USPAS “Hadron Beam Cooling in Particle Accelerators”

Problem 1: Re-distribution of decrements/increments. 25 points

Let’s consider a storage ring with plane orbit (no torsion) and absence of elements coupling horizontal and vertical degrees of freedom. In this case, transvers components of the vector potential can be set to zero with Canonical momenta coinciding with mechanical momenta:

$$\pi_x = \frac{P_x}{p_o} = x'.$$

Horizontal and longitudinal oscillations in a storage ring remain coupled in all places where dispersion is non-zero. As we discuss in class, in the absence of dissipative processes,

$$\frac{dX}{ds} = \mathbf{D}(s)X;\ X = \begin{bmatrix} x \\ x' \\ \tau \\ \delta \end{bmatrix}; \mathbf{D}(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K_1(s) & 0 & 0 & K_o(s) \\ -K_o(s) & 0 & 0 & 1/(\gamma\beta)^2 \\ 0 & 0 & -U(s) & 0 \end{bmatrix}$$

and slow synchrotron oscillations ($Q_s << 1$) it is described by two actions and two phases with associated periodic eigen vectors eq. (**M3.33**):

$$X = \mathrm{Re}\left(a_x Y_x(s) e^{i(\psi_x+\varphi_x)} + a_s Y_s(s) e^{i(\psi_s+\varphi_s)}\right);\ \psi_x{}' = \frac{1}{\mathrm{w}_x(s)^2} \equiv \frac{1}{\beta_x(s)};\ \psi_s{}' \cong 2\pi Q_s \frac{s}{C};$$

$$Y_x = \begin{bmatrix} \mathrm{w}_x \\ \mathrm{w}_x' + \dfrac{i}{\mathrm{w}_x} \\ \eta\left(\mathrm{w}_x' + \dfrac{i}{\mathrm{w}_x}\right) - \eta' \mathrm{w}_x \\ 0 \end{bmatrix};\ Y_s = \begin{bmatrix} \dfrac{i}{\mathrm{w}_s}\eta \\ \dfrac{i}{\mathrm{w}_s}\eta' \\ \mathrm{w}_s \\ \dfrac{i}{\mathrm{w}_s} \end{bmatrix};\ \mathrm{w}_s = \sqrt{|\eta_\tau / \mu_3|};$$

Let’s add a distributed weak cooling process with linear drag forces:

$$\frac{dX}{ds} = (\mathbf{D}(s) + \delta\mathbf{D}(s))X;\ \delta\mathbf{D}(s) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\chi_x(s) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_{xs}(s) & 0 & 0 & -\chi_s(s) \end{bmatrix};$$

$$Trace(\delta\mathbf{D}) = -(\chi_x(s) + \chi_s(s)) < 0.$$

Assuming that this is a weak perturbation, calculate one turn decrements. Assuming that both friction forces are dissipative $\chi_{x,y}(s) > 0$ find if it is possible that one degree of freedom experiences growth instead of damping. In other words what an off-diagonal tear $g_{xs}(s)$ (asymmetric, i.e. non-Hamiltonian, x-δ coupling) contributes to distribution of decrements?

Problem 2:
Ion energy loss in the electron beam and cooling forces for electron cooling, 25 points

1. **5 points.** Start with energy loss for a single collision (assuming small deflection angle)

$$\Delta E_{loss}(b) = \frac{2Z^2e^4}{m_e v_{ei}^{\;2}(4\pi\varepsilon_0)^2 b^2}$$

Show that the energy loss rate of an ion passing through electron beam (with cylindrical geometry in lecture) with multiple collision can be expressed as

$$\frac{dE}{ds} = 2\pi n_e \int_{b_{\min}}^{b_{\max}} b\Delta E_{loss}(b)\,db = \frac{4\pi n_e Z^2 e^4}{m_e v_{ei}^{\;2}(4\pi\varepsilon_0)^2}\ln\left(\frac{b_{\max}}{b_{\min}}\right)$$

For a Gaussian electron velocity distribution

$$\frac{e^{-\left[\frac{v_{e\parallel}^2}{2\sigma_{ve\parallel}^2}+\frac{v_{e\perp}^2}{2\sigma_{ve\perp}^2}\right]}}{(2\pi)^{3/2}\sigma_{ve\perp}^2\sigma_{ve\parallel}}$$

the friction force becomes

$$\vec{F} = \frac{4\pi n_e Z^2 e^4 L_c}{m_e(4\pi\varepsilon_0)^2}\int_{-\infty}^{\infty}\frac{\vec{v}_e-\vec{v}_i}{\left|\vec{v}_e-\vec{v}_i\right|^3} f(\vec{v}_e)\,d^3v_e$$

.

Consider the case when transverse temperature is much higher than the longitudinal temperature, e.g., $\sigma_{ve\parallel} << \sigma_{ve\perp}$.

2. **10 points.** Show that the cooling forces (in both longitudinal and transverse directions) scale with $1/v_i^2$, for large ion velocities (v_i is ion velocity in the co-moving frame). Thus, the electron cooling is less efficient for ion beam with large velocity spread.

3. **10 points.** Show that the cooling force in both longitudinal and transverse directions scale linearly with v_i, for small ion velocities.

Problem 3: Damping rates of Optical Stochastic Cooling (Total: 25 points)

Let us consider an Optical Stochastic Cooler (OSC) and find the horizontal and longitudinal damping rates of OSC. Assume a particle motion that is only coupled between longitudinal and horizontal planes such that the vertical motion is uncoupled and can be safely omitted.

1. (5 points) The relative longitudinal momentum change of a particle can be expressed as $\delta p/p = \kappa \sin(ks)$ with $k = 2\pi/\lambda$. Show that the linearized longitudinal kick takes the following form:

$$\delta p/p = \kappa k \left(M_{51}^{\mathrm{PK}} x + M_{52}^{\mathrm{PK}} \theta_x + M_{56}^{\mathrm{PK}} \Delta p/p \right), \tag{1}$$

where $\mathbf{M}^{\mathrm{PK}}$ is a pick-up to kicker transfer matrix.

2. (10 points) We will use the result obtained in part 1 to define the matrix $\mathbf{M}^{\mathrm{C}}$ such as:

$$\mathbf{M}^{\mathrm{C}} = \kappa k \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{51}^{\mathrm{PK}} & M_{52}^{\mathrm{PK}} & 0 & M_{56}^{\mathrm{PK}} \end{bmatrix}. \tag{2}$$

The perturbation theory for the case of symplectic unperturbed motion yields that the tune shifts are:

$$\delta Q_k = \frac{1}{4\pi} \mathbf{v}_k^{\dagger} \mathbf{U} \mathbf{M}^{C} \mathbf{U} (\mathbf{M}^{\mathrm{PK}})^T \mathbf{U} \mathbf{v}_k = \frac{k\kappa}{4\pi} \mathbf{v}_k^{\dagger} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{26}^{\mathrm{PK}} & -M_{16}^{\mathrm{PK}} & 0 & M_{56}^{\mathrm{PK}} \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{v}_k \tag{3}$$

Find the damping rate of the betatron motion $\lambda_x = -2\pi \mathrm{Im} \delta Q_1$ if the eigenvector equals to:

$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{\beta_2} \\ -(i + \alpha_2)/\sqrt{\beta_2} \\ -(iD_2(1 - i\alpha_2) + D_2'\beta_2)/\sqrt{\beta_2} \\ 0 \end{bmatrix} \tag{4}$$

3. (10 points) Find the damping rate of the synchrotron motion $\lambda_s = -2\pi \mathrm{Im} \delta Q_2$ if the eigenvector is:

$$\mathbf{v}_2 = \begin{bmatrix} -iD_2/\sqrt{\beta_s} \\ -iD_2'/\sqrt{\beta_s} \\ \sqrt{\beta_s} \\ -i/\sqrt{\beta_s} \end{bmatrix}, \text{ with } \beta_s = R\eta/\nu_s \tag{5}$$

Problem 4: CeC related problem – 25 points

FEL amplifier – 8 points

a) In the FEL amplifier, the electrons lose energy while emitting photons during the FEL process. One way to compensate for the energy loss while keeping the resonant condition for the amplification process is by varying the wiggler field along the FEL, i.e. tapering. If we consider a helical wiggler for the FEL amplifier and assume that the energy of the electrons along the FEL evolves as

$$\gamma(z) = \gamma_0\left[1-\delta_{loss}(z)\right],$$

derive the dependance of the wigger field along the FEL so that the resonant condition can be kept during the amplification process (Hint: assume $\delta_{loss}(z) << \frac{K^2}{1+K^2} < 1$ and keep up to the linear terms in $\delta_{loss}(z)$).

Kicker response – 17 points

b) The energy kick due to the cooling force at the kicker section can be approximated by the following expression:

$$\Delta\delta\gamma_c(\Delta z) = -\Delta\gamma_0 \sin(k_0\Delta z)\exp\left(-\frac{\Delta z^2}{2\sigma^2}\right),$$

where $\Delta z = R_{56}\delta\gamma/\gamma$ is the slippage of the ion with respect to center of the cooling field. For an ion with synchrotron oscillation amplitude $I = \frac{1}{2}\left(P^2+\phi^2\right)$ where $P \equiv -h\frac{|\eta|}{\nu_s}\frac{\Delta p}{p}$ and $\phi = \omega_{rf}\tau$.

Derive the cooling rate of the CeC system with linear approximation of cooling force for

$$\Delta z << \sigma, 1/k_0$$

Derive the reduction of cooling rate with synchrotron oscillation. (Hint: follow the procedures developed in the electron cooling lectures for averaging cooling over the synchrotron oscillation. The final expression can be expressed into a 1-D integral).