\approx eE\theta_s l_w\left\langle\int_0^1 \Delta\psi(\psi_0,\hat{z})\sin\left[\psi_0 + \hat{C}\hat{z}\right]d\hat{z}\right\rangle - \frac{eE\theta_s l_w}{2\pi}\int_0^1 d\hat{z}\int_0^{2\pi}\cos\left[\psi_0 + \hat{C}\hat{z}\right]d\psi_0$$

$$\left\langle\int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z}\right]\sin\left[\psi_0 + \hat{C}\hat{z}\right]d\hat{z}\right\rangle = 0$$

$$= \frac{eE\theta_s l_w}{2\pi}\frac{\hat{u}}{\hat{C}^2}\int_0^1 d\hat{z}\left\{\hat{C}\hat{z}\cos\left(\hat{C}\hat{z}\right)\int_0^{2\pi}\sin^2\psi_0 d\psi_0 - \sin\left(\hat{C}\hat{z}\right)\int_0^{2\pi}\cos^2\psi_0 d\psi_0\right\}$$

$$= -eE\theta_s l_w\frac{\hat{u}}{\hat{C}^3}\left(1 - \frac{\hat{C}}{2}\sin\hat{C} - \cos\hat{C}\right)$$

Low Energy Regime: Derivation of FEL Gain

Growth in the amplitude of radiation field:

$$\Delta E = -\frac{j_0 \langle P \rangle}{2c\varepsilon_0 E_{ext} e} = \frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma} \frac{l_w^3 E_{ext}}{I_A} \frac{2}{\hat{C}^3} \left(1 - \frac{\hat{C}}{2}\sin\hat{C} - \cos\hat{C}\right)$$

$$\hat{u} = \frac{l_w^2 e E_{ext} \theta_s \omega}{\gamma_z^2 c \gamma m c^2}$$

$$I_A = \frac{4\pi\varepsilon_0 m c^3}{e}$$

The gain is defined as the relative growth in radiation power:

$$g_s = \frac{\left(E_{ext} + \Delta E\right)^2 - E_{ext}^2}{E_{ext}^2} \approx \frac{2\Delta E}{E_{ext}} = \tau \cdot f\left(\hat{C}\right)$$

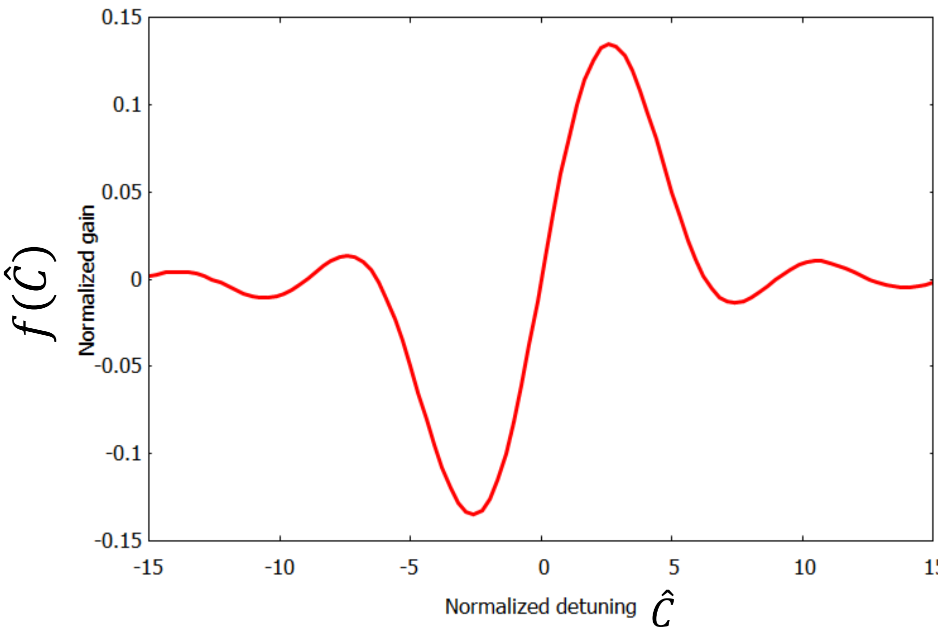
As observed earlier, there is no gain if the electrons has resonant energy.

$$\tau \equiv \frac{2\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma} \frac{l_w^3}{I_A}$$

Cubic in FEL length

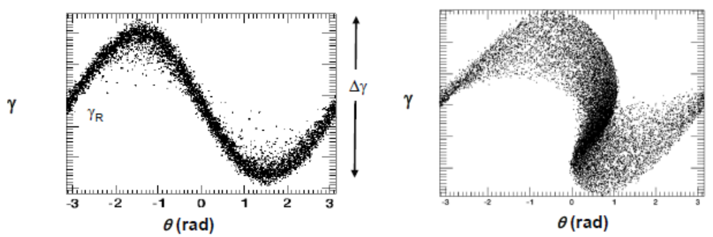
$$f\left(\hat{C}\right) = \frac{2}{\hat{C}^3}\left(1 - \cos\hat{C} - \frac{\hat{C}}{2}\sin\hat{C}\right)$$

$$= -2\frac{d}{d\hat{C}} \frac{\sin^2\left(\hat{C}/2\right)}{\hat{C}^2}$$



High Gain Regime: Concept

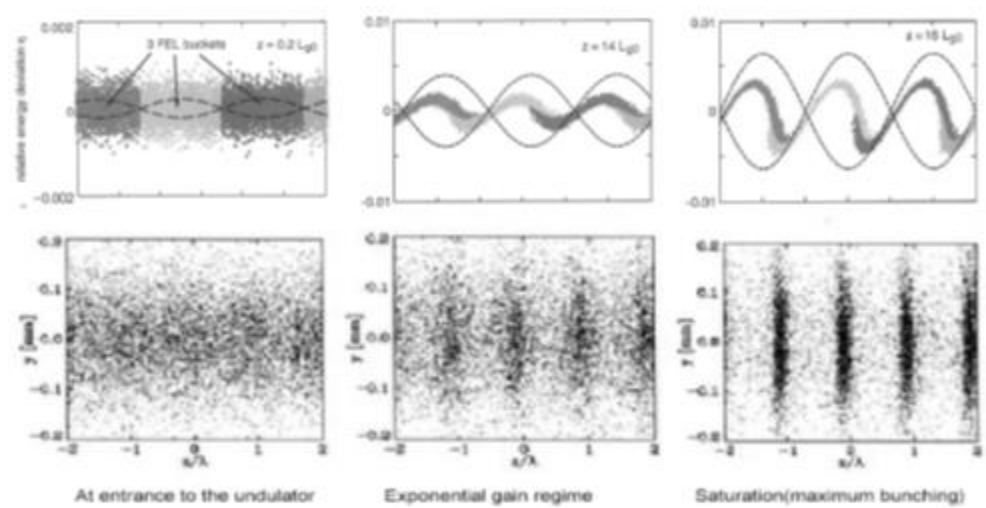
1. Energy kick from radiation field + dispersion/drift -> electron density bunching;

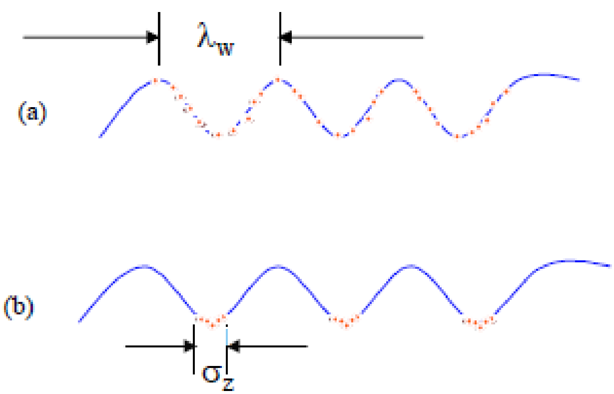


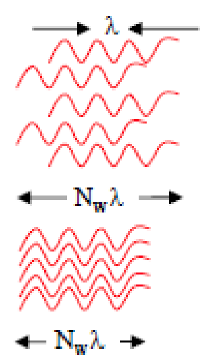
*The plots are for illustration only. The right plot actually shows somewhere close to saturation.

2. Electron density bunching makes more electrons radiates coherently -> higher radiation field;

3. Higher radiation fields leads to more density bunching through 1 and hence closes the positive feedback loop -> FEL instability.







$$|E| \propto \sqrt{N_e}$$

$$I_{incoherent} \propto N_e$$

$$|E| \propto N_e$$

$$I_{coherent} \propto N_e^2$$

The positive feedback loop between radiation field and electron density bunching is the underlying mechanism of high gain FEL regime.

1-D Model for cold beam without detuning

$$B(z)=\left\langle e^{-i\psi}\right\rangle=\frac{1}{N}\sum_{j=1}^{N}e^{-i\psi_j} \qquad D(z)=\left\langle Pe^{-i\psi}\right\rangle=\frac{1}{N}\sum_{j=1}^{N}P_j e^{-i\psi_j}$$

Assuming that $C=0$, it follows $\frac{d}{dz}\psi=C+\frac{\omega}{\gamma_z^2 c\mathrm{E}_0}P=\frac{\omega}{\gamma_z^2 c\mathrm{E}_0}P$

$$\frac{d}{dz}B(z)=-i\left\langle e^{-i\psi}\frac{d}{dz}\psi\right\rangle=-i\frac{\omega}{c\gamma_z^2\mathrm{E}_0}\left\langle e^{-i\psi}P\right\rangle=-i\frac{\omega}{c\gamma_z^2\mathrm{E}_0}D(z)$$

$$\frac{dP}{dz}=-e\theta_s E(z)\cos(\psi)$$

$$\frac{d}{dz}D(z)=\left\langle e^{-i\psi}\frac{d}{dz}P\right\rangle-i\left\langle e^{-i\psi}P\frac{d}{dz}\psi\right\rangle\approx\left\langle e^{-i\psi}\frac{d}{dz}P\right\rangle=-\left\langle e^{-i\psi}eE\theta_s\cos(\psi)\right\rangle\approx-\frac{1}{2}e\theta_s E$$

Wave Equation

$$\psi = k_w z + k\left(z - ct\right)$$

1-D theory and hence $\partial/\partial x = 0$ and $\partial/\partial y = 0$

Wave equation for transverse vector potential: $$\frac{\partial^2 \vec{A}_\perp}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \vec{A}_\perp}{\partial t^2} = -\mu_0 \vec{j}_\perp \qquad (1)$$

Transverse current perturbation: $$j_x + i j_y = \frac{1}{v_z}\left(v_x + i v_y\right) j_z = \theta_s e^{-ik_w z} j_z \qquad (2)$$

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos\left(k_u z\right)\hat{x} - \sin\left(k_u z\right)\hat{y}\right]$$

We seek the solution for vector potential of the form:

$$A_{x,y}(z,t) = \tilde{A}_{x,y}(z) e^{i\omega(z/c - t)} + \tilde{A}_{x,y}^{\,*}(z) e^{-i\omega(z/c - t)} \qquad (3)$$

Inserting eq. (2) and (3) into eq. (1) yields

$$e^{i\omega(z/c-t)}\left\{\frac{2i\omega}{c}\frac{\partial}{\partial z}\begin{pmatrix}\tilde{A}_x\\ \tilde{A}_y\end{pmatrix} + \frac{\partial^2}{\partial z^2}\begin{pmatrix}\tilde{A}_x\\ \tilde{A}_y\end{pmatrix}\right\} + C.C. = -\mu_0\theta_s\begin{pmatrix}\cos(k_w z)\\ -\sin(k_w z)\end{pmatrix} j_z$$

Multiplying both sides by $e^{ik_w z}$ and neglecting terms proportional to $e^{ik_w z - ik(z-ct)}$ since they will change fast over the FEL (same as the helicity argument).

$$\left\{\frac{2i\omega}{c}\frac{\partial}{\partial z}\begin{pmatrix}\tilde{A}_{tot,x}\\ \tilde{A}_{tot,y}\end{pmatrix} + \frac{\partial^2}{\partial z^2}\begin{pmatrix}\tilde{A}_{tot,x}\\ \tilde{A}_{tot,y}\end{pmatrix}\right\} = -\frac{\mu_0 N\theta_s}{2}\begin{pmatrix}e^{ik_w z} + e^{-ik_w z}\\ ie^{ik_w z} - ie^{-ik_w z}\end{pmatrix}\left\langle j_z e^{-i\psi}\right\rangle e^{ik_w z}$$

1. Ignoring fast oscillating term ~ $e^{2ik_w z}$

2. Ignoring second derivative by assuming that the variation of $\tilde{A}_x'$ is negligible over the optical wave length.

Wave Equation

After neglecting the fast oscillation terms, we get the following relation between the current perturbation and the vector potential of the radiation field:

$$\frac{\partial}{\partial z}\tilde{A}_{tot,x} = -\frac{c\mu_0 N\theta_s}{4i\omega}\left\langle j_z e^{-i\psi}\right\rangle \qquad \frac{\partial}{\partial z}\tilde{A}_{tot,y} = \frac{\mu_0 N c\theta_s}{4\omega}\left\langle j_z e^{-i\psi}\right\rangle$$

In order to relate the vector potential to the electric field, we use the Maxwell equation:

$$\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \Rightarrow \nabla\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0 \Rightarrow \left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=\vec{\nabla}\varphi \Rightarrow E_{x,y}=-\frac{\partial A_{x,y}}{\partial t}$$

1D: $\frac{\partial}{\partial x}=0\,,\quad \frac{\partial}{\partial y}=0$

$$\Rightarrow E_x + iE_y = Ee^{i\omega(z/c-t)} = -\frac{\partial}{\partial t}\left[\left(\tilde{A}_{tot,x}+i\tilde{A}_{tot,y}\right)e^{i\omega(z/c-t)}\right]$$

$$\Rightarrow E = i\omega\left(\tilde{A}_{tot,x}+i\tilde{A}_{tot,y}\right)$$

$$\vec{E}_\perp(z,t)=E\left[\cos\left(k(z-ct)\right)\hat{x}+\sin\left(k(z-ct)\right)\hat{y}\right]$$

Finally, the relation between the radiation field and the current modulation is obtained:

$$\frac{d}{dz}E = i\omega\left(\frac{\partial}{\partial z}\tilde{A}_{tot,x}+i\frac{\partial}{\partial z}\tilde{A}_{tot,y}\right) = -\frac{c\mu_0 N\theta_s}{2}\left\langle j_z e^{-i\psi}\right\rangle = \frac{ec^2 N\mu_0\theta_s}{2V}B = \frac{ec^2 n\mu_0\theta_s}{2}B$$

$$\left\langle j_z e^{-i\psi}\right\rangle = -\frac{ec}{NV}\sum_{k=1}^{N} e^{-i\psi_k} = -\frac{ecB}{V}$$

$$n = N/V$$

1-D High Gain FEL Equation for Cold Beam and Zero Detuning

$$\frac{d}{dz}B(z) = -i\frac{\omega}{c\gamma_z^2 E_0}D(z)$$

$$\frac{d}{dz}D = -\frac{1}{2}e\theta_s E$$

$$\frac{d}{dz}E = \frac{ec^2 n\mu_0\theta_s}{2}B$$

$$\frac{d^3}{d\hat{z}^3}E = iE$$

$\hat{z} \equiv \Gamma z$ is normalized longitudinal location along wiggler,

$$\Gamma \equiv \left[\frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma I_A}\right]^{1/3}$$ is the 1-D Gain rate parameter

$$I_A = \frac{4\pi\varepsilon_0 mc^3}{e}$$ is called Alfven current

$$E(\hat{z}) = \sum_{k=1}^{3} B_k e^{\lambda_k \hat{z}}$$

$$\lambda^3 = i \Rightarrow$$

$\lambda_1 = e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ ← Growing mode

$\lambda_2 = e^{i\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$ ← Damping mode

$\lambda_3 = e^{-i\frac{\pi}{2}} = -i$ ← Oscillating mode

1D Gain Length

- At high gain limit, i.e. $\hat{z} >> 1$, the radiation field is given by

$$E(\hat{z}) \approx B_1 e^{\lambda_k \hat{z}} = B_1 \exp\left[\frac{\sqrt{3}}{2}\Gamma z\right] \exp\left[i\frac{1}{2}\Gamma z\right]$$

and the radiation power is

A: cross section of the radiation field

$$P(\hat{z}) = \varepsilon_0 c \left| E(\hat{z})^2 \right| A = \varepsilon_0 c \left| B_1 \right|^2 \exp\left(\sqrt{3}\Gamma z\right) = \varepsilon_0 c \left| B_1 \right|^2 A \exp\left(\frac{z}{L_G}\right)$$

and the 1-D power gain length is

$$L_G \equiv \frac{1}{\sqrt{3}\Gamma} = \frac{\lambda_w}{4\pi\sqrt{3}\rho}$$

Pierce Parameter

$$\rho \equiv \frac{\gamma_z^2 \Gamma c}{\omega} = \frac{\Gamma}{2k_w}$$

1-D amplitude gain length is $$L_{GA} = 2L_G \equiv \frac{2}{\sqrt{3}\Gamma} = \frac{\lambda_w}{2\pi\sqrt{3}\rho}$$

Solution for Cold Beam with Nonzero Detuning

For non-vanishing detuning, the differential equation becomes

$$C \equiv k_w + k - \frac{\omega}{v_z(\mathcal{E}_0)}$$

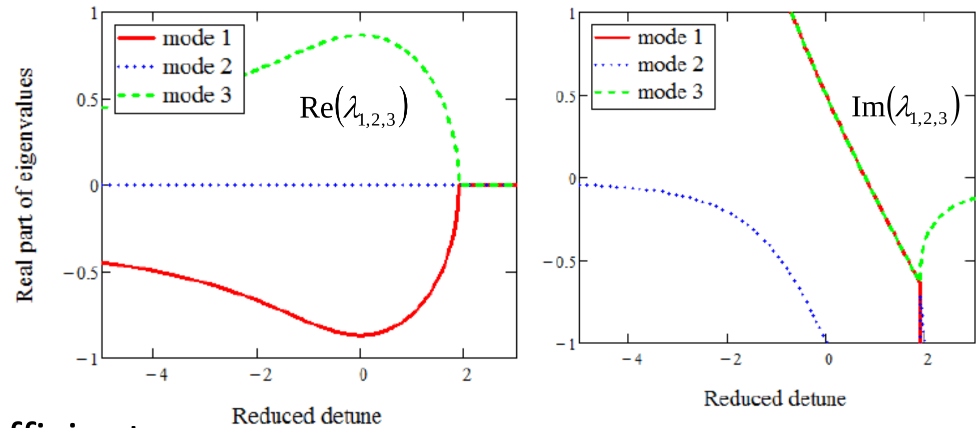
$$\hat{C} = C / \Gamma$$

$$\frac{d^3}{d\hat{z}^3} E(\hat{z}) + 2i\hat{C} \frac{d^2}{d\hat{z}^2} E(\hat{z}) - \hat{C}^2 \frac{d}{d\hat{z}} E(\hat{z}) = iE(\hat{z})$$

The general solution of the ODE reads:

$$E(\hat{z}) = \sum_{k=1}^{3} B_k e^{\lambda_k \hat{z}}$$

$$\lambda^3 + 2i\hat{C}\lambda^2 - \hat{C}^2\lambda = i$$



Applying initial condition to get the coefficients

$$\begin{pmatrix} E(0) \\ E'(0) \\ E''(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \Rightarrow \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}^{-1} \begin{pmatrix} E(0) \\ E'(0) \\ E''(0) \end{pmatrix}$$

For $E(0) = E_{ext}$ and $E'(0) = E''(0) = 0$, the solution can be explicitly written as

$$E(\hat{z}) = E_{ext}\left[\frac{\lambda_2\lambda_3 e^{\lambda_1\hat{z}}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2\hat{z}}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3\hat{z}}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right]$$

Low Gain Limit of High Gain Solution

Can we reproduce the previously obtained low gain solution by taking the proper limit of the high gain solution?

$$\tau \equiv \frac{2\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma} \frac{l_w^3}{I_A} = 2\Gamma^3 l_w^3$$

$$g_l = \frac{(E_{ext}+\Delta E)^2 - E_{ext}^2}{E_{ext}^2} \approx \frac{2\Delta E}{E_{ext}} = \tau \cdot f(\hat{C}_l) = 2\Gamma^3 l_w^3 f_l(\hat{C}_l)$$

$$f_l(\hat{C}_l) = \frac{2}{\hat{C}_l^{\,3}}\left(1-\cos\hat{C}_l - \frac{\hat{C}_l}{2}\sin\hat{C}_l\right)$$

$$\hat{C}_l = C l_w$$

$$g_h(\hat{C}_l) = \frac{\tilde{E}^2 - E_{ext}^2}{E_{ext}^2} = \left|\frac{\lambda_2\lambda_3 e^{\lambda_1 \hat{l}_w}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2 \hat{l}_w}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3 \hat{l}_w}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right|^2 - 1$$

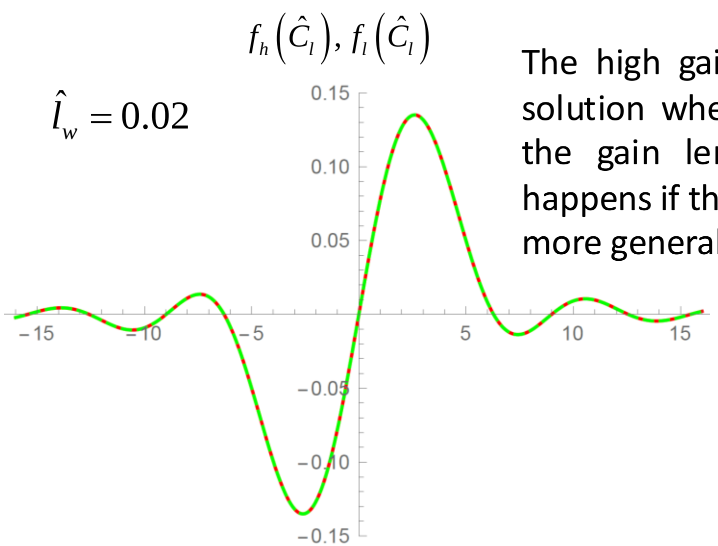
$$= 2\Gamma^3 l_w^3 f_h(\hat{C}_l) \qquad \hat{l}_w = l_w \Gamma$$

$$f_h(\hat{C}_l) = \frac{1}{2\hat{l}_w^3}\left\{\left|\frac{\lambda_2\lambda_3 e^{\lambda_1 \hat{l}_w}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2 \hat{l}_w}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3 \hat{l}_w}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right|^2 - 1\right\}$$

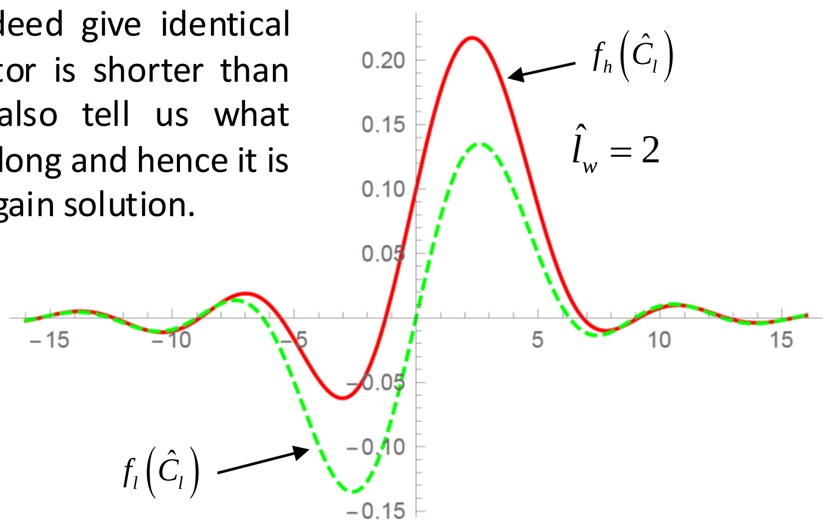
The normalization factor for high gain is different from that of low gain:

$$\hat{C}_h = C/\Gamma = C l_w/\hat{l}_w = \hat{C}_l/\hat{l}_w$$

$$\lambda^3 + 2i\frac{\hat{C}_l}{\hat{l}_w}\lambda^2 - \left(\frac{\hat{C}_l}{\hat{l}_w}\right)^2 \lambda = i$$

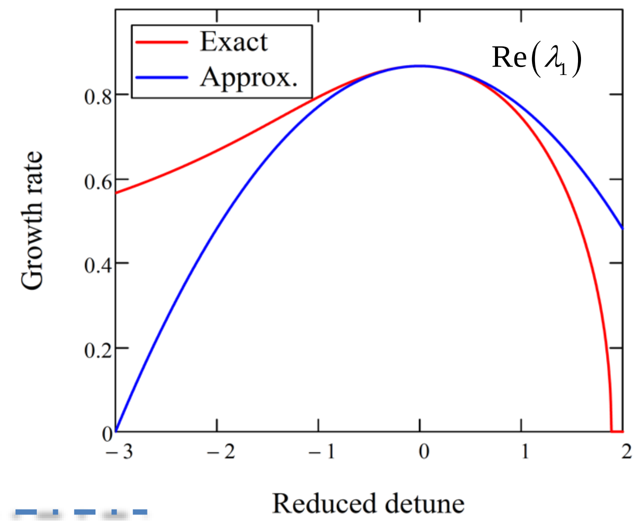


The high gain solution indeed give identical solution when the undulator is shorter than the gain length. But it also tell us what happens if the undulator is long and hence it is more general than the low gain solution.



Bandwidth at High Gain Limit I

It is sometimes hard to extract insights from the exact solution of the 3rd order polynomial equation for the eigenvalue. Therefore, it is useful to get the approximate solution which is simpler but gives accurate results for the region that we are interested in.



$$\lambda^3 + 2i\hat{C}\lambda^2 - \hat{C}^2\lambda = i \qquad \boxed{\lambda = a_0 + a_1\hat{C} + a_2\hat{C}^2}$$

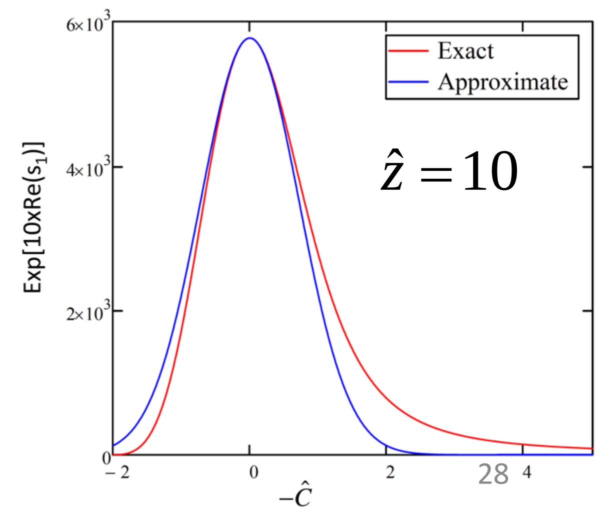
$$f\left(\hat{C}\right) = \left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right)^3 + 2i\hat{C}\left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right)^2 - \hat{C}^2\left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right) - i = 0$$

$$f\left(\hat{C}\right) = f_0\left(a_0, a_1, a_2\right) + f_1\left(a_0, a_1, a_2\right)\hat{C} + f_2\left(a_0, a_1, a_2\right)\hat{C}^2 = 0$$

Zeroth order equation: $f(0) = 0 \Rightarrow \boxed{a_0 = \frac{\sqrt{3}}{2} + i\frac{1}{2}}$

First order equation: $\left.\frac{d}{d\hat{C}} f\left(\hat{C}\right)\right|_{\hat{C}=0} = 0 \Rightarrow \boxed{a_1 = -i\frac{2}{3}}$

Second order equation: $\left.\frac{d^2}{d\hat{C}^2} f\left(\hat{C}\right)\right|_{\hat{C}=0} = 0 \Rightarrow \boxed{a_2 = -\frac{1}{9}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)}$



Bandwidth at High Gain Limit II

After taking the approximate eigenvalue, the radiation field in frequency domain is

$$E(\hat{C}) : \exp\left[a_0\hat{z} + a_1\hat{C}\hat{z} + a_2\hat{C}^2\hat{z}\right] : \exp\left[-\frac{\hat{C}^2}{2\sigma_{\hat{C}}^{\,2}}\right] \Rightarrow \sigma_{\hat{C}} = \sqrt{-\frac{1}{2\,\mathrm{Re}(a_2)\,\hat{z}}}$$

$$\mathrm{Re}(a_2) = -\frac{\sqrt{3}}{18} \qquad \sigma_{\hat{C}} = 3\sqrt{\frac{1}{\sqrt{3}\Gamma z}} \qquad \hat{C} \equiv \frac{1}{\mathrm{G}}\left(k_w - \frac{w}{2cg_z^2}\right) \qquad \Gamma = \rho\frac{\omega}{\gamma_z^2 c}$$

1D FEL bandwidth for radiation field:

$$\sigma_\omega = \Gamma 2c\gamma_z^2\sigma_{\hat{C}} = 6c\gamma_z^2\sqrt{\frac{\Gamma}{\sqrt{3}z}} = 3\omega_0\sqrt{\frac{2\rho}{\sqrt{3}k_w z}}$$

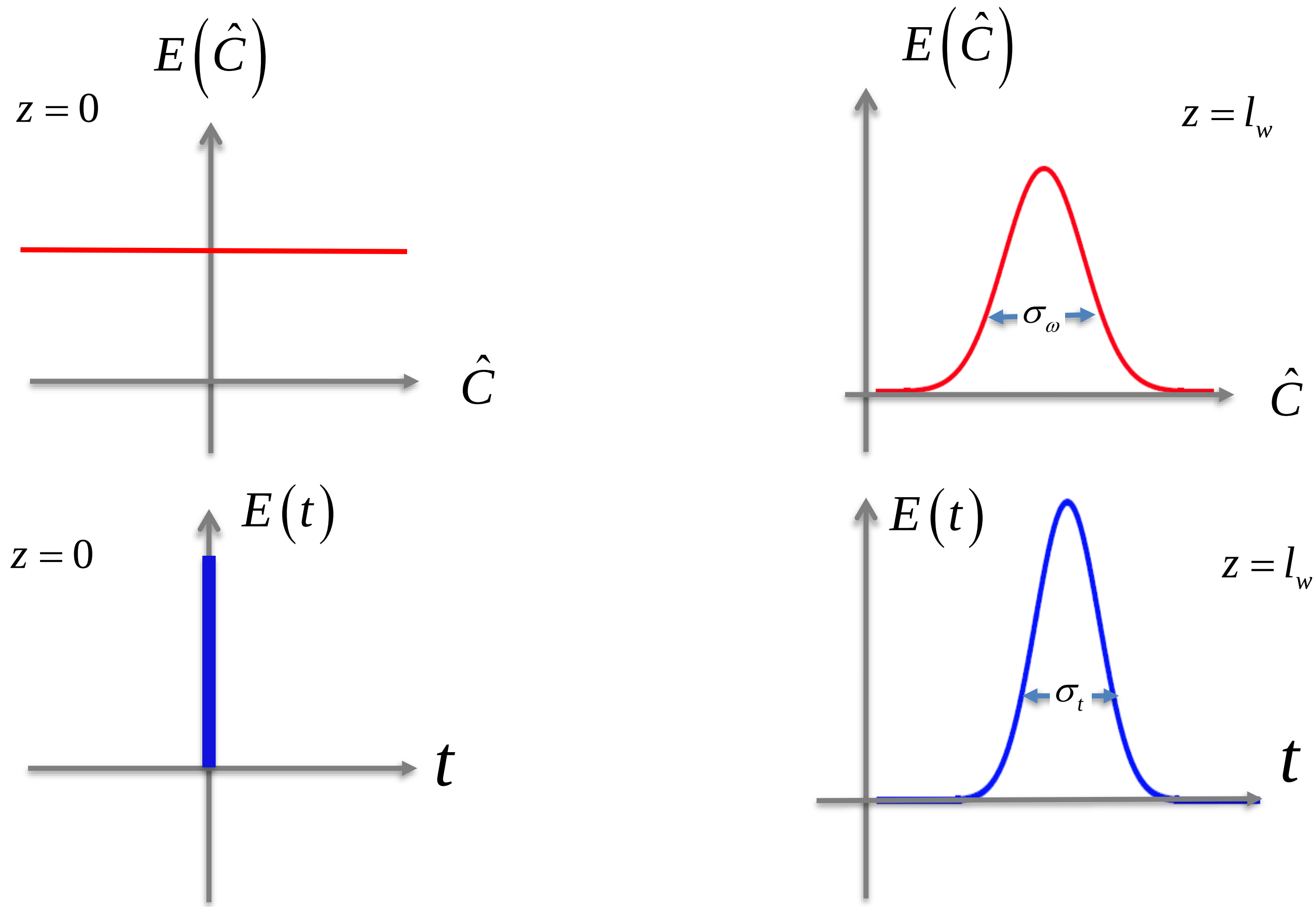
1D FEL bandwidth for radiation power:

$$\sigma_A = \frac{\sigma_\omega}{\sqrt{2}} = \omega_0\sqrt{\frac{3\sqrt{3}\rho}{k_w z}}$$

Pierce Parameter

$$\rho = \frac{\gamma_z^2\Gamma c}{\omega}$$

Coherent Length

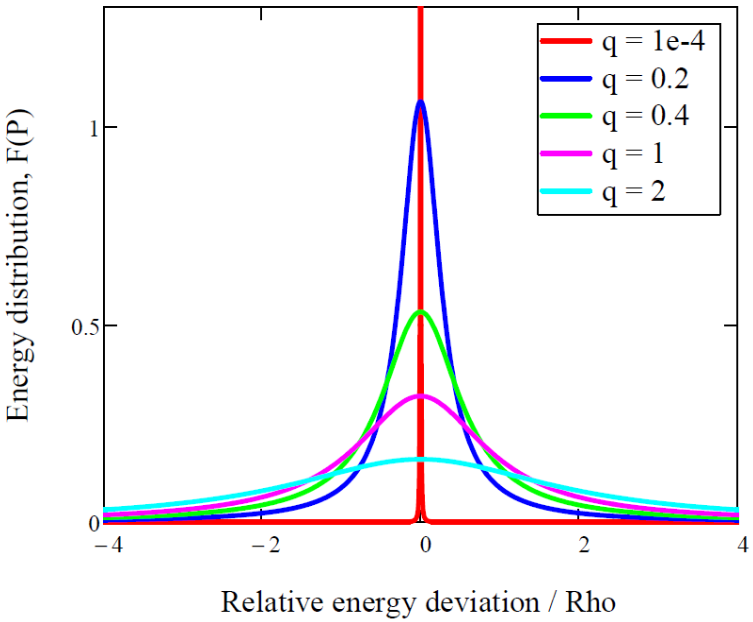


Coherent length is the width of the radiation wave-packet generated by a delta-like excitation.

$$E(\omega):\ \exp\left[-\frac{\omega^2}{2\sigma_\omega^2}\right] \Rightarrow E(t):\ \exp\left[-\frac{t^2}{2\sigma_t^2}\right]$$

$$\sigma_t = \frac{|a_2|}{k_0 c}\sqrt{\frac{-k_w z}{\rho\,\mathrm{Re}(a_2)}} = \frac{1}{3k_0 c}\sqrt{\frac{2k_w z}{\rho\sqrt{3}}} = \frac{2}{\sqrt{3}\sigma_\omega}$$

FEL Gain for warm Beam with Lorentzian Energy Distribution

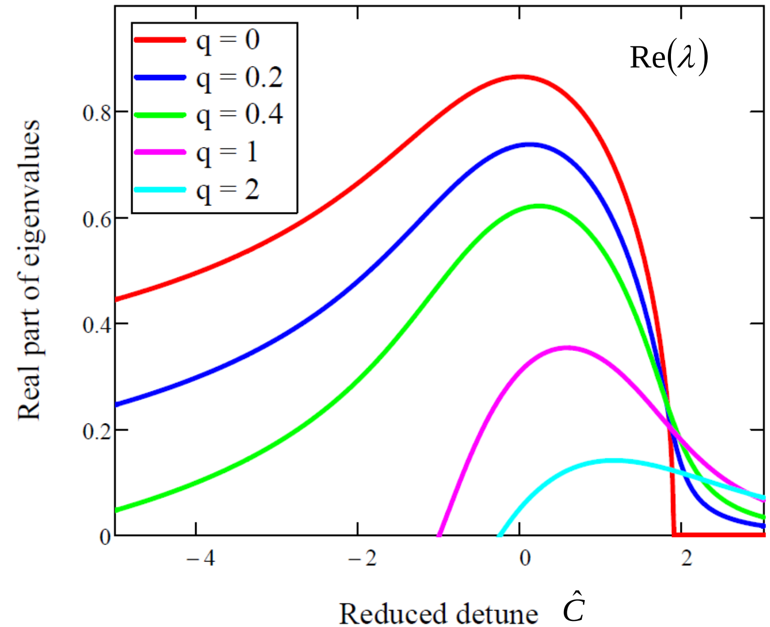


$$F\left(\hat{P}\right)=\frac{1}{\pi\hat{q}}\frac{1}{1+\frac{\hat{P}^2}{\hat{q}^2}}$$

$$\hat{P}=\frac{E-E_0}{E_0\rho}$$

Pierce Parameter

$$\rho=\frac{\gamma_z^2\Gamma c}{\omega}$$



If there is no initial modulation in the electron beam:

$$\frac{d^3}{d\hat{z}^3}E\left(\hat{z}\right)+2\left(i\hat{C}+\hat{q}\right)\frac{d^2}{d\hat{z}^2}E\left(\hat{z}\right)+\left(i\hat{C}+\hat{q}\right)^2\frac{d}{d\hat{z}}E\left(\hat{z}\right)=iE\left(\hat{z}\right)$$

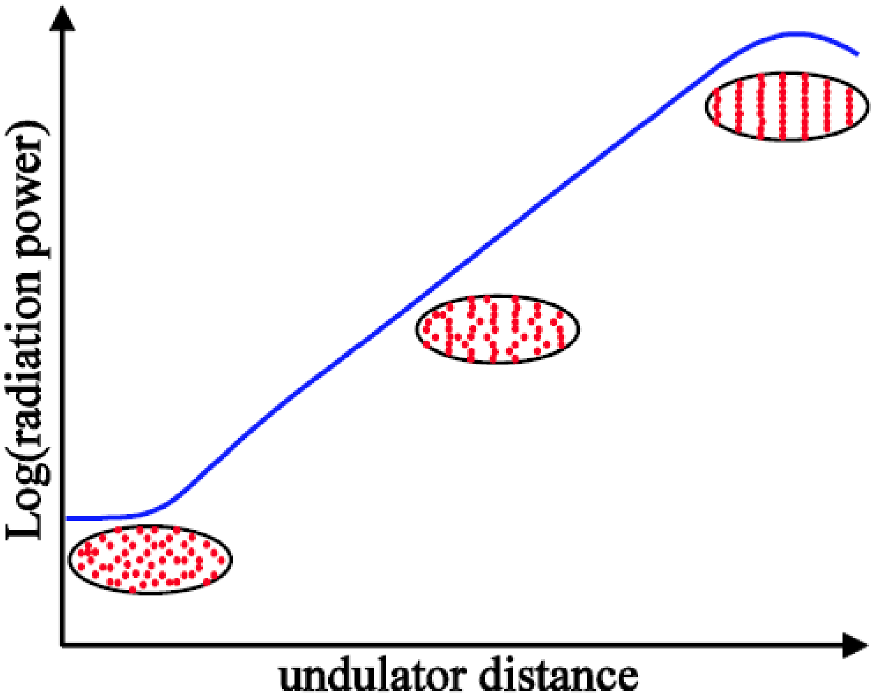
The eigenvalues are determined by : $\lambda\left(\lambda+\hat{q}+i\hat{C}\right)^2=i$

- FEL gain reduced substantially when the relative energy spread become comparable or larger than the Pierce parameter.

FEL Saturation I

Like any other amplification mechanism, the exponential growth of FEL radiation can not continue forever. One of the criteria to determine the onset of saturation is when there is no electrons to be bunched further, i.e. $\delta n / n_0 \sim 1$, which happens to be the point where nonlinear effects starts to take over.

$$n(\psi) = n_0 + \delta n(\psi)$$



For FEL process starts from shot noise, i.e. SASE, the maximal gain can be derived as

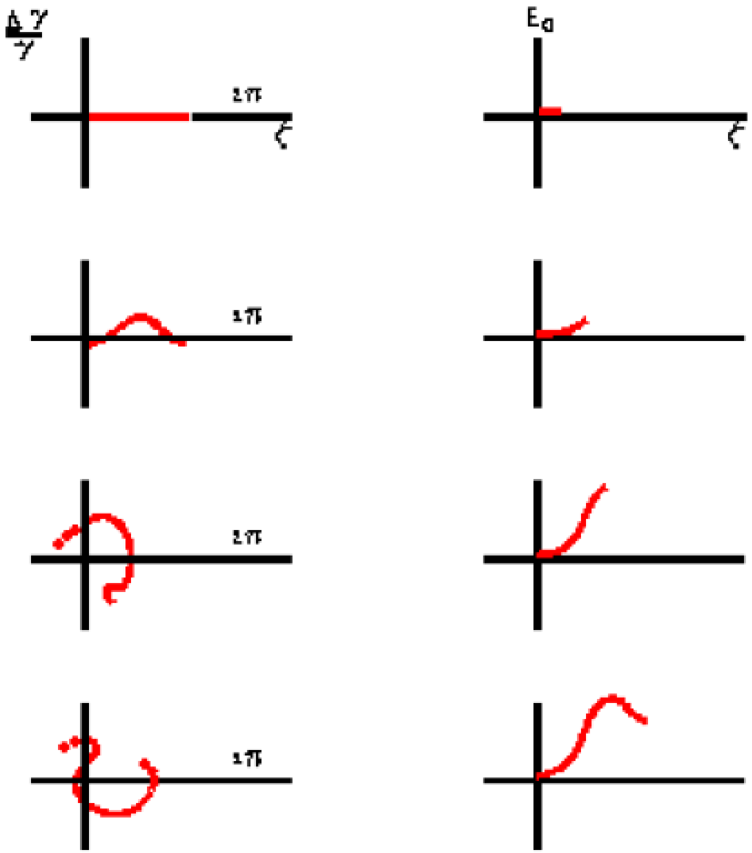
$$\delta n / n_0 \sim 1 \Rightarrow g_{\max} \le \sqrt{\frac{M_e}{N_c}}$$

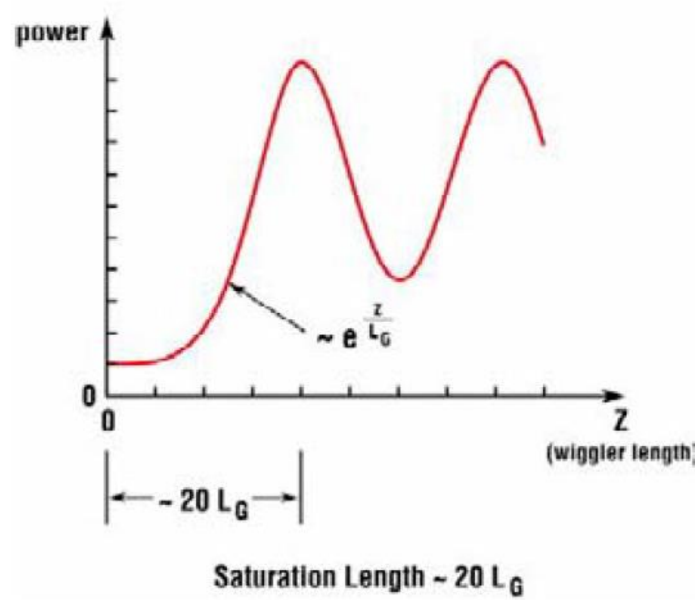
$N_c = L_c / \lambda_{opt}$ is the ratio between coherent length and the radiation wavelength.

M_e is the number of electrons in a radiation wavelength.

FEL Saturation II

There are other criteria which give similar results for the maximal Gain in SASE:





$$\frac{d^2}{d\hat{z}^2}\left(y+\frac{\rho}{2}\right)+\hat{u}\sin\left(y+\frac{\rho}{2}\right)=0$$

$$\Omega_p=\frac{\sqrt{\hat{u}}}{l_w}=\sqrt{\frac{eE\theta_s\omega}{\gamma_z^2 c\mathrm{E}_0}}\approx\frac{1}{L_G}=\sqrt{3}\Gamma$$

$$\boxed{P_{sat}=\varepsilon_0 cE_{sat}{}^2 A=\chi\cdot\rho\cdot\frac{\mathrm{E}_0}{e}I_e}$$

A: cross section of the beam (and the radiation field)

χ: a numerical factor in the order of one.

Hence the Pierce parameter is also called efficiency parameter.

FEL Saturation III

- If we use the result that FEL typically saturates at 20 power gain length, the FEL bandwidth at saturation is given by

$$\sigma_{\omega,sat} = 3\omega_0\sqrt{\frac{2\rho}{\sqrt{3}k_w z_{sat}}} \approx 3\omega_0\sqrt{\frac{2\rho}{\sqrt{3}k_w 20L_G}} \qquad L_G \equiv \frac{1}{\sqrt{3}\Gamma} = \frac{\lambda_w}{4\pi\sqrt{3}\rho}$$

FEL bandwidth for radiation amplitude at saturation:

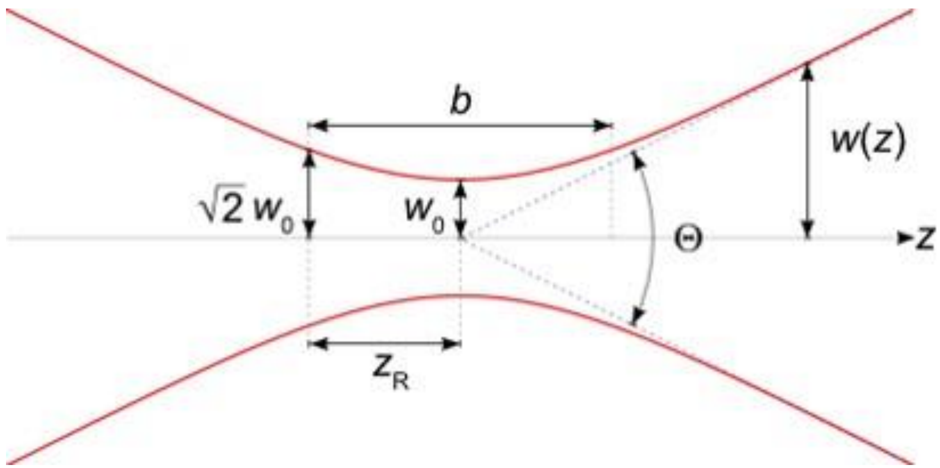
$$\frac{\sigma_{\omega,sat}}{\omega_0} = 3\omega_0\sqrt{\frac{2\rho}{\sqrt{3}k_w z_{sat}}} \approx \rho\sqrt{1.8}$$

FEL bandwidth for radiation power at saturation:

$$\boxed{\frac{\sigma_{A,sat}}{\omega_0} = \frac{\sigma_{\omega,sat}}{\sqrt{2}\omega_0} = \sqrt{0.9}\rho \approx \rho}$$

Pierce parameter is roughly equal to the bandwidth of the FEL at saturation.

3D Effects: Diffraction



The radius of the radiation at a given distance is given by $w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R} \right)^2}$

The Rayleigh length or Rayleigh range is the distance along the propagation direction of a beam from the waist to the place where the area of the cross section is doubled.

For a Gaussian radiation beam: $z_R = \frac{\pi w_0^2}{\lambda_{opt}}$

The size of the electron beam and the seeding radiation field optics have to be properly chosen so that the interaction efficiency between radiation fields and electrons can be optimized.

Three Dimensional Effects: 3D Gain

- In reality, the gain length will be longer than the 1D gain length due to diffraction, electron emittance, and electron beam energy spread. It is difficult to obtain a general analytical expression for the gain length with all these effects taken into account.
- The analytical approach typically involves expansion over a series of transverse modes.
- For the dominant transverse mode, there is a fitting formula derived by Ming Xie, which is typically of the accuracy of 10% compared with simulation results.

Ming Xie's fitting formula for 3D gain length

$$\boxed{L_{3D} = L_{1D}\left(1+\Lambda\right)}$$

$$\Lambda = 0.45\eta_d^{0.57} + 0.55\eta_\varepsilon^{1.6} + 3\eta_\gamma^2 + 0.35\eta_\varepsilon^{2.9}\eta_\gamma^{2.4} + 51\eta_d^{0.95}\eta_\gamma^3 + 0.62\eta_d^{0.99}\eta_\varepsilon^{1.1}$$
$$+5.3\eta_d^{0.76}\eta_\varepsilon^{2.3}\eta_\gamma^{2.7} + 120\eta_d^{2.1}\eta_\varepsilon^{2.9}\eta_\gamma^{2.8} + 3.7\eta_d^{0.43}\eta_\varepsilon\eta_\gamma$$

Energy spread effects ↓

$$\eta_\gamma = \left(\frac{L_{1D}4\pi}{\lambda_w}\right)\frac{\delta\gamma}{\gamma}$$

Electron emittance effects ↓

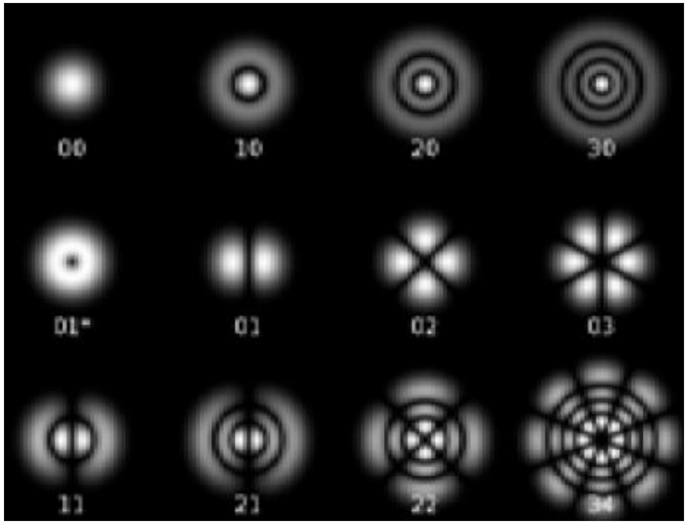
$$\eta_\varepsilon = \left(\frac{L_{1D}4\pi}{\beta_b\gamma\lambda}\right)\varepsilon_n$$

Diffraction effects ↓

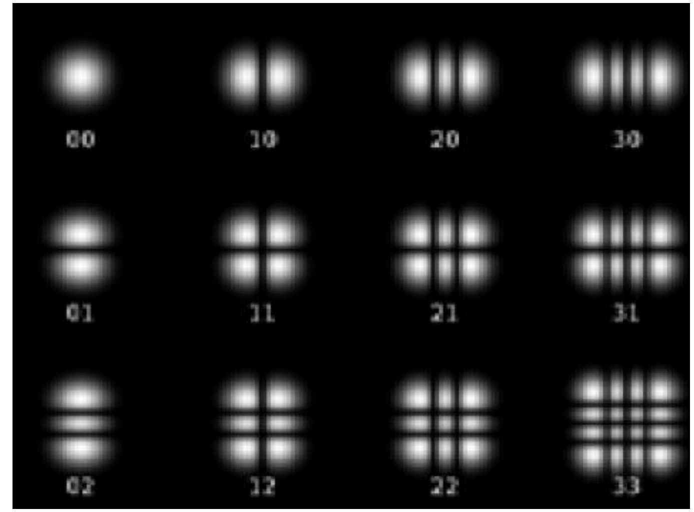
$$\eta_d = \frac{L_{1D}}{Z_R}$$

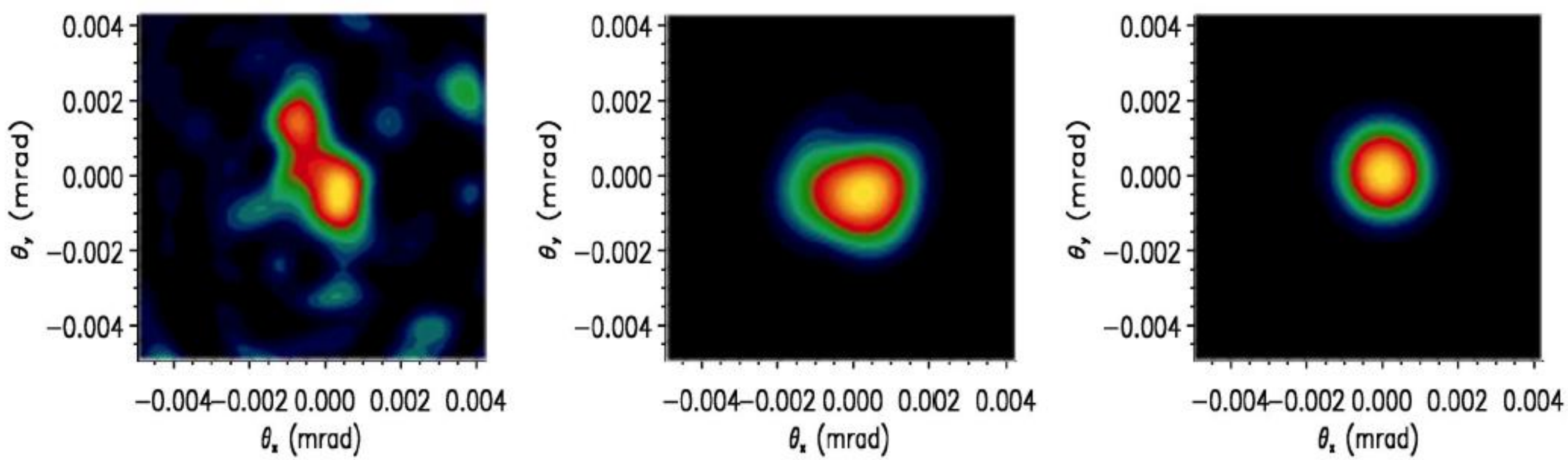
Three-Dimensional Effects: transverse modes

Cylindrical coordinates, Laguerre-Gaussian modes



Cartesian coordinates, Hermite-Gaussian modes





(a) $z = 25$ m (b) $z = 50$ m (c) $z = 75$ m

FIG. 9. (Color) Evolution of the LCLS transverse profiles at different z locations (courtesy of Sven Reiche, UCLA).