Homework 5. Due September 23

Problem 1. 5 points. Following up HW4: you proved that simple combination of field multipoles cannot describe the edge of a magnet. You also learned that we can used Laplacian equation on effective field potential:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \varphi = 0$$

Let expand the potential in transverse direction while keeping arbitrary dependence along the beam propagaton axis (s=z)

$$\varphi = \sum_{n+m=k}^\infty a_{nm}(z)x^ny^m$$

Derive the condition (connections) between functions $a_{nm}(z)$.

Problem 2. 8 points. Prove that

$$\det[I + \epsilon A] = 1 + \epsilon \cdot \text{Trace}[A] + O(\epsilon^2)$$

where I is unit $n \times n$ matrix, A is an arbitrary $n \times n$ matrix and $\epsilon$ is infinitesimally small real number. Term $O(\epsilon^2)$ means that it contains second and higher orders of $\epsilon$.

Hint: first, look on the diagonal elements $\prod_{m=1}^n (1 + \epsilon a_{mm})$ first, then see what contribution to determinant comes from non-diagonal terms $a_{km}; k \neq m$. 

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\varphi = \sum_{n+m=k}^{\infty} a_{nm}(z) x^n y^m
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Derive the condition (connections) between functions \(a_{nm}(z)\).
Solution:
\[
\varphi = \sum_{n+m=k}^{\infty} a_{nm}(z) x^n y^m
\]
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \sum_{n+m=k}^{\infty} a_{nm}(z) x^n y^m =
\]
\[
\sum_{n+m=k}^{\infty} (a''_{nm} + (n+2)(n+1)a_{n+2,m} + (m+2)(m+1)a_{n,m+2}) x^n y^m = 0
\]
\[
(n+2)(n+1)a_{n+2,m} + (m+2)(m+1)a_{n,m+2} = -a''_{nm}
\]
It means that a “multipole” of k-th order will generate terms \(a_{n,k-n+2}\) where \(n=1, \ldots, k+2\)
No lower order terms are generated!

Problem 2. 8 points. Prove that
\[
\text{det}[I + \varepsilon A] = 1 + \varepsilon \cdot \text{Trace}[A] + O(\varepsilon^2)
\]
where I is unit \(n \times n\) matrix, A is an arbitrary \(n \times n\) matrix and \(\varepsilon\) is infinitesimally small real number. Term \(O(\varepsilon^2)\) means that it contains second and higher orders of \(\varepsilon\).

Hint: first, look on the diagonal elements \(\prod_{m=1}^{n}(1 + \varepsilon a_{mm})\) first, then see what contribution to determinant comes from non-diagonal terms \(a_{km}; k \neq m\).

Solution: The contribution to determinant from the diagonal elements is
\[
\prod_{m=1}^{n}(1 + \varepsilon a_{mm}) = 1 + \varepsilon \sum_{m=1}^{n} a_{mm} + O(\varepsilon^2) = 1 + \varepsilon \cdot \text{Trace}[A] + O(\varepsilon^2)
\] (1)
A generic term containing a non-diagonal element \(a_{km}; k \neq m\), excludes from the product at least two diagonal elements \(1 + \varepsilon a_{mm}\) and \(1 + \varepsilon a_{kk}\).
\[
\pm \varepsilon_{m \ldots k} \varepsilon a_{m,k} \prod_{i \neq m, j \neq k} a_{i,j} (\delta_{ij} + \varepsilon a_{i,j})
\]
Since the total number of elements in the product is \( n \), such term contains at least two non-diagonal elements, each of which contains \( \varepsilon \). This proves that non-diagonal terms can contribute only second and higher order term into \( O(\varepsilon^2) \). Combining it with (1) finishes the proof.