Homework 9.

Problem 1. 10 points. Two frequency RF system. Consider a storage ring negative η_τ and the RF system operating at two frequencies:

$$\frac{d\mathrm{E}}{ds}=\frac{eV_o}{C}\left(\sin\left(h_{rf}k_o\tau\right)-\sin\left(2h_{rf}k_o\tau\right)\right)$$

Find stationary point on the phase diagram, draw characteristic phase-space trajectories (approximately is fine) and show the direction of the motion by arrows.

Problem 2. 4x5 points.

For a single frequency RF system with Hamiltonian with α indicating an energy loss/gain,

$$\left\langle \mathcal{H}_s\right\rangle=\eta_\tau\frac{\pi_\tau^{\,2}}{2}+\frac{1}{C}\frac{eV_{RF}}{p_oc}\frac{\cos\left(k_oh_{rf}\tau\right)}{k_oh_{rf}}+\alpha\cdot\tau;\quad \eta_\tau<0.$$

1. Define the stationary points (RF phases) in the phase space and indicate level of α when stationary points are no longer exists.
2. Draw phase space trajectories for $\alpha=\frac{1}{2}\cdot\frac{1}{C}\frac{eV_{RF}}{p_oc}$. Show the direction of the motion by arrows.
3. Define the depth of the "RF bucket", e.g. the difference between the maximum and minimum π_τ staying within a single RF separatrix (e.g. being localized). Express it through the RF voltage, the slip factor and the value of stationary phase. Note – consider the central separatrix around $\tau=0$.
4. Find period of the oscillation as function of $\left\langle \mathcal{H}_s\right\rangle$ inside the central separatrix (around $\tau=0$).

Solution:

Problem 1. Adding the corresponding term into the longitudinal Hamiltonian gives:

$$\left\langle \mathcal{H}_s\right\rangle=\eta_\tau\frac{\pi_\tau^{\,2}}{2}+\frac{eV_o}{p_ocC}\left(\frac{\cos\left(h_{rf}k_o\tau\right)}{h_{rf}k_o}-\frac{\cos\left(2h_{rf}k_o\tau\right)}{2h_{rf}k_o}\right)$$

Fig. 1 shows that graph of the RF potential with minima at nπ and maxima at ±1.0472 rad. The correspond to the stationaty points

$$\frac{d\tau}{ds}=\eta_\tau\pi_\tau=0\Rightarrow\pi_\tau=0;\quad \varphi=h_{rf}k_o\tau;$$

$$\frac{d\pi_\tau}{ds}=\frac{eV_o}{p_ocC}\left(\sin\varphi-\sin 2\varphi\right)=0\Rightarrow\varphi=n\pi;\varphi=\pm 1.0472\ rad.$$

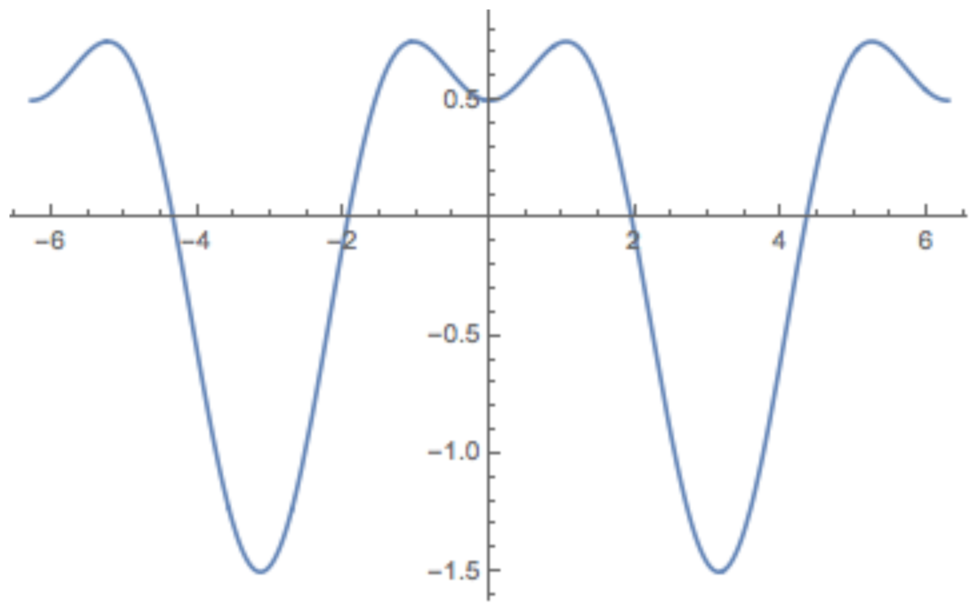


Fig.1. RF potential. It has minima at zero and ±π. It has maxima at ±1.0472 rad.

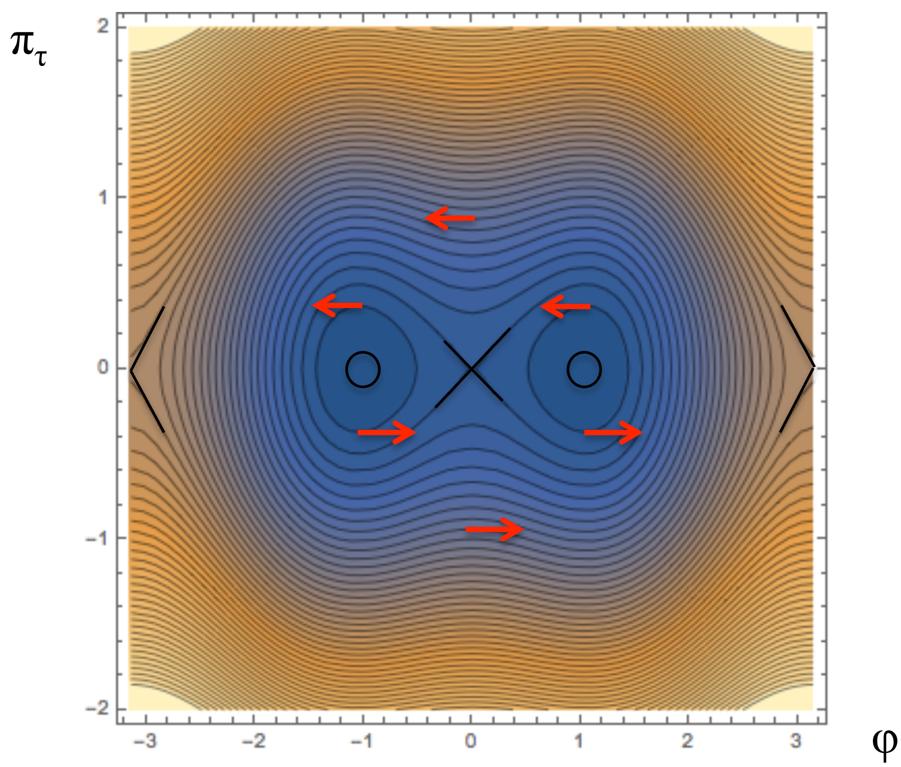


Fig. 2. RF separatrices for negative slip factor.

Expanding the Hamiltonian around the stationary points we can find if the motion is stable around it or unstable (separatrix point). Be we know that for positive mass (slip-factor) minima will be stable, but for negative mass (slip-factor) – the maxima will be stable. Hence, Fig.2 show the RF separatrices with stable point at ±1.0472 rad and unstable at zero and ±π.

Problem 2. With given Hamiltonian

$$\langle \mathcal{H}_s \rangle = \frac{\eta_\tau}{C}\frac{\pi_\tau^2}{2} + \frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}} + \alpha\cdot\tau; \quad \eta_\tau < 0.$$

we can easily find stationary points:

$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau = 0 \Rightarrow \pi_\tau = 0; \;\; \varphi = h_{rf} k_o \tau;$$

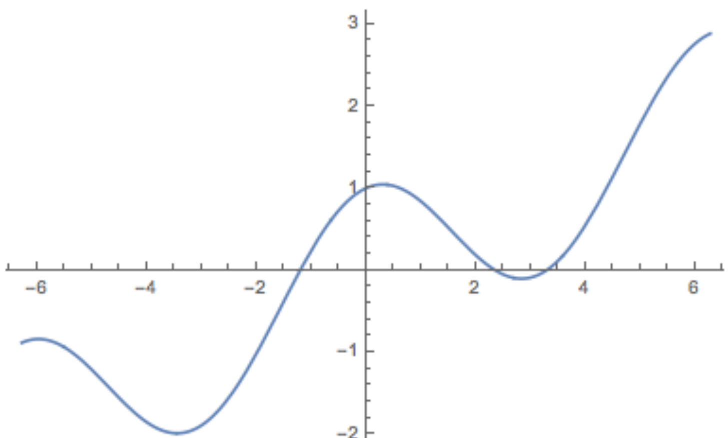
$$\frac{d\pi_\tau}{ds} = \frac{eV_o}{p_o c C}\sin\varphi - \alpha = 0 \Rightarrow \sin\varphi_o = l_f = C\cdot p_o c\left|\frac{\alpha}{eV_o}\right|.$$

$$\varphi_{o\pm} = \left\{ \begin{array}{c} 2n\pi + \varphi_o \\ (2n+1)\pi - \varphi_o \end{array} \right\}; \varphi_o = \sin^{-1}\left(l_f\right);$$

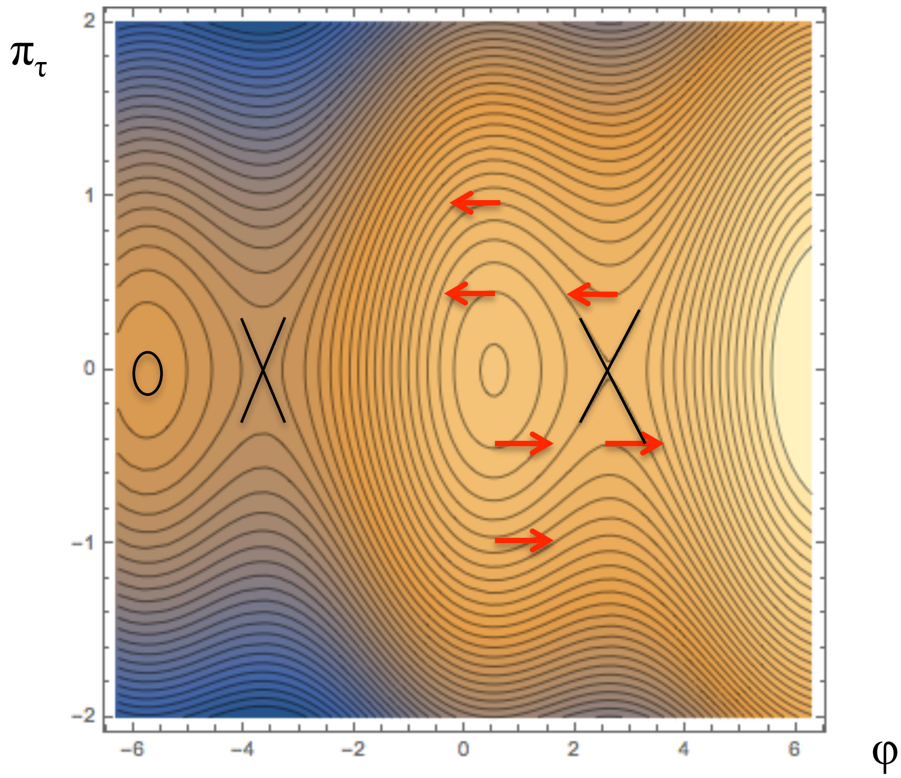
Stationary point do not exist (e.g. it requires impossible $|\sin\varphi_o| > 1$) if the energy loss (gain) per turn exceed the maximum energy change in the RF cavity:

$$|\Delta E| = |\alpha| C\cdot p_o c \geq |eV_o|$$

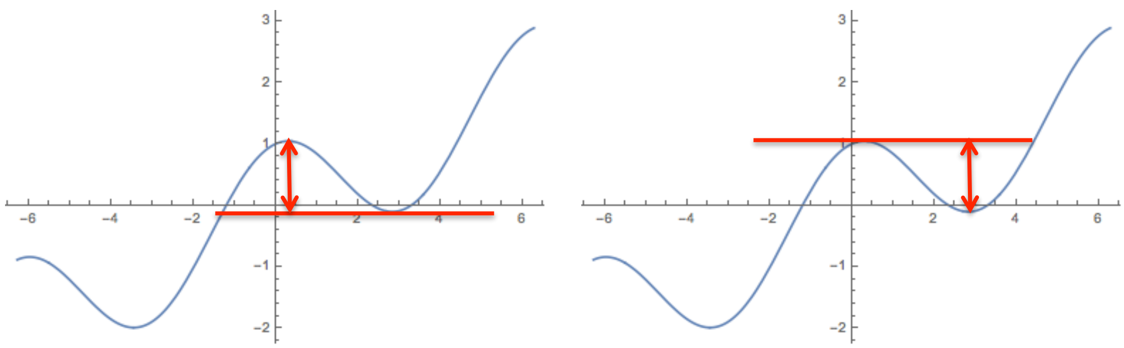
We now assume that the energy loss α is positive (typical for synchrotron radiation losses or acceleration), we can see that the potential in the Hamiltonian has maxima at $\varphi_o = 2n\pi + \sin^{-1}\left(l_f\right)$ and minima at $\varphi_o = (2n+1)\pi - \sin^{-1}\left(l_f\right)$. It means that for negative slip factor (typical for electron storage rings) $\varphi_o = 2n\pi + \sin^{-1}\left(l_f\right)$ will be stable.



For positive slip factor, $\varphi_o = (2n+1)\pi - \sin^{-1}\left(l_f\right)$ will be stable point. For l_f =1/2 $\varphi_{o+} = \sin^{-1}(0.5) = \frac{\pi}{6}$, e.g. 30 degrees. For a negative slip factor the phase-space trajectories will be as shown below.



The depth of the RF bucket depends on the difference between the maxima and minima of the potential – see fig. below.



Potential depth for negative (left) and positive (right) slip factor – it is obvious that the just different by sign. Particles outside of the RF "bucket" are not limited in the motion and slip either toward positive or negative infinity…

Finding the difference is straightforward:

$$\frac{1}{Ck_o h_{rf}}\frac{|eV_{RF}|}{p_o c}\left|\Delta\left(\cos(\varphi)+\sin\varphi_o\cdot\varphi\right)\Big|_{\varphi_{o-}}^{\varphi_{o+}}\right| = \frac{1}{Ck_o h_{rf}}\frac{|eV_{RF}|}{p_o c}\left(2\cos\varphi_o+\sin\varphi_o\cdot(2\varphi_o-\pi)\right);$$

and the maximum π_τ can be calculated by simply equating the depth of the potential well with kinetic energy $|\eta_\tau|\dfrac{\pi_\tau^{\ 2}}{2C}$:

$$\pi_{\tau acc} = \left|\frac{2}{\eta_\tau k_o h_{rf}}\frac{eV_{RF}}{p_o c}\left(2\cos\varphi_o + \sin\varphi_o\cdot(2\varphi_o - \pi)\right)\right|^{1/2}$$

It can be rewritten in many forms. Without energy loss the RF bucket depth is maximums and equal to

$$\pi_{\tau acc} = 2\left|\frac{1}{C\eta_\tau k_o h_{rf}}\frac{eV_{RF}}{p_o c}\right|^{1/2}$$

To find the small amplitude oscillation frequency, we need to expand the potential around the stationary point $\varphi_{o\pm}$ to the second order. Since $\alpha\cdot\tau$ is a linear function, it does not contribute to the second order ter. Hence, noticing that

$$\cos(\varphi_o + \delta\varphi) = \cos\varphi_o\cos\delta\varphi - \sin\varphi_o\sin\delta\varphi = \cos\varphi_o\left(1 - \frac{\delta\varphi^2}{2}\right) + \sin\varphi_o\delta\varphi + O\left(\delta\varphi^2\right)$$

we can simply conclude that our result will be different from what we derive in class by coefficient $\cos\varphi_o$:

$$\Omega = \frac{1}{C}\left|\eta_\tau k_o h_{rf}\frac{eV_{RF}}{p_o c}\cos\varphi_o\right|^{1/2}$$

We used here the fact that $\cos\varphi_{o-} = -\cos\varphi_{o+}$. For large amplitude oscillations, we can use the fact that the Hamiltonian is invariant and

$$\frac{\eta_\tau}{C}\frac{\pi_\tau^{\ 2}}{2} + \frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}} + \alpha\cdot\tau = H_o = inv$$

$$\pi_\tau = \eta_\tau\frac{d\tau}{ds} = \pm\sqrt{H_o - \frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}} - C\alpha\cdot\tau};$$

$$ds = \eta_\tau\frac{d\tau}{\pm\sqrt{H_o - \frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}} - C\alpha\cdot\tau}};$$

$$s = s_o \pm \eta_\tau\int_{\tau_1}^{\tau_2}\frac{d\tau}{\sqrt{H_o - \frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}} - C\alpha\cdot\tau}}$$

where $\tau_{1,2}$ are stopping points defined by $\pi_\tau = 0$:

$$\frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau_{1,2}\right)}{k_o h_{rf}} + \alpha\cdot\tau_{1,2} = H_o.$$

Since the period of oscillations comprises of travel back and forth, we have:

$$\frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau_{1,2}\right)}{k_o h_{rf}}+\alpha\cdot\tau_{1,2}=H_o;$$

$$P=2\left|\eta_\tau\right|\int_{\tau_1}^{\tau_2}\frac{d\tau}{\sqrt{H_o-\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}}-C\alpha\cdot\tau}}.$$