Homework 6.

Problem 1. 3 x 5 points. Function of a Jordan block

(a) Show that powers of m x m Jordan block

$$\mathbf{G}=\begin{bmatrix}\lambda & 1 & 0 & 0\\ 0 & \lambda & 1 & 0\\ 0 & 0 & \dots & 1\\ 0 & 0 & 0 & \lambda\end{bmatrix}$$

are

$$\mathbf{G}^n=\begin{bmatrix}\lambda^n & C_1^n\lambda^{n-1} & C_2^n\lambda^{n-2} & \dots & C_k^n\lambda^{n-k} & C_{k+1}^n\lambda^{n-k-1} & ..\\ 0 & \lambda^n & C_1^n\lambda^{n-1} & \dots & C_{k-1}^n\lambda^{n+1-k} & C_k^n\lambda^{n-k} & \\ .... & .... & .... & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^n\end{bmatrix};C_k^n=\frac{n!}{(n-k)!k!}\quad(1)$$

Suggestion: use $\mathbf{G}^0=\mathbf{I};\mathbf{G}^1=\mathbf{G}$ - as first step, they satisfy 1. Than use induction assuming that (1) is correct for ***n*** and show that $\mathbf{G}^{n+1}=\mathbf{G}\cdot\mathbf{G}^n$ satisfy (1) for **n+1**. Use a well know ratio $C_k^{n+1}=C_k^n+C_{k-1}^n$.

(b) For a polynomial function $f(x)=\sum_{n=0}^{N} f_n x^n$ show that

$$f(\mathbf{G})=\sum_{n=0}^{N} f_n\mathbf{G}^n=\begin{bmatrix}\sum_{n=0}^{\infty} f_n\lambda^n \dots & \sum_{n=0}^{\infty} f_n C_k^n\lambda^{n-k} & ....\\ 0 & \dots & ....\\ 0 & 0 & \sum_{n=0}^{\infty} f_n\lambda^n\end{bmatrix}$$

and $\sum_{n=0}^{N} f_n C_k^n\lambda^{n-k}=\frac{1}{k!}\frac{d^k f}{d\lambda^k}$ #

(c) Prove that for an arbitrary (well behaved function!) $f(x)=\sum_{n=0}^{\infty} f_n x^n$

(d) $$f(\mathbf{G})=\begin{bmatrix}f(\lambda) & f'(\lambda)/1! & \dots f^{(k)}(\lambda)/k! & f^{(n-1)}(\lambda)/(n-1)!\\ 0 & f(\lambda) & \dots & f^{(n-2)}(\lambda)/(n-2)!\\ \dots & .... & \dots & \dots\\ 0 & 0 & \dots & f'(\lambda)/1!\\ 0 & 0 & \dots & f(\lambda)\end{bmatrix}$$

Solution:

$$\mathbf{G}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & & 1 \end{bmatrix};\ \mathbf{G}^1 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & ... & 1 \\ 0 & 0 & & \lambda \end{bmatrix}$$

$$\mathbf{G}^2 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & ... & 1 \\ 0 & 0 & & \lambda \end{bmatrix} \cdot \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & ... & 1 \\ 0 & 0 & & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & \lambda & 1... & 0 \\ 0 & \lambda^2 & \lambda... & 0 \\ 0 & 0 & ... & \lambda \\ 0 & 0 & 0 & \lambda^2 \end{bmatrix}$$

Induction: let assume it is correct for n:

$$\mathbf{G}^n == \begin{bmatrix} \lambda^n & n\lambda^{n-1}/1! & n(n-1)\lambda^{n-1}/2!... & ... \\ 0 & \lambda^n & n\lambda^{n-1}/1! & ... \\ 0 & 0 & ... & n\lambda^{n-1}/1! \\ 0 & 0 & 0 & \lambda^n \end{bmatrix}$$

$$\mathbf{G}^{n+1} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & ... & 1 \\ 0 & 0 & & \lambda \end{bmatrix} \cdot \begin{bmatrix} \lambda^n & n\lambda^{n-1}/1! & n(n-1)\lambda^{n-1}/2!... & ... \\ 0 & \lambda^n & n\lambda^{n-1}/1! & ... \\ 0 & 0 & ... & n\lambda^{n-1}/1! \\ 0 & 0 & 0 & \lambda^n \end{bmatrix} =$$

$$\begin{bmatrix} \lambda^{n+1} & (n+1)\lambda^n/1! & (n(n-1)+2n)\lambda^{n-1}/2!... & ... \\ 0 & \lambda^{n+1} & (n+1)\lambda^n/1! & ... \\ 0 & 0 & ... & (n+1)\lambda^n/1! \\ 0 & 0 & 0 & \lambda^{n+1} \end{bmatrix}$$

$$\begin{bmatrix} \lambda^n & C_1^n\lambda^{n-1} & C_2^n\lambda^{n-2} & ... & C_k^n\lambda^{n-k} & C_{k+1}^n\lambda^{n-k-1} & .. \\ 0 & \lambda^n & C_1^n\lambda^{n-1} & ... & C_{k-1}^n\lambda^{n+1-k} & C_k^n\lambda^{n-k} & \\ .... & .... & .... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^n \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & ... & 1 \\ 0 & 0 & & \lambda \end{bmatrix} \cdot \begin{bmatrix} \lambda^n & C_1^n \lambda^{n-1} & C_2^n \lambda^{n-2} & ... & C_k^n \lambda^{n-k} & C_{k+1}^n \lambda^{n-k-1} & .. \\ 0 & \lambda^n & C_1^n \lambda^{n-1} & ... & C_{k-1}^n \lambda^{n+1-k} & C_k^n \lambda^{n-k} & \\ .... & .... & .... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^n \end{bmatrix} =$$

$$\begin{bmatrix} \lambda^{n+1} & (C_1^n+1)\lambda^n & (C_2^n + C_1^n)\lambda^{n-2} & ... & (C_k^n + C_{k-1}^n)\lambda^{n-k+1} & (C_{k+1}^n + C_k^n)\lambda^{n-k} & .. \\ 0 & \lambda^{n+1} & (C_1^n+1)\lambda^n & ... & .... & (C_k^n + C_{k-1}^n)\lambda^{n-k+1} & \\ .... & .... & .... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^{n+1} \end{bmatrix}$$

Using ration for polynomial coefficients: $C_k^{n+1} = C_k^n + C_{k-1}^n;\ C_k^n = n!/k!/(n-k)!$ proves the point for n+1. Hence, it is correct for all orders.

Hence, we can now calculate a polynomial functions or any function expandable into a Taylor series:

$$f(\mathbf{G}) = \sum_{n=0}^{\infty} f_n \mathbf{G}^n = \sum_{n=0}^{\infty} f_n \begin{bmatrix} \lambda^n & C_1^n \lambda^{n-1} & ....C_k^n \lambda^{n-k} ... & ... \\ 0 & & ... & .... \\ 0 & & ... & \lambda^n \end{bmatrix}^i = \begin{bmatrix} \sum_{n=0}^{\infty} f_n \lambda^n ... & \sum_{n=0}^{\infty} f_n C_k^n \lambda^{n-k} & .... \\ 0 & ... & .... \\ 0 & 0 & \sum_{n=0}^{\infty} f_n \lambda^n \end{bmatrix}$$

The final stroke is noting that

$$\sum_{n=0}^{\infty} f_n C_k^n \lambda^{n-k} = \sum_{n=0}^{\infty} f_n \cdot \frac{n!\lambda^{n-k}}{k!\cdot(n-k)!} = \frac{1}{k!}\sum_{n=0}^{\infty} f_n \cdot \frac{n!\lambda^{n-k}}{(n-k)!} = \frac{1}{k!}\sum_{n=0}^{\infty} f_n \cdot \lambda^{n-k} \prod_{j=0}^{k-1}(n-j)$$

$$= \frac{1}{k!}\frac{d^k}{d\lambda^k}\sum_{n=0}^{\infty} f_n \cdot \lambda^n = \frac{1}{k!}\frac{d^k f}{d\lambda^k} \ \#$$

which gives final:

$$f(\mathbf{G}) = \begin{bmatrix} f(\lambda) & f'(\lambda)/1! & ...f^{(k)}(\lambda)/k! & f^{(n-1)}(\lambda)/(n-1)! \\ 0 & f(\lambda) & ... & f^{(n-2)}(\lambda)/(n-2)! \\ ... & .... & ... & ... \\ 0 & 0 & ... & f'(\lambda)/1! \\ 0 & 0 & ... & f(\lambda) \end{bmatrix}$$