Problem 1

We are trying to find the approximate solution of the form

$$\lambda = a_0 + a_1\hat{C} + a_2\hat{C}^2 , \tag{1}$$

for the polynomial equation

$$\lambda^3 + 2i\hat{C}\lambda^2 - \hat{C}^2\lambda = i \ . \tag{2}$$

Inserting eq. (1) into eq. (2) yields

$$f\left(\hat{C}\right) \equiv \left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right)^3 + 2i\hat{C}\left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right)^2 - \hat{C}^2\left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right) - i = 0 \ . \tag{3}$$

Requiring eq. (3) to be satisfied in the zeroth order in $\hat{C}$ leads to

$$f\left(0\right) \equiv {a_0}^3 - i = 0 \ ,$$

which leads to (we are searching for the growing mode, i.e. $\mathrm{Re}\left(a_0\right) > 0$)

$$a_0 = \frac{\sqrt{3}}{2} + i\frac{1}{2} \ . \tag{4}$$

Requiring eq. (3) to be satisfied in the first order in $\hat{C}$ leads to

$$\frac{d}{d\hat{C}} f\left(\hat{C}\right)\bigg|_{\hat{C}=0} = 0 \Rightarrow 3a_1{a_0}^2 + 2i{a_0}^2 = 0 \Rightarrow a_1 = -i\frac{2}{3} \ . \tag{5}$$

Similarly, requiring eq. (3) to be satisfied in the second order in $\hat{C}$ leads to

$$\begin{aligned} \frac{d^2}{d\hat{C}^2} f\left(\hat{C}\right)\bigg|_{\hat{C}=0} = 0 &\Rightarrow \frac{d}{d\hat{C}}\left[3\lambda^2\lambda' + 2i\lambda^2 + 4i\hat{C}\lambda\lambda' - 2\hat{C}\lambda - \hat{C}^2\lambda'\right]\bigg|_{\hat{C}=0} = 0 \\ &\Rightarrow 6a_0{a_1}^2 + 3{a_0}^2 2a_2 + 8ia_0a_1 - 2a_0 = 0 . \\ &\Rightarrow a_2 = \frac{-3{a_1}^2 - 4ia_1 + 1}{3a_0} = -\frac{1}{9}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) \end{aligned} \tag{6}$$

Problem 2

From the criteria of FEL saturation,

$$\Omega_p L_G = 1 \ , \tag{7}$$

we obtain

$$\Omega_p \equiv \sqrt{\frac{eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0}} = \frac{1}{L_G} = \sqrt{3}\Gamma \ . \tag{8}$$

Taking the fourth power of eq. (8) yields

$$\left(\frac{eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0}\right)^2 = 9\Gamma^4 = 9\Gamma\Gamma^3 = 9\Gamma\frac{\pi j_0\theta_s^2\omega}{c\gamma_z^2\gamma I_A} \ , \tag{9}$$

where we used

$$\Gamma \equiv \left[\frac{\pi j_0\theta_s^2\omega}{c\gamma_z^2\gamma I_A}\right]^{1/3} . \tag{10}$$

Eq. (9) can be rewritten into

$$E^2 = 9\Gamma\frac{c\gamma_z^2}{\omega}\frac{\pi j_0}{e^2\gamma I_A}\mathcal{E}_0^2 = 9\rho\frac{\pi j_0}{e^2\gamma I_A}\mathcal{E}_0^2 \ . \tag{11}$$

Inserting the definition of Alfven current,

$$I_A = \frac{4\pi\varepsilon_0 mc^3}{e} \ , \tag{12}$$

into eq. (11) yields

$$E^2 = 9\rho\frac{\pi j_0}{e\gamma 4\pi\varepsilon_0 mc^3}\mathcal{E}_0^2 = \frac{9}{4}\rho\frac{j_0}{e\varepsilon_0 c}\mathcal{E}_0 \ , \tag{13}$$

where we used

$$\mathcal{E}_0 = m\gamma c^2 . \tag{14}$$

Consequently, the radiation power at saturation is

$$P_{sat} = \varepsilon_0 c E^2 A = \frac{9}{4}\rho\frac{j_0 A}{e}\mathcal{E}_0 = \frac{9}{4}\rho\frac{\mathcal{E}_0}{e}I_e \ , \tag{15}$$

i.e. $\chi = \dfrac{9}{4}$.