Homework 15. Due November 8

Problem 1. 30 points. Statistical definition of beam emittance.

We consider a statistical distribution of non-interacting particles in phase space (x, x'). Let $\rho(x, x')$ be the distribution function with

$$\int \rho(x, x')dxdx' = 1$$

the first and second moments of beam distribution are

$$\langle x \rangle = \int x\rho(x, x')dxdx' \qquad \langle x' \rangle = \int x'\rho(x, x')dxdx'$$

$$\langle x^2 \rangle = \int x^2\rho(x, x')dxdx' \qquad \langle x'^2 \rangle = \int x'^2\rho(x, x')dxdx'$$

$$\sigma_x{}^2 = \langle x^2 \rangle - \langle x \rangle^2 \qquad \sigma_{x'}{}^2 = \langle x'^2 \rangle - \langle x' \rangle^2$$

$$\sigma_{xx'}{}^2 = \langle xx' \rangle - \langle x \rangle\langle x' \rangle \stackrel{\text{def}}{=} r\sigma_x\sigma_{x'}$$

here σ_x and $\sigma_{x'}$ are rms beam widths, and r is the correlation coefficient. The rms emittance is therefore defined as

$$\varepsilon_{rms}{}^2 = \sigma_x{}^2\sigma_{x'}{}^2 - \sigma_{xx'}{}^2 = \sigma_x{}^2\sigma_{x'}{}^2(1 - r^2)$$

a) Assuming that particles are uniformly distributed in an ellipse

$$\frac{x^2}{a^2} + \frac{x'^2}{b^2} = 1$$

show that the total phase space area is $A = \pi ab = 4\pi\varepsilon_{rms}$

b) Show that the rms emittance defined above is invariant under a coordinate rotation

$$X = x\cos\theta + x'\sin\theta \qquad X' = -x\sin\theta + x'\cos\theta$$

and show that the correlation coefficient r is 0 if we choose the rotation angle to be

$$\tan 2\theta = \frac{2\sigma_x\sigma_{x'}r}{\sigma_x{}^2 - \sigma_{x'}{}^2}$$

show that σ_X and $\sigma_{X'}$ reach extrema at this rotation angle.

c) In accelerators, particles are distributed in Courant-Snyder ellipse

$$I(x, x') = \gamma x'^2 + 2\alpha xx' + \beta x^2$$

where $1 + \alpha^2 = \beta\gamma$. Apply the coordinate rotation you did above to this invariant to show that

$$\varepsilon_{rms} = \frac{\sigma_x{}^2}{\beta} = \frac{\sigma_{x'}{}^2}{\gamma}$$

and $$r = -\frac{\alpha}{\sqrt{\beta\gamma}}$$

or $$\begin{pmatrix} \sigma_x{}^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'}{}^2 \end{pmatrix} = \varepsilon_{rms}\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

show that $$\boldsymbol{x}^T\boldsymbol{\sigma}\boldsymbol{x} = \frac{\mathbf{1}}{\varepsilon_{rms}}(\gamma x'^2 + 2\alpha xx' + \beta x^2)$$

where $\boldsymbol{x} = \begin{pmatrix} x \\ x' \end{pmatrix}$ thus $\boldsymbol{x}^T\boldsymbol{\sigma}\boldsymbol{x}$ is invariant.

d) For a linear Hamiltonian, particle motion in accelerator obeys Hamiltonian dynamics

$$\frac{dx'}{ds} = -\frac{\partial H}{\partial x} = -Kx$$

where K(s) is focusing function. Show that the rms emittance is conserved (hint: write what is $\frac{d\varepsilon_{rms}{}^2}{ds}$ in terms of $\frac{dx'}{ds}$ and $\frac{\partial H}{\partial x}$)