PHY 564
Advanced Accelerator Physics
Lecture 25

Nonlinear elements and nonlinear dynamics. Part I

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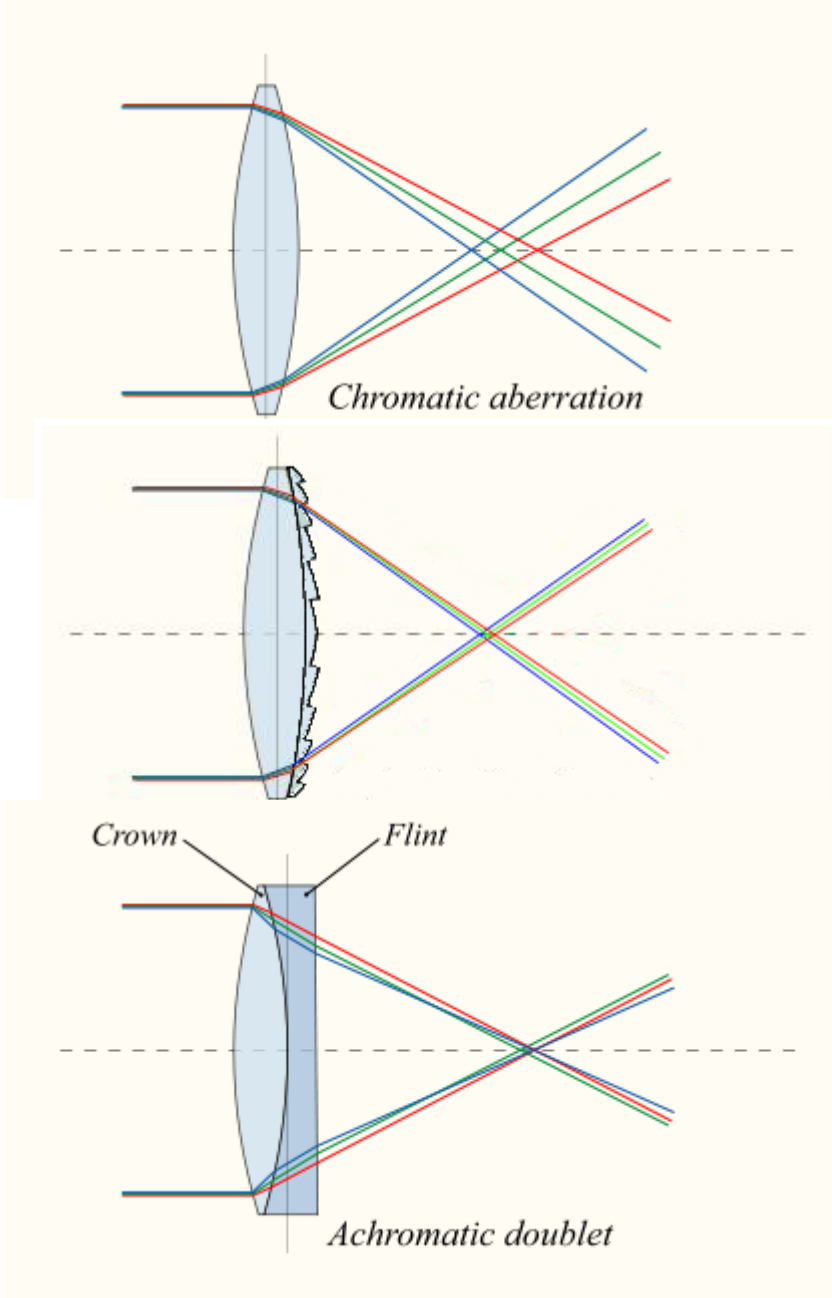
Chromaticity and correction



Nonlinear effects in particle's motion arise from various sources: high order kinematic terms in Hamiltonian expansion, spatial and temporal inhomogeneity of EM fields, edge effects, bending (e.g. bending plus gradient generates third order term), collective fields (space charge, wake-fields, beam-beam collisions). Typical methods include Hamiltonian perturbation methods or numerical tracking of many types (from particles tracking to particle-in-cell codes). A novel approach, exploiting symmetries of Hamiltonian systems and power of Lie algebraic tools, is the most comprehensive approach to the non-linear beam dynamics. Hence, a short introduction to this method.

But first, let us start by discussing a typical – and very important – nonlinear effect called chromaticity. It is nothing else than dependence of the betatron tune on particle's energy. While you can do this for fully coupled motion using our well-developed parameterization and perturbation methods, here –for compactness - we will consider just an uncoupled betatron motion with Hamiltonian in transverse magnetic field:

$$\tilde{h} = -P_s = -(1+Kx)\sqrt{p^2 - p_x{}^2 - p_y{}^2} + \frac{e}{c}A_s; \quad p = p_o(1+\delta);$$

$$\frac{d(x,y)}{ds} = \frac{\partial \tilde{h}}{\partial p_{x,y}} = -(1+Kx)\frac{p_{x,y}}{\sqrt{p^2 - p_x{}^2 - p_y{}^2}} \cong -(1+Kx)\frac{p_{x,y}}{p_o(1+\delta)} + O\left(p_{x,y}{}^3\right); \quad (25\text{-}1)$$

$$\frac{dp_x}{ds} = -\frac{\partial \tilde{h}}{\partial x} = K\sqrt{p^2 - p_x{}^2 - p_y{}^2} - \frac{e}{c}\frac{\partial A_s}{\partial x}; \frac{dp_y}{ds} = -\frac{\partial \tilde{h}}{\partial y} = -\frac{e}{c}\frac{\partial A_s}{\partial y};$$

with

$$A_2 = \sum_{n=1}^{\infty}\left\{\partial_x^{n-1}\left((1+Kx)B_y\right)_{ro}\frac{x^n}{n!} - \partial_y^{n-1}\left((1+Kx)B_x\right)_{ro}\frac{y^n}{n!}\right\} \quad (25\text{-}2)$$

If we consider easiest scenario for a storage ring using pure dipole and quadrupole field we get:

$$\frac{eA_2}{c}=\frac{eB_y}{c}x+\frac{eG}{c}\frac{x^2-y^2}{2}+p_oK^2\frac{x^2}{2};\;\; K=\frac{eB_{yro}}{p_oc}. \tag{25-3}$$

expression which does not contain any nonlinear terms (cubic or higher). Remember that linear term in (25-3) disappears because of the condition on the reference orbit. We can see that angle is x',y' inverse proportional to the particle's momentum $p=p_o(1+\delta)$ while the force $p'_{x,y}$ does not depend on the particle's momentum. Hence, the lowest order (cubic) term in the Hamiltonian expansion are $\delta\cdot p_{x,y}{}^2$.

Since here we are considering constant energy of our particles (p=const) and betatron oscillations, we also can rewrite (25-1) in more traditional form

$$h=-(1+Kx)\sqrt{1-\pi_x{}^2-\pi_y{}^2}+\frac{e}{pc}A_s(x,y,s);\;\; \pi_{x,y}=\frac{p_{x,y}}{p}=\frac{p_{x,y}}{p_o(1+\delta)};$$

$$\frac{d(x,y)}{ds}=\frac{\partial h}{\partial\pi_{x,y}}=-(1+Kx)\frac{\pi_{x,y}}{\sqrt{1-\pi_x{}^2-\pi_y{}^2}}\cong-(1+Kx)\pi_{x,y}+O\left(\pi_{x,y}{}^3\right) \tag{25-4}$$

$$\frac{d\pi_{x,y}}{ds}=-\frac{\partial h}{\partial(x,y)}=-\frac{e}{pc}\frac{\partial A_s}{\partial(x,y)}=-\frac{e}{p_oc(1+\delta)}\frac{\partial A_s}{\partial(x,y)}.$$

which clearly indicates that with fixed magnetic field, its affect on the particle is inverse proportional to particle's momentum p_o. This is traditional way of consider chromatic effect. Naturally, both descriptions are identical and gave exactly the same result! But this is always lost in description of chromatic effects that its origin is purely geometrical – for the same "so-called normalized" transverse emittance, angle of trajectory is inverse proportional to the particle's longitudinal momentum. In the Hamiltonian (25-4), the lowest (cubic) terms are $\delta\cdot x^2,\delta\cdot y^2$.

From our Hamiltonians it is obvious that there are nonlinear kinematic terms $\sim \pi_{x,y}{}^4$, $\pi_x{}^2\pi_y{}^2$ and higher in the Hamiltonian expansion. Furthermore, there are always third order $Kx\pi_{x,y}{}^2$ terms. While this term can cause third order resonance (we will look at them later) its role is not as important as that of the chromaticity of betatron oscillations. Hence, let's leave in the Hamiltonian (25-4) only linear (up to quadratic term) for transverse motion while keeping particle momentum arbitrary:

$$H = \frac{\pi_x{}^2 + \pi_y{}^2}{2} + \frac{1}{1+\delta}\left(\left(K_1 + K^2\right)\frac{x^2}{2} - K_1\frac{y^2}{2}\right);$$
$$H = H_o + \Delta H; \quad \Delta H = -\frac{\delta}{1+\delta}\left(\left(K_1 + K^2\right)\frac{x^2}{2} - K_1\frac{y^2}{2}\right). \tag{25-5}$$

Note, that similar (but much-much longer) expression can be derived for arbitrary magnetic and electric fields. While possible, it does not bring any new physics into what we considering here.

We already found what (in first order of perturbation) the tune shift will result from variation of the Hamiltonian (using our perturbation method):

$$\Delta Q_x \cong -\frac{1}{4\pi}\frac{\delta}{1+\delta}\oint \beta_x\left(K_1 + K^2\right)ds;\ \Delta Q_y \cong \frac{1}{4\pi}\frac{\delta}{1+\delta}\oint \beta_y K_1 ds; \tag{25-6}$$

e.g. the betatron tunes in such storage ring depend on the particle momentum (energy). Note that keeping $1+\delta$ in the denominator is overestimation of accuracy in (25-6) – there are other terms of order δ^2 and higher. The linear term in (25-6) is called chromaticity

$$C_x = \frac{\Delta Q_x}{\delta} \cong -\frac{1}{4\pi}\oint \beta_x\left(K_1 + K^2\right)ds;\ C_y = \frac{\Delta Q_y}{\delta} \cong \frac{1}{4\pi}\oint \beta_y K_1 ds; \tag{25-7}$$

Note that the linear chromatic term is strictly speaking is result of non-linear (third order) term in the Hamiltonian. Still, there is tradition to call it linear chromaticity and call the higher orders - higher order chromaticity. One important observation is that natural chromaticity (25-7) usually has negative values ("focusing of higher energy particles is weaker") for practically all storage rings. While statement in brackets is not strictly rigorous, it is true that for very high energy particles tunes will go to zero. A better explanation is coming from observation that in strong-focusing storage rings beta-functions are reaching maxima in focusing elements (e.g. β_x reaches maxima in focusing quadrupoles, while β_y reaches maxima in horizontally de-focusing quadrupoles where K_1<0) and therefore this tendency is correct. Furthermore, expectation for their values is that of the storage ring tune: $C_{x,y} \sim -(1 \div 2) \cdot Q_{x,y}$. Still, it is impossible to prove this rule explicitly in general case.

Chromaticity has multiplicity of effects on particle's dynamics in storage rings. In modern storage rings with Q ~ 10-100, chromatic effects are very important. Chromaticity can generates spread of betatron tunes (for a typical energy spread ~10^{-3}-10^{-4}), which can move particles onto linear and non-linear resonances. It also can impede injection into the storage rings as dynamics aperture (e.g. limit amplitudes of stable oscillations). Hence, chromaticity is usually corrected (by sextupoles, as we discuss it later in the lecture) to few units.

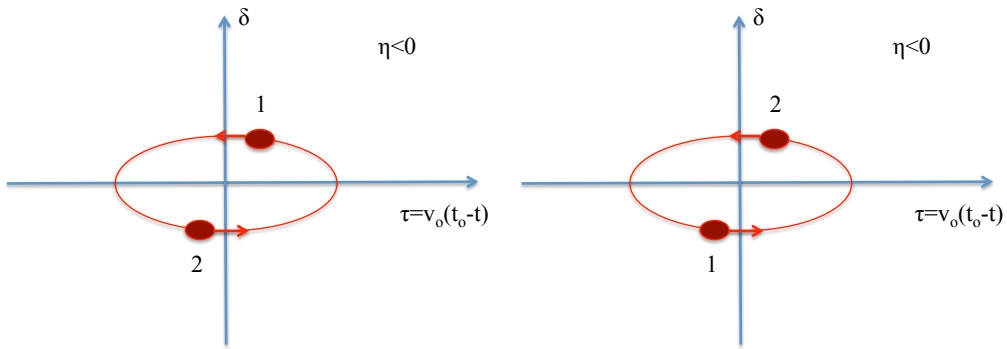
But the most important problem that natural (e.g. negative) chromaticity creates is so called head-tail instability, which occurs at energies above the critical, e.g. when the slip factor is negative. Head-tail instability is one of few major menaces in storage rings, which can simply kill the beam if not taken care off.

Incomplete list of major instabilities in a storage ring:

1. Wrong lattice, where motion is unstable;
2. Robertson instability (operating RF cavities with wrong sign of frequency detuning - we are not discussing it in this course);
3. Integer and parametric resonances (and frequently 3rg and 4^{th} order resonances);
4. Head-tail instability.

While a nasty microwave instability would fortunately saturate by inducing growth of energy spread, but not head-tail instability. It could be major killer of the beams.

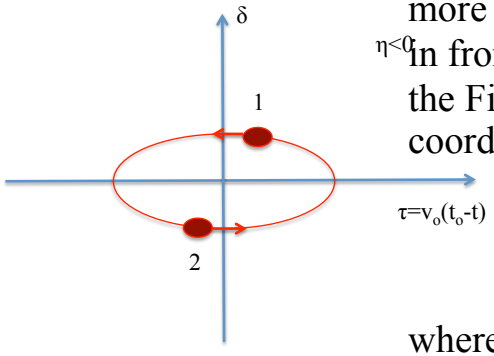
Let's consider this menace using a simple two-macro-particle model, which was originally used to describe this experimentally observed phenomena. In Fig. 1 we depict this simple model when two macro-particles execute slow synchrotron oscillations 180-degrees out of phase – hence the name, head and tail: when one particle is in front of the bunch, the other is at the tail, and vice-versa.



(a) (b)

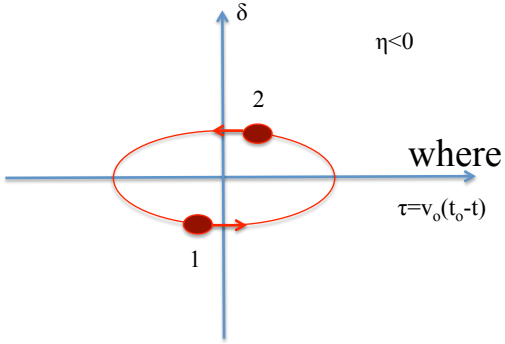
Fig. 1 Two macro-particles executing synchrotron oscillations. (a) particle 1 is in front of particles 2, (b) 180-degrees later – particles exchange the positions.

Since instability is sensitive to the chromaticity, details (such as strength of the wake-field and value of the amplitude of the oscillations) are not important. It is also an indication of universality of this problem – it just occurs if chromaticity is on a wrong sign! (e.g. negative for negative slippage). We will simply use some arbitrary values assuming that synchrotron oscillations are much slower that betatron ones. Finally, one more important fact you learned from the class on wake-fields and instabilities: particles in front of the bunch generate wake sensed by those in the tail, not vice-versa! Hence: for the Fig. 1 (a) we can write equations of motions as (we use y as generic transverse coordinate):



$$\begin{aligned} y_1'' + K_1(s)\left(1-\delta_1\right)y_1 &= 0; \\ y_2'' + K_1(s)\left(1-\delta_2\right)y_2 &= W\cdot y_1; \end{aligned} \tag{25-7}$$

where W is a transverse focusing (or defocusing) induced by macro-particle ahead. 180-degrees later, it changes to



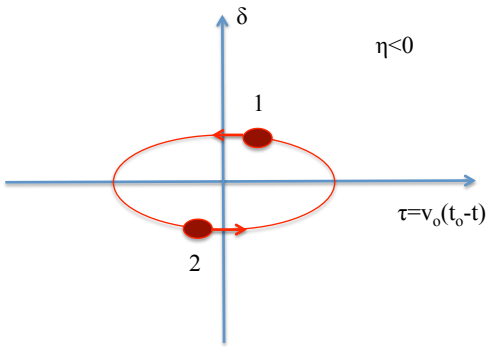
$$\begin{aligned} y_1'' + K_1(s)\left(1-\delta_1\right)y_1 &= W\cdot y_2; \\ y_2'' + K_1(s)\left(1-\delta_2\right)y_2 &= 0; \end{aligned} \tag{25-7}$$

where we just need to add

$$\delta_1 = \delta\cos\varphi_s;\delta_2 = -\delta\cos\varphi_s;\ \ \varphi_s = \Omega_s s;\ S = \frac{\pi}{\Omega_s};$$

$$\tau_1 = \tau\sin\varphi_s;\tau_2 = -\tau\sin\varphi_s;$$

$$\begin{aligned} y_1'' + K_1(s)\left(1-\delta_1\right)y_1 &= W\frac{1-sign(\tau_2-\tau_1)}{2}\cdot y_2; \\ y_2'' + K_1(s)\left(1-\delta_2\right)y_2 &= W\frac{1+sign(\tau_2-\tau_1)}{2}\cdot y_1; \end{aligned} \tag{25-8}$$



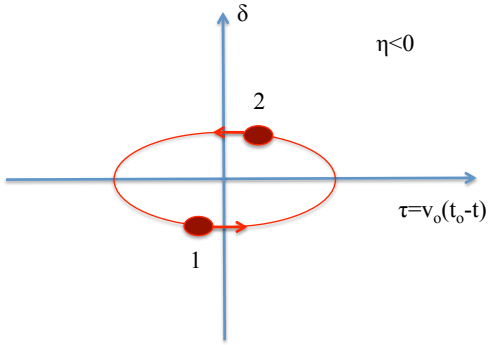
Let's consider particle (1) and (2) having complex amplitudes of oscillations a_1 and a_2 starting at zero s. For the first $\varphi_s = \{0,180\}$ degrees in picture (a).

$$y_1 = w_y \operatorname{Re} a_{10} e^{i(\psi+\Delta\psi)}; y_2 = w_y \operatorname{Re} a_{20}\, e^{i(\psi-\Delta\psi)};$$

$$\Delta\psi(s) = 2\pi C_y \delta \int_0^s \cos\Omega_s s\, ds = \frac{2\pi C_y \delta}{\Omega_s} \sin\Omega_s s; \quad S = \frac{\pi}{\Omega};$$

$$a_2(s) = a_{20} + \frac{a_{10}}{2i}\int_o^s W w_y{}^2 e^{2i\Delta\psi} ds;\ a_{21} = a_2(S) = a_{20} + a_{10}\left\langle W w_y{}^2 \right\rangle \frac{1}{2i}\int_o^S e^{2i\Delta\psi}\, ds; \qquad (25\text{-}9)$$

$$\begin{pmatrix} a_1(S) \\ a_2(S) \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix}; \ u = \left\langle W w_y{}^2 \right\rangle \frac{1}{2i}\int_o^S e^{2i\Delta\psi}\, ds.$$



Now, let's look how the amplitude of oscillation of first particles changes during next half of the synchrotron oscillation:

$$y_1 = w_y \,\mathrm{Re}\, a_1(s) e^{i(\psi-\Delta\psi)}; y_2 = w_y \,\mathrm{Re}\, a_{21} \mathrm{e}^{i(\psi+\Delta\psi)};$$

$$a_1(s) = a_{10} + a_{21}\frac{1}{2i}\int_S^s W w_y{}^2 e^{2i\Delta\psi} ds;\ a_{11} = a_1(2S) = a_{10} + a_{21}\left\langle W w_y{}^2 \right\rangle \frac{1}{2i}\int_S^{2S} e^{2i\Delta\psi}\, ds; \quad (25\text{-}10)$$

$$\begin{pmatrix} a_1(2S) \\ a_2(2S) \end{pmatrix} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \begin{pmatrix} a_{10} \\ a_{21} \end{pmatrix};\ u = \left\langle W w_y{}^2 \right\rangle \frac{1}{2i}\int_o^S e^{2i\Delta\psi}\, ds.$$

The overall matrix for a single synchrotron oscillation period is

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix} = \begin{bmatrix} 1+u^2 & u \\ u & 1 \end{bmatrix} \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix};$$

$$\det \begin{bmatrix} 1+u^2-\lambda & u \\ u & 1-\lambda \end{bmatrix} = (1-\lambda)(1+u^2-\lambda) - u^2 = (1-\lambda)^2 - \lambda u^2; \quad (25\text{-}11)$$

$$\lambda_{1,2} = 1 + \frac{u^2}{2} \pm u\sqrt{1+\frac{u^2}{4}}; u = \left\langle W w_y{}^2 \right\rangle \frac{1}{2i} \int_o^S e^{2i\Delta\psi}\, ds.$$

Note that determinant of matrix is 1, hence is one solution is growing, the other is damped. Since we are considering weak wake-field, we can write the eigen values as

$$u \approx \left\langle W w_y^2 \right\rangle \left(\frac{S}{2i} + \frac{2\pi C_y \delta}{\Omega_s} \right); \quad \lambda_{1,2} \cong e^{\pm \left\langle W w_y^2 \right\rangle \left(\frac{S}{2i} + \frac{2\pi C_y \delta}{\Omega_s} \right)} \quad (25\text{-}12)$$

and the growth rate is proportional to the chromaticity and its value should be limited. The detailed studies (which we have to skip) show that + (sign instability corresponding to positive chromaticity) is much weaker and that having a small positive chromaticity for storage ring above transition (negative slip factor) is required for stability of the beam – this is the mode in which most of electron storage ring and high energy hadron colliders do operate.

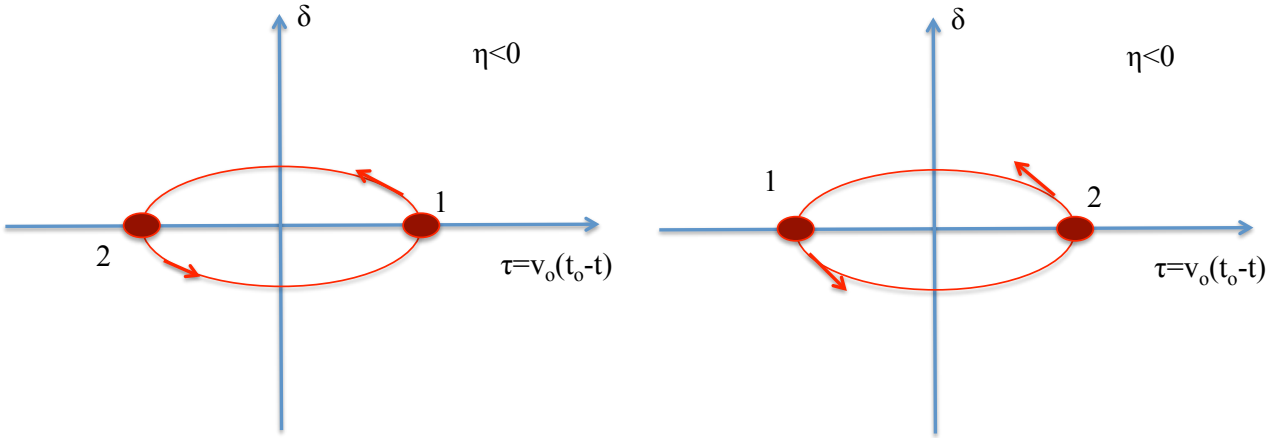


Fig.2. Two extreme positions.

Intuitively this can be described as follows. Let's consider negative chromaticity and the fact that the strength of the transverse wakefield is increasing with the distance between particles, e.g. most of the impact will come from the head particle (1) when it is in extreme forward position. (1) excites the tail particle (2) resonantly it oscillates in phase.

$$\tilde{y}_2 = \tilde{y}_{20} + T\varepsilon\tilde{y}_{10}$$

When they exchange position, the exited the tail particle (2) oscillates with higher betatron frequency (it passes through lower energy) than the (1) having higher energies. It means that particle (2) comes to head position with positive phase advance – it corresponds to an effective response from the future

$$\Delta\tilde{y}_1(t) = T\varepsilon\tilde{y}_{20}(t+\tau) + T\varepsilon^2\tilde{y}_{10}(t+\tau)$$

We can equivalently write

$$y_1''(t) + \omega^2 y_1(t) = T\varepsilon^2\tilde{y}_1(t+\tau) \approx T\varepsilon^2\left(\tilde{y}_1(t) + \tau\tilde{y}_1'(t)\right); \qquad (25\text{-}13)$$

generating growth rate of $\tau\, T\varepsilon^2$. We should note that $\tau = -\dfrac{4\pi C_y\delta}{\Omega_s c}$ is proportional to the chromaticity and has opposite sign for negative slippage. For positive slippage (below transition), natural sign of chromaticity is favored for head-tail stability.

These exercises were to establish a need for chromaticity compensations. Naturally, linear element cannot do this (they are introducing the chromaticity, not compensating it!). Hence, lest consider sextupole fields with

$$\frac{eA_2}{c} = \frac{eS}{c}\frac{x^3 - 3xy^2}{3!} \tag{25-14}$$

you can easily check that it satisfies 2D Maxwell equation. We are aware that in storage ring closed orbit depends on particle's momentum as

$$x_\delta = \eta_x(s)\delta \tag{25-15}$$

and introduction of sextupoles in (25-5) will result is:

$$H = \frac{\pi_x^{\,2} + \pi_y^{\,2}}{2} + \frac{1}{1+\delta}\left(\left(K_1 + K^2\right)\frac{x_\beta^{\,2}}{2} - K_1\frac{y^2}{2}\right) + \frac{1}{1+\delta}\frac{K_2}{3!}\left(x^3 - 3xy^2\right)$$

$$x = \eta_x(s)\delta + x_\beta;\; K_2 = \frac{eS}{p_o c};$$

$$H = H_o + \Delta H_1 + \Delta H_{NL} \tag{25-16}$$

$$\Delta H_1 = -\delta\left(\left(K_1 + K^2\right)\frac{x^2}{2} - K_1\frac{y^2}{2}\right) + \delta\cdot\eta_x\cdot K_2\frac{x_\beta^{\,2} - y^2}{2}$$

$$\Delta H_{NL} = \frac{K_2}{3!}\left(x_\beta^{\,3} - 3x_\beta y^2\right) + O\left(\delta^2\right);$$

We can calculate the linear chromaticity in the same fashion we deed above:

$$C_x = \frac{\Delta Q_x}{\delta} \cong -\frac{1}{4\pi}\oint\beta_x\left(K_1 + K^2 - \eta_x K_2\right)ds;$$

$$C_y = \frac{\Delta Q_y}{\delta} \cong \frac{1}{4\pi}\oint\left(\beta_y K_1 - \eta_x K_2\right)ds; \tag{25-17}$$

To zero chromaticities we should find distribution of sextupole fields (as function of s) such that that

$$\oint K_2(s)\eta_x(s)\beta_x(s)ds = \oint \beta_x(s)\left(K_1(s)+K(s)^2\right)ds;$$
$$\oint K_2(s)\eta_x(s)\beta_y(s)ds = \oint \beta_y(s)K_1(s)ds; \qquad (25\text{-}18)$$

Assuming positive dispersion (which is usual) it can be done by placing focusing sextupoles ($K_2>0$) in areas where β_x is large and β_y is small, and vice versa for defocusing quadrupoles with $K_2<0$. For most of know strong focusing lattices this can be done. The only exception is weak-focusing lattice where all terms are constants the compensation

$$\langle K_2\rangle\eta_x = \left(K_1+K^2\right);$$
$$\langle K_2\rangle\eta_x = -K_1; \qquad (25\text{-}19)$$

could be possible only when $K_1 = -K^2/2$, which is exactly on the top the coupling resonances $Q_x = Q_y$. Hence, in general, in a weak-focusing storage ring chromaticity could be compensated only in one plane.

How the chromaticity compensation works: particle's average orbits shifts as function of energy and displacement is sextupoles generates effective gradient (quadrupole) field, which compensate change of the focusing from regular quadrupoles. This process is called feed-down – displacement of a high (n-th) order multipole generates lower orders multipoles, from dipole up to (n-1).

One important notion – compensating chromaticity requires orbit dependence on energy, which comes only as result of bending magnet. It means, that it is impossible to compensate chromaticity in a perfectly linear accelerator (no bends!) since transverse dispersion is always equal zero.

Thus, we established that in a modern storage rings chromaticity could be compensated using sextupoles. What is not obvious is that this can create significant problems. Indeed, modern light sources in order to generate high brightness beams reducing emittance (22-25)

$$\left\langle a_x^2 \right\rangle = \frac{55}{32\sqrt{3}} \gamma^2 \frac{\hbar}{mc} \frac{\left\langle |K|^3 \left(\left(\mathrm{w}_x \eta_x' - \mathrm{w}_x' \eta_x \right)^2 + \left(\frac{\eta_x}{\mathrm{w}_x} \right)^2 \right) \right\rangle}{\left(1 - \xi_{xy}\right) \left\langle K^2 \right\rangle}$$

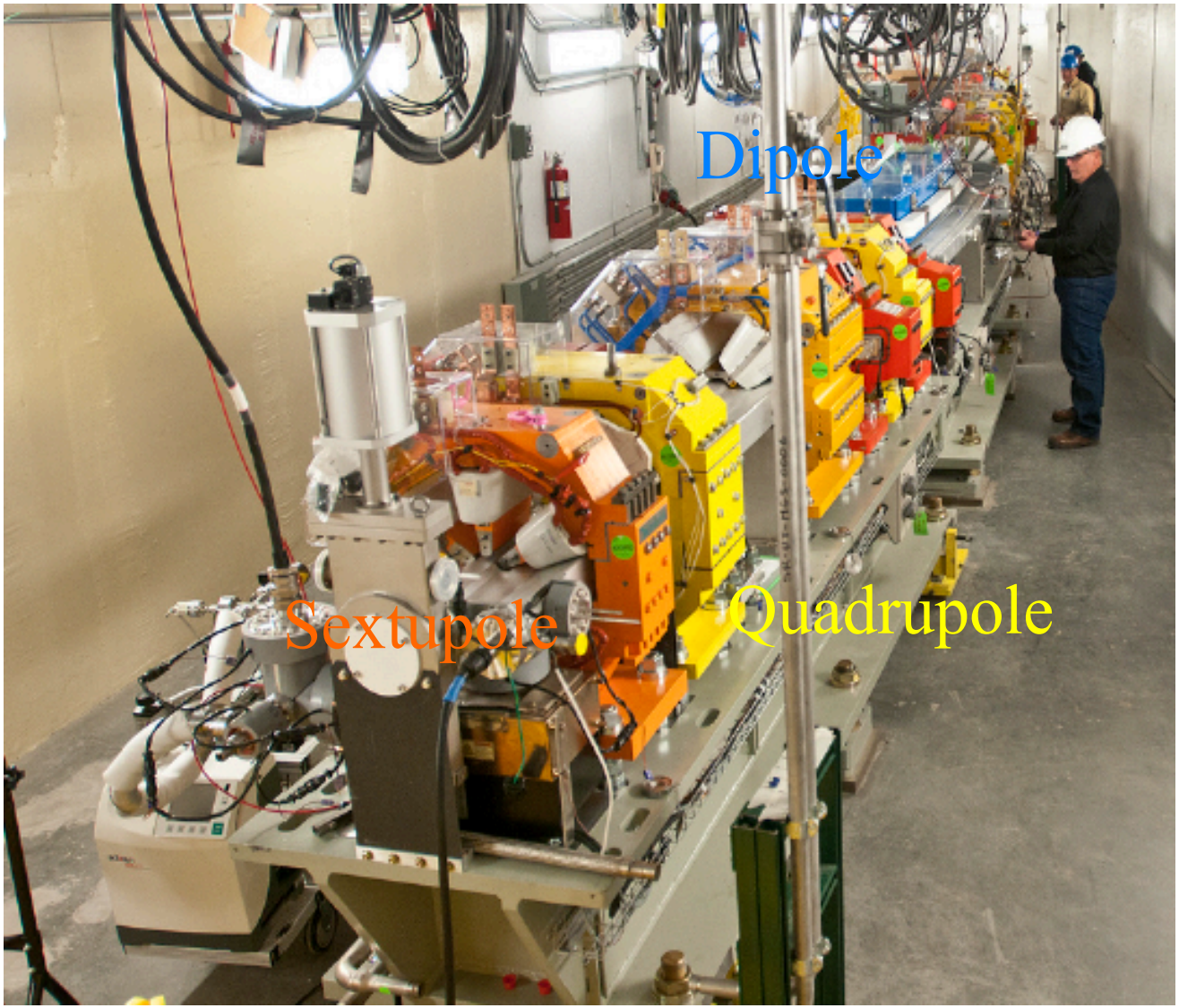
strong focusing resulting in very large betatron tunes (~30) and very small beta-functions $\beta \sim R/Q$ and dispersion $\eta \sim R/Q^2$ measured in few cm. As follows from (25-17), we will need sextupole strength

$$\left\langle |K_2| \right\rangle \sim \left\langle |K_1| \right\rangle / \eta_x;$$

e.g. field inside the aperture of accelerator is very nonlinear and particle oscillating with large amplitudes can become unstable.

Since we introduced sextupoles, we should notice that equations of motion become non-linear. Even though the kick is locally proportional to a square of the transverse displacement, we cannot assume that it will generate some kind of expansion to a map of second order. One can simply observe that there is no analytical solution for equation of motion in a sextupole, or that two short sextupoles will already generate forth order terms in the transformations. Needless to say that multiple thick non-linear elements making the map tractable only by computers. But there is a BIG BUT – there is still a lot we can do to describe and to understand this nonlinear map – beyond just staring on them helplessly.

NSLS II arc lattice



1. Tune shift, or tune spread, due to chromatic aberration:

$$\Delta\nu_x = \left[-\frac{1}{4\pi}\oint\beta_x(s)K_x(s)ds\right]\delta \equiv C_x\delta, \qquad C_x = d\nu_x/d\delta$$

$$\Delta\nu_z = \left[-\frac{1}{4\pi}\oint\beta_z(s)K_z(s)ds\right]\delta \equiv C_z\delta, \qquad C_z = d\nu_z/d\delta$$

The chromaticity induced by quadrupole field error is called natural chromaticity. For a simple FODO cell, we find

$$\Delta\nu_x = \left[-\frac{1}{4\pi}\oint\beta_x(s)K_x(s)ds\right]\delta \approx -\frac{1}{4\pi}\sum\frac{\beta_{xi}}{f_i}\ \delta$$

$$C_{y,\mathrm{nat}}^{\mathrm{FODO}} = -\frac{1}{4\pi}N\left(\frac{\beta_{\max}}{f}-\frac{\beta_{\min}}{f}\right) = -\frac{\tan(\Phi/2)}{\Phi/2}\nu_y \approx -\nu_y$$

We define the specific chromaticity as $\xi_x = C_x/\nu_x, \qquad \xi_z = C_z/\nu_z$

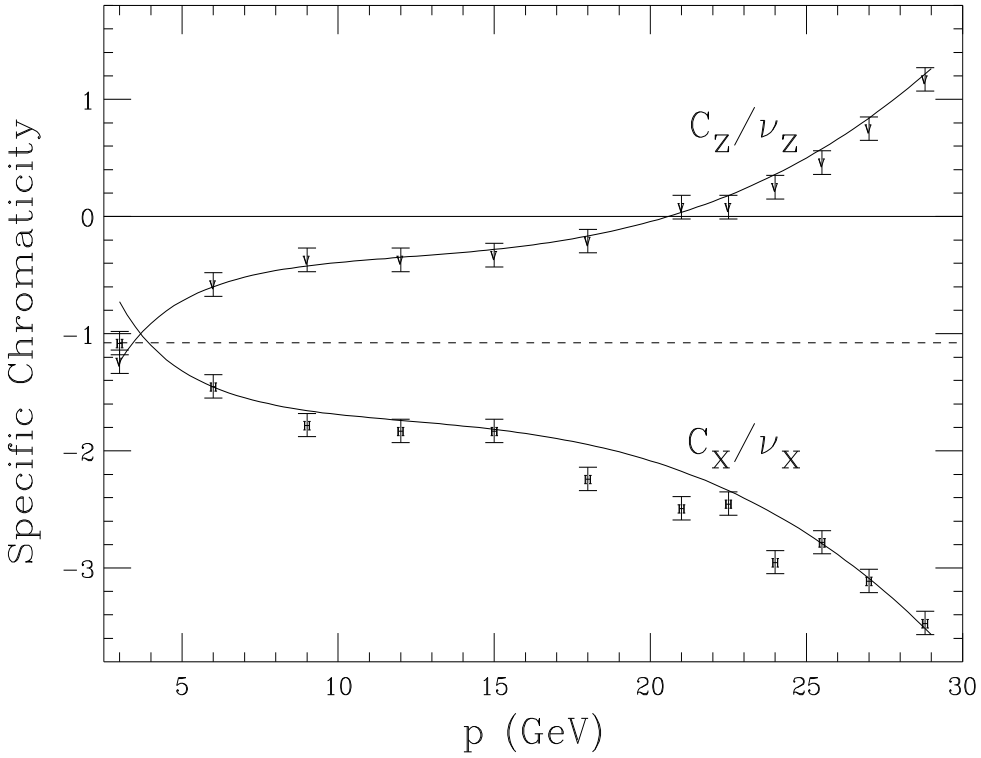
The **specific chromaticity is about −1 for FODO cells**, and can be as high as -4 for high luminosity colliders and high brightness electron storage rings.

$$\sin\frac{\Phi}{2} = \frac{L_1}{2f} \qquad \beta_{\max} = \frac{2L_1(1+\sin(\Phi/2))}{\sin\Phi}, \quad \beta_{\min} = \frac{2L_1(1-\sin(\Phi/2))}{\sin\Phi}$$

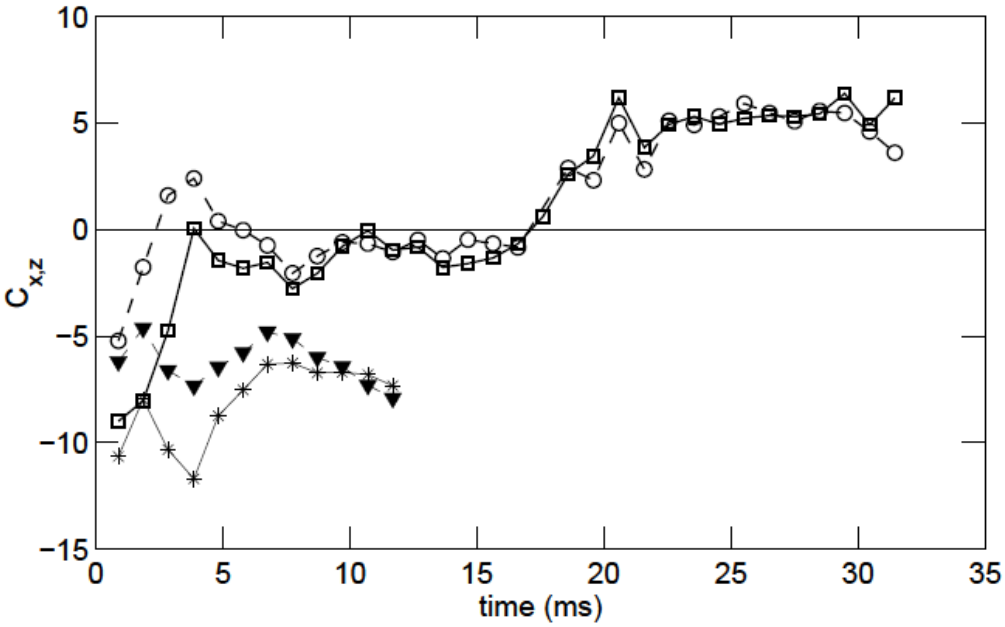
Examples:

BNL AGS (E. Blesser 1987): Chromaticities measured at the AGS.

$$C_{y,\mathrm{nat}}^{\mathrm{FODO}} = -\frac{\tan(\Phi/2)}{\Phi/2}\nu_y \approx -\nu_y$$



Fermilab Booster (X. Huang, Ph.D. thesis, IU 2005): The measured horizontal chromaticity C_x when SEXTS is on (triangles) or off (stars), and the measured vertical chromaticity C_z when SEXTS is on (dash, circles) or off (squares). The error bar is estimated to be 0.5. The natural chromaticities are $C_{nat,z}$=–7.1 and $C_{nat,x}$=–9.2 for the entire cycle.



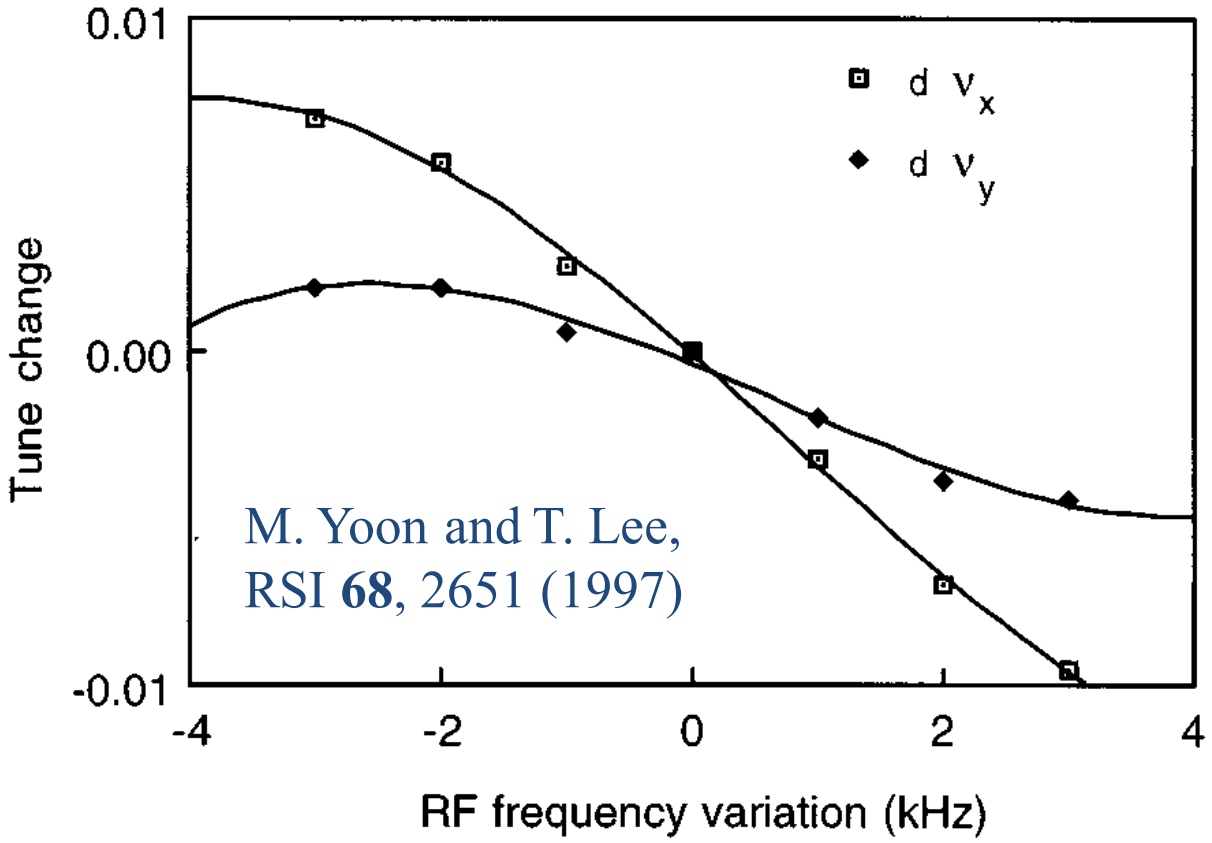
Chromaticity measurement:

The chromaticity can be measured by measuring the betatron tunes vs the rf frequency f, i.e.

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = (\alpha_c - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} = \eta\delta,$$

$$\Delta f / f_0 = -\eta\delta,$$

$$C = \frac{d\nu}{dp/p} = -\eta f_{rf}\frac{d\nu}{df_{rf}}$$

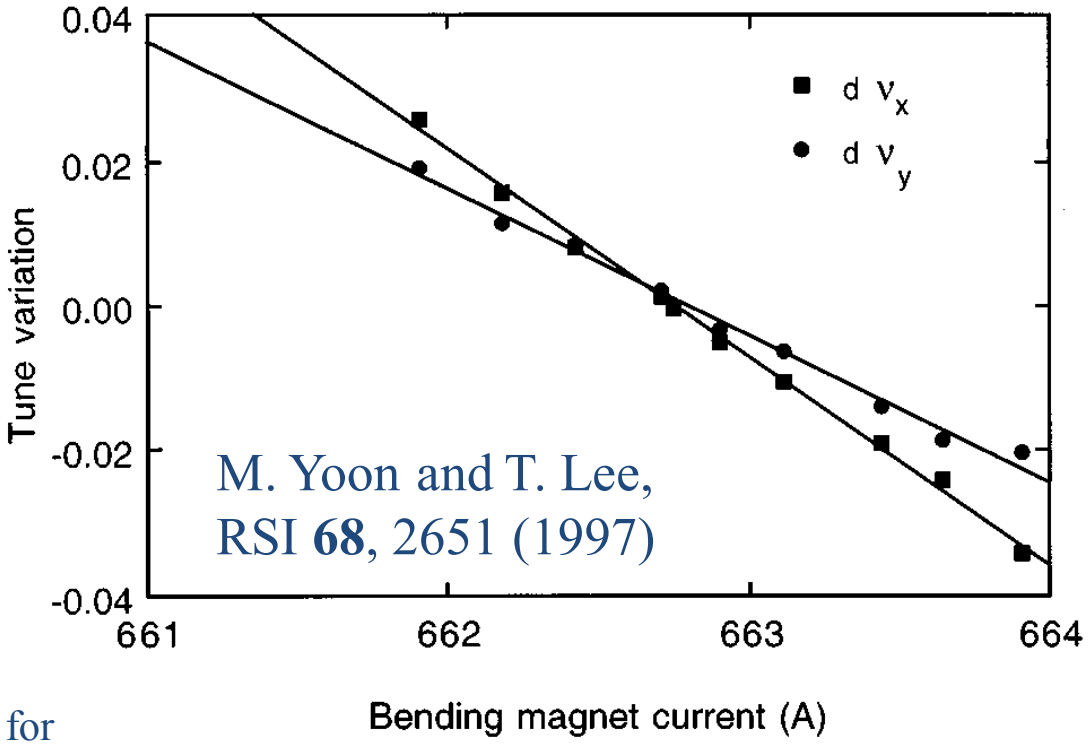


The chromaticites are Cx=+2.9, Cy=+1.4.

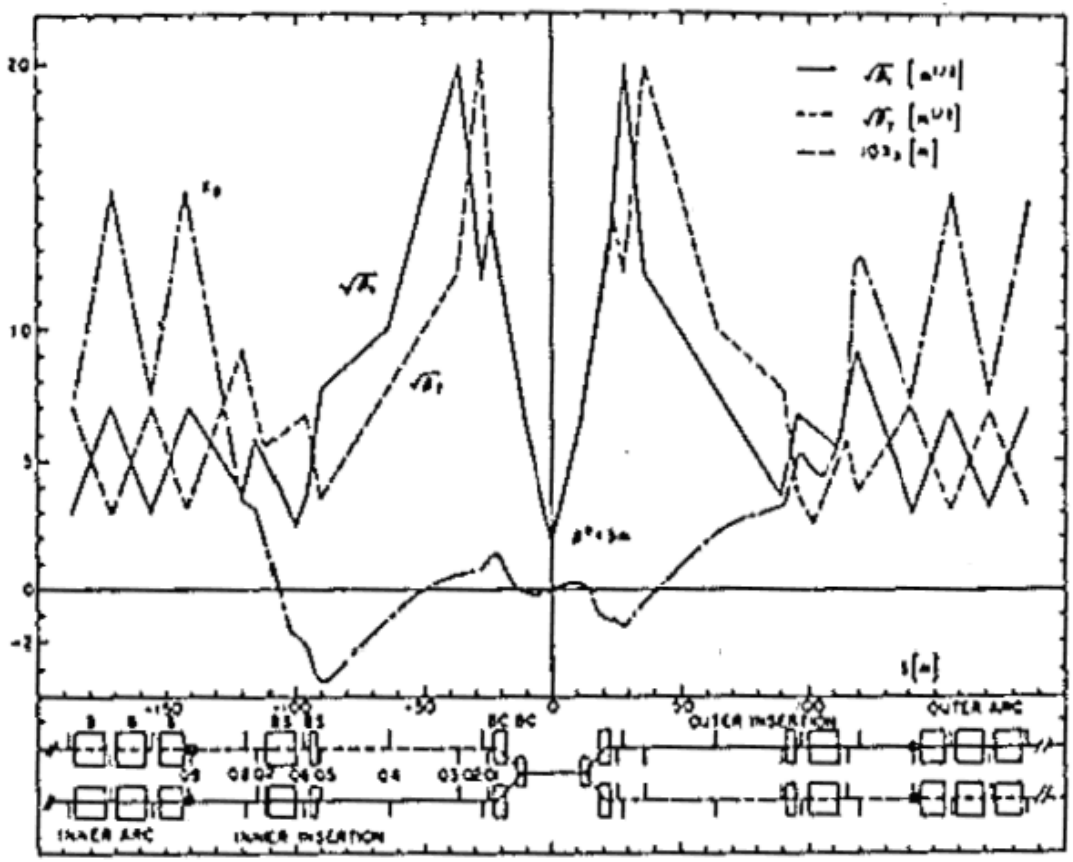
The **Natural chromaticity** can be obtained by measuring the tune variation vs the bending-magnet current at a **constant rf frequency**. Change of the bending-magnet current is equivalent to the change of the beam energy. Since the orbit is not changed, the effect of the sextupole magnets on the beam motion can be neglected. The Figure shows the horizontal and vertical tune vs the bending-magnet current in the PLS storage ring.

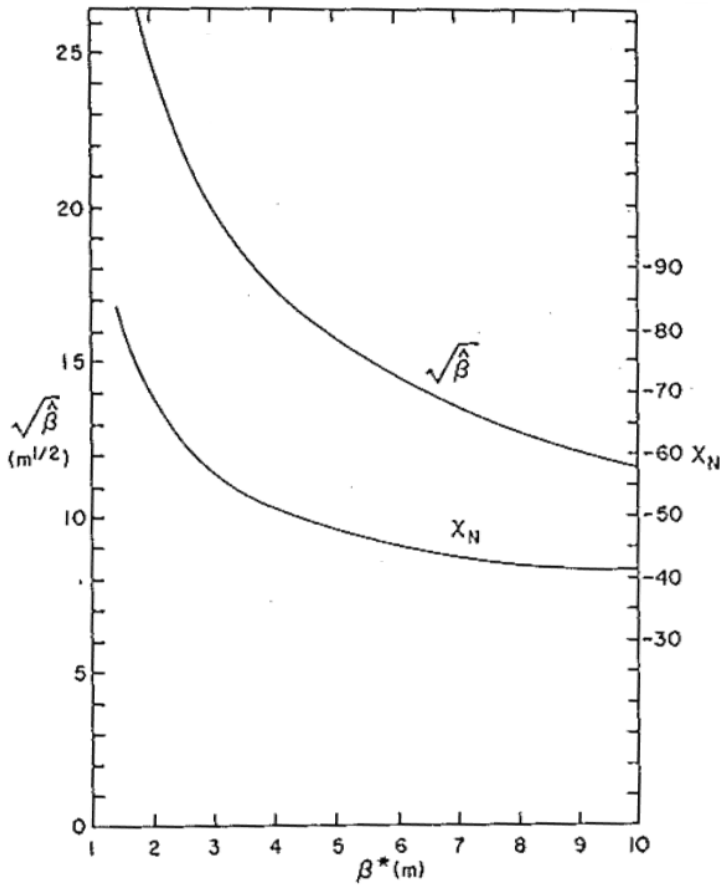
$$C = \frac{d\nu}{dp/p} = \frac{d\nu}{dB/B} = \frac{d\nu}{dI/I}$$

The data give
C_x=−18.96, C_z=−13.42;
vs theory:
C_x=−23.36, C_z=−16.19.



Note that this method may not applicable for combined function dipoles.



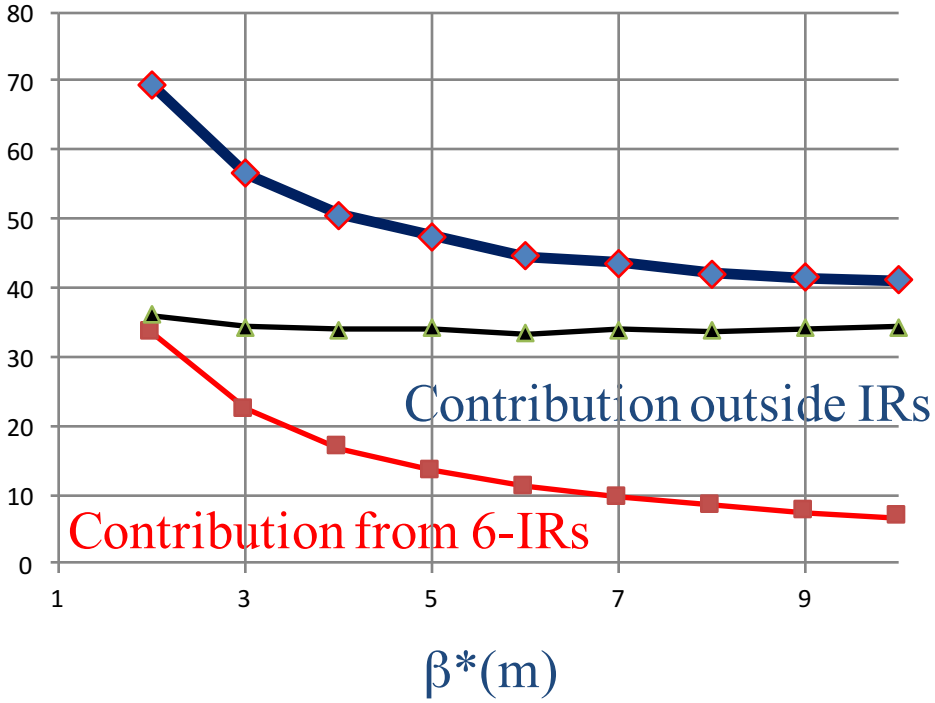


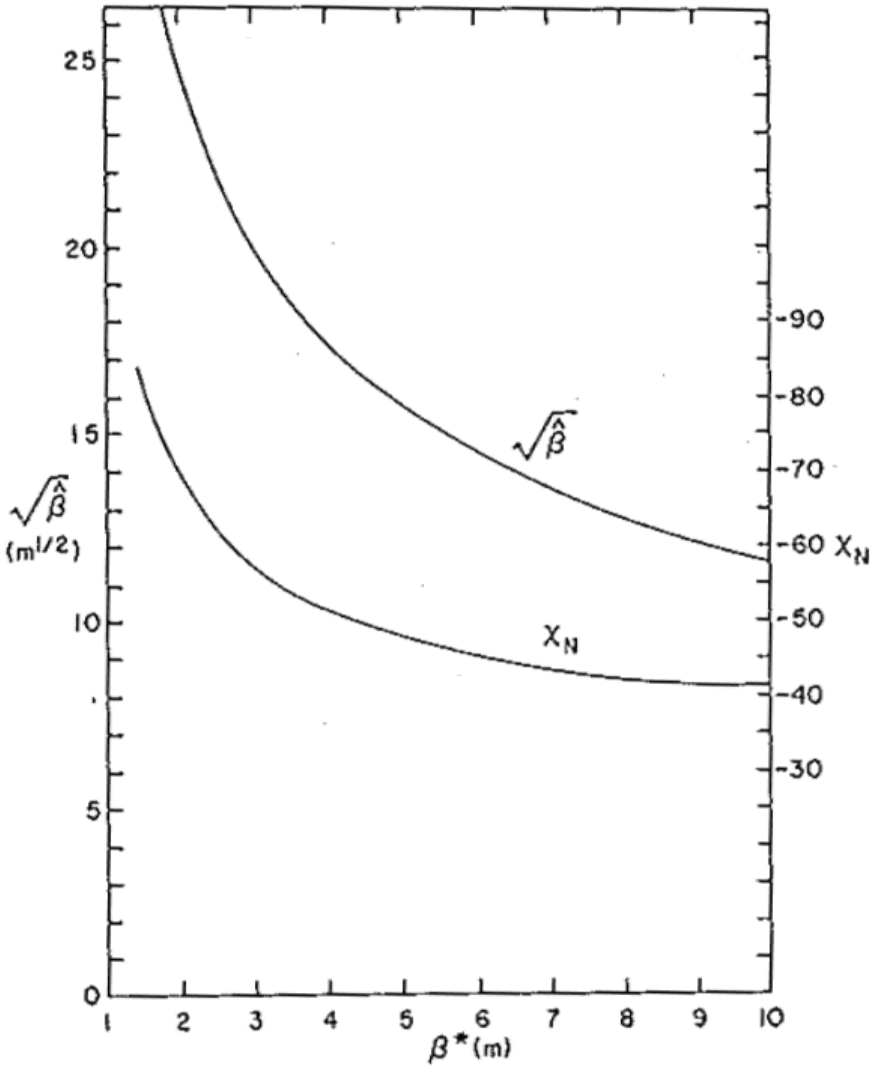
Contribution of low β triplets in an IR to the natural chromaticity is (exercise 2.5.2)

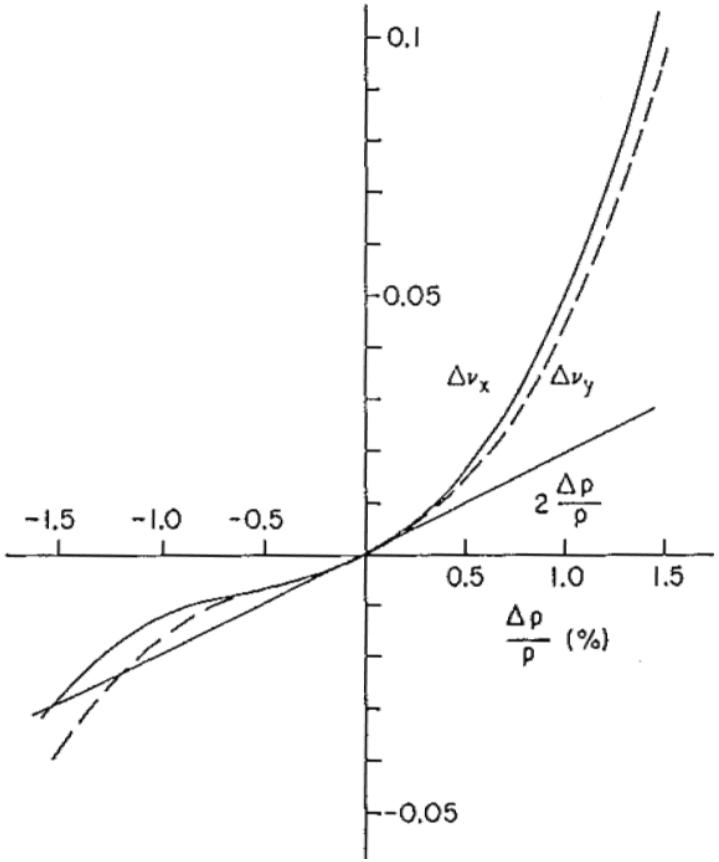
$$C_{total} = N_{IR} C_{IR} + C_{bare\ machine}$$

$$C_{IR} = -\frac{2\Delta s}{4\pi\beta^*} \approx -\frac{1}{2\pi}\sqrt{\frac{\beta_{max}}{\beta^*}}$$

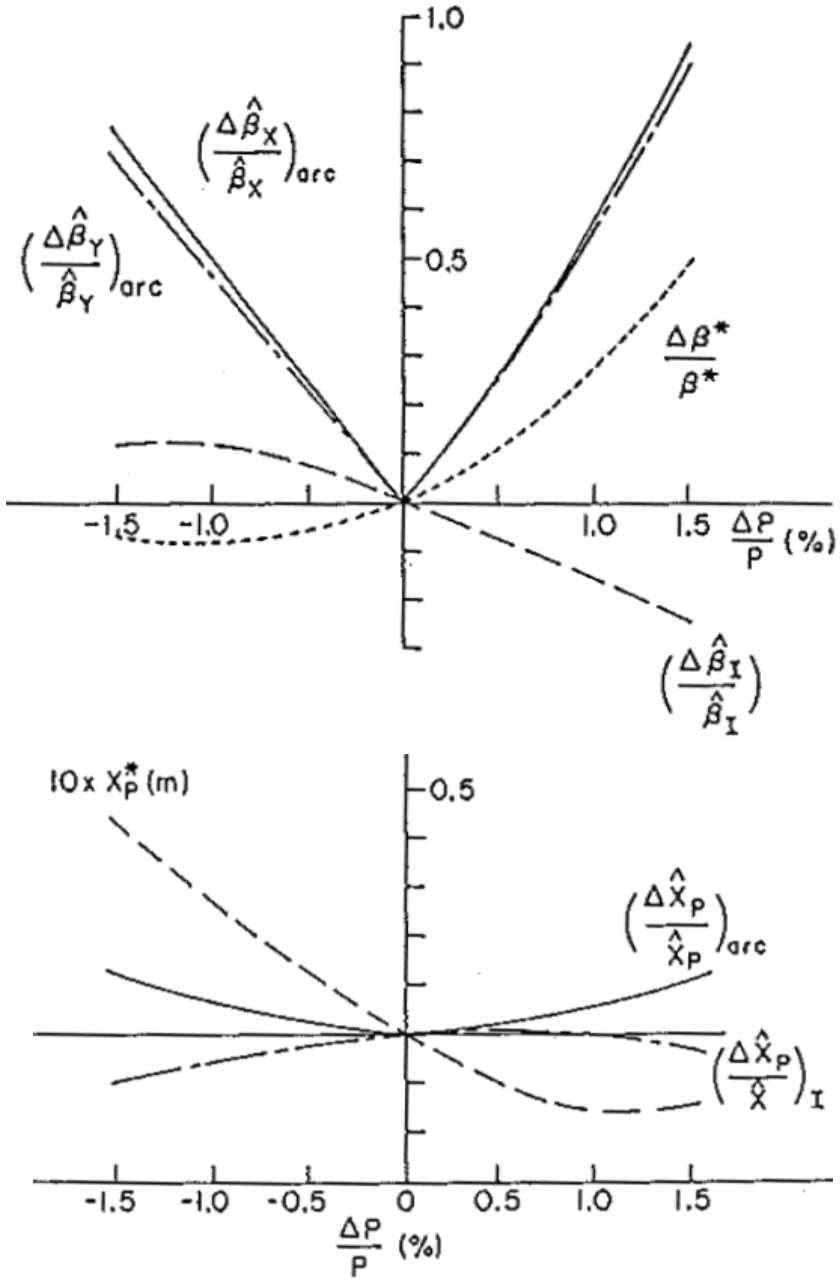
The total chromaticity is composed of contributions from the low β-quads and the rest of accelerators that is made of FODO cells. The decomposition to fit the data is **Δs≈35** m in RHIC.







ν vs Δp/p



β and D vs Δp/p

2. Chromaticity correction:

The chromaticity can cause tune spread to a beam with momentum spread $\Delta\nu=C\delta$. For a beam with C=-100, δ=0.005, $\Delta\nu$=0.5. The beam is not stable for most of the machine operation. Furthermore, there exists collective (head-tail) instabilities that requires positive chromaticity for stability! To correct chromaticity, we need to find magnetic field that provide stronger focusing for off-(higher)-momentum particles. We first try sextupole with

$$\Delta B_z + j\Delta B_x = B_0 b_2 (x+jz)^2, \qquad A_s = \tfrac{1}{3}\mathrm{Re}\{B_0 b_2 (x+jz)^3\}$$

$$x'' + K_x(s)x = \frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = -\frac{\Delta B_x}{B\rho} \qquad x = x_\beta + D\delta \qquad z = z_\beta$$

$$\Delta B_z = B_0 b_2 (x^2 - z^2) = B_0 b_2 (2x_\beta D\delta + D^2\delta^2 + x_\beta^2 - z_\beta^2)$$

$$\Delta B_x = B_0 b_2 2xz = B_0 b_2 2z_\beta D\delta + B_0 b_2 2x_\beta z_\beta$$

Let $K_2=-2B_0b_2/B\rho=-B_2/B\rho$, we obtain:

$$x_\beta'' + (K_x(s) + K_2 D\delta)x_\beta = 0, \quad z_\beta'' + (K_z(s) - K_2 D\delta)z_\beta = 0$$

$$\frac{\Delta B_z}{B\rho} = \frac{B_2}{2B\rho}(x^2 - z^2), \quad \frac{\Delta B_x}{B\rho} = \frac{B_2}{B\rho}xz, \qquad x = x_\beta(s) + D(s)\delta$$

$$\begin{cases} \dfrac{\Delta B_z}{B\rho} = -[S(s)D(s)\delta]x_\beta - \dfrac{S(s)}{2}(x_\beta^2 - z_\beta^2) - \dfrac{S(s)}{2}D^2(s)\delta^2, \\ \dfrac{\Delta B_x}{B\rho} = -[S(s)D(s)\delta]z_\beta - S(s)x_\beta z_\beta, \qquad S(s) = -B_2/B\rho \end{cases}$$

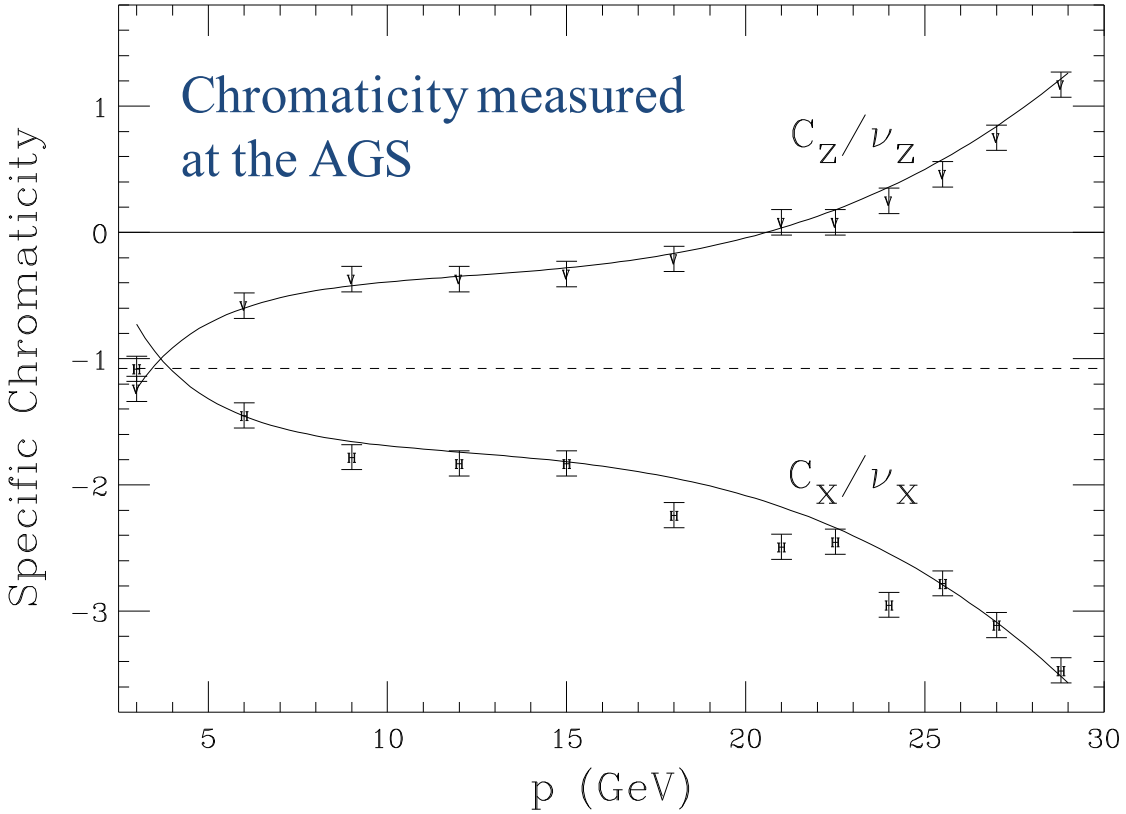
$$\Delta K_x = S(s)D(s)\delta, \quad \Delta K_z = -S(s)D(s)\delta$$

$$C_x = \frac{-1}{4\pi}\oint \beta_x[K_x(s) - S(s)D(s)]ds,$$

$$C_z = \frac{-1}{4\pi}\oint \beta_z[K_z(s) + S(s)D(s)]ds.$$

- In order to minimize their strength, the chromatic sextupoles should be located near quadrupoles, where $\beta_x D_x$ and $\beta_z D_x$ are maximum.
- A large ratio of β_x/β_z for the focusing sextupole and a large ratio of β_z/β_x for the defocussing sextupole are needed for optimal independent chromaticity control.
- The families of sextupoles should be arranged to minimize the systematic half-integer stopbands and the third-order betatron resonance strengths.

To model the AGS, we assume that the sextupole fields arise from systematic error at the ends of each dipole, the eddy current sextupole due to the vacuum chamber wall, and the iron saturation sextupole at high field.



The systematic error is independent of the beam momentum; the eddy current sextupole field depends inversely on the beam momentum; and the saturation sextupole field depends on a higher power of the beam momentum. The solid lines represent theoretical calculations with the integrated body and end sextupole strengths

$$
\begin{aligned}
S_{\mathrm{b}} &= -5.2\times10^{-4} + 5.8\times10^{-2}/p \\
&\quad -(3.6\times10^{-4}p - 7.0\times10^{-5}p^2 + 2.8\times10^{-6}p^3)\ (\mathrm{m}^{-2}), \\
S_{\mathrm{e}} &= -0.017\ (\mathrm{m}^{-2}),
\end{aligned}
$$

With sextupoles, the chromaticities becomes

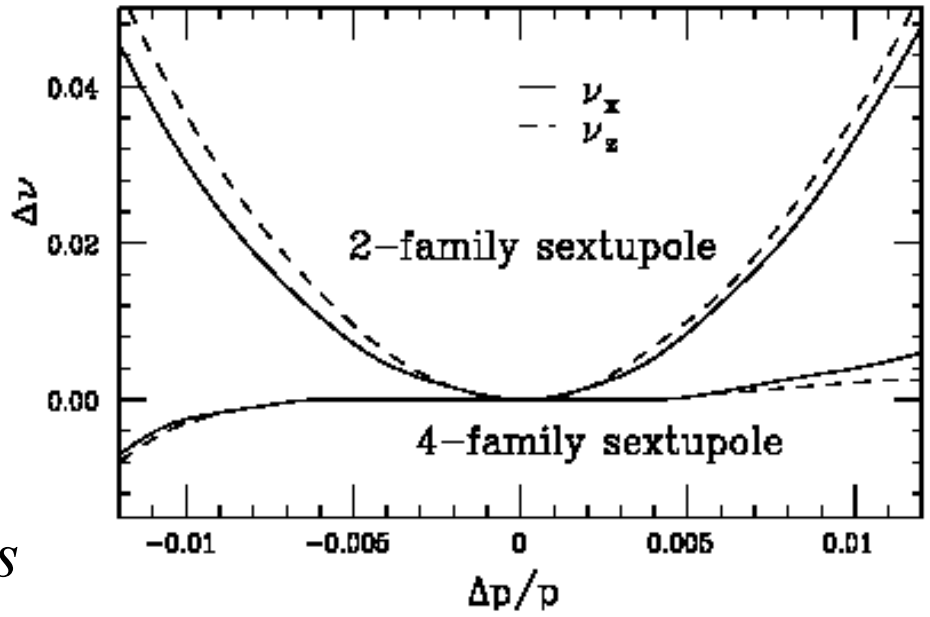
$$C_x = -\frac{1}{4\pi}\oint \beta_x(s)[K_x(s) - K_2(s)D(s)]ds$$

$$C_z = -\frac{1}{4\pi}\oint \beta_z(s)[K_z(s) + K_2(s)D(s)]ds$$

For FODO cells, the integrated sextupole strength is

$$S_F \equiv K_2 \ell_{SF} = \frac{\sin(\Phi/2)}{2f^2\theta(1+\frac{1}{2}\sin(\Phi/2))}, \quad S_D \equiv K_{2D}\ell_{SD} = -\frac{\sin(\Phi/2)}{2f^2\theta(1-\frac{1}{2}\sin(\Phi/2))}$$

For high energy colliders and high brightness synchrotron light sources, the sextupole strength can be much higher. Even more important is the effect of the systematic half-integer stopbands.



$$J_{y,p} = \frac{1}{2\pi}\oint\left[\beta_y(s)\Delta K_y(s)\right]e^{-jp\phi_y(s)}ds$$

Chromatic Aberration and Correction

Defining the betatron amplitude difference functions A and B as

$$A = \frac{\alpha_1\beta_0 - \alpha_0\beta_1}{\sqrt{\beta_0\beta_1}}, \quad B = \frac{\beta_1 - \beta_0}{\sqrt{\beta_0\beta_1}}$$

$$\frac{dB}{ds} = -A\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right), \quad \frac{dA}{ds} = +B\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right) + \sqrt{\beta_0\beta_1}\,\Delta K,$$

where $\Delta K = K_1 - K_0$ is the gradient error; the betatron amplitude functions β_0 and β_1 satisfy the Floquet equation

$$\beta_0' = -2\alpha_0, \quad \alpha_0' = K_0\beta_0 - \gamma_0, \quad d\psi_0/ds = 1/\beta_0,$$
$$\beta_1' = -2\alpha_1, \quad \alpha_1' = K_1\beta_1 - \gamma_1, \quad d\psi_1/ds = 1/\beta_1,$$

and ψ_0 and ψ_1 are the unperturbed and perturbed betatron phase functions.

$A^2 + B^2 =$ constant in regions where $\Delta K = 0$.

The change of A across a quadrupole is

$$\Delta A = \int \sqrt{\beta_0\beta_1}\,\Delta K\, ds \approx -\frac{\beta_0}{f}\frac{\Delta p}{p_0}$$

$$A = \frac{\alpha_1\beta_0 - \alpha_0\beta_1}{\sqrt{\beta_0\beta_1}}, \quad B = \frac{\beta_1 - \beta_0}{\sqrt{\beta_0\beta_1}}$$

$$\frac{dB}{ds} = -A\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right), \quad \frac{dA}{ds} = +B\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right) + \sqrt{\beta_0\beta_1}\,\Delta K,$$

$$\frac{d^2B}{d\bar{\phi}^2} + 4\bar{\nu}^2 B = -4\bar{\nu}^2 \frac{(\beta_0\beta_1)^{3/2}}{\beta_0 + \beta_1}\Delta K \qquad \bar{\phi} = \frac{1}{2\bar{\nu}}\int_{s_0}^{s}\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right) ds,$$

$$\bar{\nu} = \frac{1}{4\pi}\int_{s_0}^{s_0+C}\left(\frac{1}{\beta_0} + \frac{1}{\beta_1}\right) ds,$$

$$\begin{aligned}\frac{\Delta\beta(s)}{\beta(s)} &= -\frac{1}{2\sin\Phi_0}\int_s^{s+C} k(s_1)\beta(s_1)\cos[2\nu_0(\pi + \phi - \phi_1)]\, ds_1 \\ &= -\frac{\nu_0}{2\sin\Phi_0}\int_\phi^{\phi+2\pi} k(\phi_1)\beta^2(\phi_1)\cos[2\nu_0(\pi + \phi - \phi_1)]\, d\phi_1,\end{aligned}$$

where $\phi = (1/\nu_0)\int_0^s ds/\beta$. It is easy to verify that $\Delta\beta/\beta$ satisfies

$$\frac{d^2}{d\phi^2}\left[\frac{\Delta\beta(s)}{\beta(s)}\right] + 4\nu_0^2\left[\frac{\Delta\beta(s)}{\beta(s)}\right] = -2\nu_0^2\beta^2 k(s).$$

$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{\nu_0}{2\sin\Phi_0}\int_{\phi}^{\phi+2\pi} d\phi_1 \beta^2(\phi_1) k(\phi_1) \sin 2\nu_0(\pi+\phi-\phi_1)$$

$$\frac{d^2}{d\phi^2}\frac{\Delta\beta(s)}{\beta(s)} + 4\nu_0^2\frac{\Delta\beta(s)}{\beta(s)} = -2\nu\beta^2 k(s)$$

$$\left[\nu_0\beta^2 k(s)\right] = \sum_{p=-\infty}^{\infty} J_p e^{jp\varphi},$$

$$J_p = \frac{1}{2\pi}\oint\left[\beta k(s)\right]e^{-jp\varphi} ds$$

Half integer stopband

$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{\nu_0}{2}\sum_{p=-\infty}^{\infty}\frac{J_p}{\nu_0^2 - (p/2)^2} e^{jp\phi}$$

What symmetry can do to stopbands?

Systematic chromatic half-integer stopband width

The effect of systematic chromatic gradient error on betatron amplitude modulation can be analyzed by using the chromatic half-integer stopband integrals

$$\begin{cases} J_{p,x} = \dfrac{1}{2\pi}\oint \beta_x \Delta K_x e^{-jp\phi_x} ds, \\ J_{p,z} = \dfrac{1}{2\pi}\oint \beta_z \Delta K_z e^{-jp\phi_z} ds. \end{cases}$$

We consider a lattice made of P superperiods, where L is the length of a superperiod with K(s + L) = K(s), β(s + L) = β(s). Let C = PL be the circumference of the accelerator. The integral becomes

$$\begin{aligned} J_{p,y} &= -\left\{\frac{\delta}{2\pi}\int_0^L \beta_y K_y e^{-jp\phi} ds\right\}\left[1 + e^{-jp\frac{2\pi}{P}} + e^{-j2p\frac{2\pi}{P}} + e^{-j3p\frac{2\pi}{P}} + \cdots\right] \\ &= -\left\{\frac{\delta}{2\pi}\int_0^L \beta_y K_y e^{-jp\phi} ds\right\} \zeta_P(\frac{p}{P})\, e^{-j\pi p\frac{P-1}{P}}, \end{aligned}$$

$$\zeta_P(u) = \frac{\sin(Pu\pi)}{\sin(u\pi)} \qquad J_{p,y} = 0, \quad \text{unless} \quad p = 0 \ (\text{Mod } P)$$

At $p = 0$ (Mod P), the half-integer stopband integral increases by a factor of P, i.e. each superperiod contributes additively to the chromatic stopband integral.

Since the perturbation of betatron functions is most sensitive to the chromatic stopbands near $p \approx [2\nu_x]$ and $[2\nu_z]$, a basic design principle of strong-focusing synchrotrons is to avoid important systematic chromatic stopbands. This can be achieved by choosing the betatron tunes such that $[2\nu_x]$ and $[2\nu_z]$ are not divisible by the superperiod P. For example, the AGS lattice has $P = 12$, and the betatron tune should avoid a value of 6, 12, 18, etc. The actual betatron tunes at $\nu_{x/z} = 8.8$ are indeed far from systematic half-integer stopbands at $p = 6$ and 12, and the resulting chromatic perturbation is small. In fact, the AGS lattice can be approximated by a lattice made of 60 FODO cells. The important stopbands are located at $p = 30, 60, 90 \cdots$, which are far from the betatron tunes. Similarly, the TEVATRON has a super-periodicity of $P = 6$, and the betatron tune should avoid 18, 24, 30, etc.[90]

Generally, it is beneficial to design an accelerator with high super-periodicity so that the betatron tunes can be located far from the important chromatic stopbands. Some examples of high superperiod machines are $P = 12$ for the ALS, $P = 40$ for the APS, $P = 16$ for the ESRF, and $P = 22$ for the SPRING-8 at JSRF. However, a high energy accelerator or storage ring with large super-periodicity is costly. Thus the goal is to design an accelerator such that the chromatic stopband integral of each module is zero, or the stopband integrals of two modules cancel each other.

Chromatic stopband integrals of FODO cells

The chromatic stopband integral of the arc, which is composed of N FODO cells, in thin-lens approximation is

$$
\begin{aligned}
J_p &= -\frac{\delta}{2\pi}\left(\frac{\beta_{\max}}{f}-\frac{\beta_{\min}}{f}e^{-jp\frac{\Phi}{2\nu}}\right)\left[1+e^{-jp\frac{\Phi}{\nu}}+e^{-j2p\frac{\Phi}{\nu}}+e^{-j3p\frac{\Phi}{\nu}}+\cdots\right] \\
&= -\frac{2\delta}{\pi\cos\frac{\Phi}{2}}\left(\sin\frac{\Phi}{2}\cos\frac{p\Phi}{4\nu}+j\sin\frac{p\Phi}{4\nu}\right)\ \zeta_N\left(\frac{p\Phi}{2\pi\nu}\right)\ e^{-j\frac{(2N-1)p\pi}{2N}},
\end{aligned}
$$

where Φ is the phase advance per cell, $\beta_{\max}$ and $\beta_{\min}$ are values of the betatron amplitude function at the focusing and defocussing quadrupoles respectively, f is the focal length of each quadrupole, and the diffracting function $\zeta_N(u)$ is given by Eq. (2.333). If $p\Phi/2\pi\nu = 0$ (Mod N), the diffracting function is equal to N. This means that each FODO cell contributes additively to the stopband integral. Fortunately, since $\Phi/2\pi$ is normally about $1/4$ (90^0 phase advance) so that $p\Phi/2\pi\nu \approx p/4\nu \approx 1/2$, the chromatic stopband integral at $p \approx 2\nu$ due to N FODO cells is small. In particular, if $N\Phi =$ integer $\times\pi$, the chromatic stopband of the arc adds up to zero at harmonics $p \approx 2\nu$, i.e. the stopband integrals at $p \approx [2\nu]$ resulting from N FODO cells in the arcs is small if the total phase advance of these FODO cells is $N\Phi =$ integer $\times\ \pi$, where the transfer matrix of the arc becomes a unit matrix I or a half-unit matrix $-I$.

The chromatic stopband integral of insertions

Let $\Phi^{\rm ins}$ and $J_p^{\rm ins}$ be respectively the phase advance and the chromatic stopband integral of an insertion. The total contribution of two adjacent insertions becomes

$$J_p = J_p^{\rm ins}\left[1+\exp\left(j\frac{p\Phi^{\rm ins}}{\nu}\right)\right]$$

At the harmonic $p \approx [2\nu]$, we obtain $J_p = 0$ if $\Phi^{\rm ins} = (2n+1)\pi/2$. Thus, if the insertion is a quarter-wave module, the chromatic stopband integrals of two adjacent insertions cancel each other. This cancellation principle remains valid when two insertions are separated by a unit transfer matrix. Such a procedure was extensively used in the design of the RHIC lattice[92] and the SSC lattice.[93]

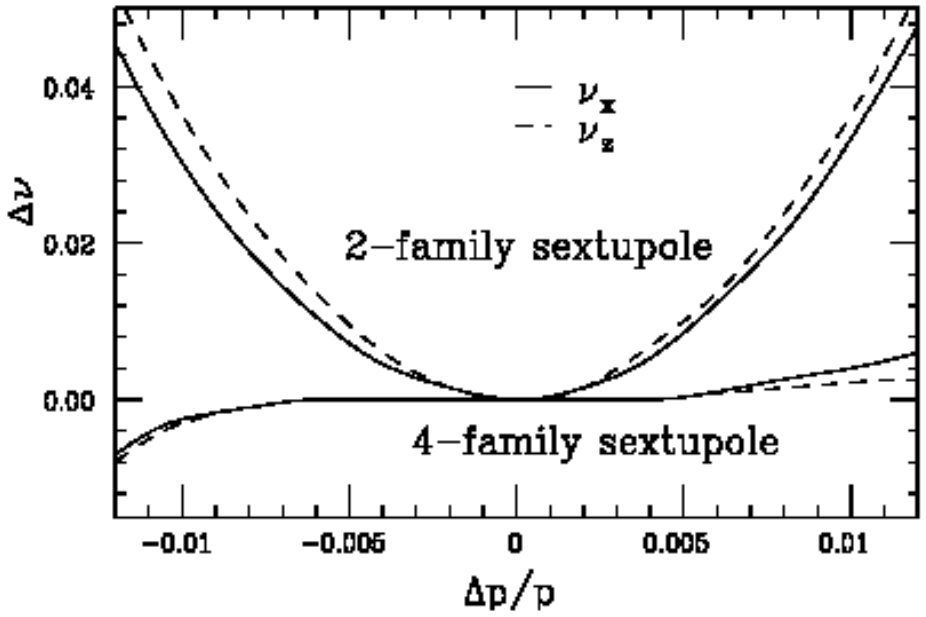
Effect of the chromatic stopbands on chromaticity

$$\frac{\Delta\beta}{\beta} \approx -\frac{|J_p|\cos(p\phi+\chi)}{2(\nu-p/2)}$$

$$\Delta\nu_y = C_y^{(1)}\delta + C_y^{(2)}\delta^2 + \cdots$$

$$C_y^{(1)} = -\frac{1}{4\pi}\oint \beta_y(K_y - S_y D)ds,$$

$$C_y^{(2)} = -C_y^{(1)} - \frac{|J_{p,y}|^2/\delta^2}{4(\nu_y - p/2)}.$$



Effect of sextupoles on the chromatic stopband integrals

First we evaluate the stopband integral due to the chromatic sextupoles. Let $S_{\rm F}$ and $S_{\rm D}$ be the integrated sextupole strength at QF and QD of FODO cells in the arc. The p-th harmonic stopband integral from these chromatic sextupoles is

$$J_{p,\text{sext}} = \frac{\delta}{2\pi}\zeta_N\left(\frac{p\Phi}{2\pi\nu}\right)\left[\beta_{\rm F}S_{\rm F}D_{\rm F} + \beta_{\rm D}S_{\rm D}D_{\rm D}e^{-jp\Phi/2\nu}\right]e^{-j(N-1)p\Phi/2\nu},$$

The stopband integral is zero or small if $N\Phi/\pi$ = integer, i.e. the chromatic sextupole does not contribute significantly to the chromatic stopband integral if the transfer matrix of the arc is I or −I.

To obtain a nonzero chromatic stopband integral, sextupoles are organized in families. We consider an example of a four-family scheme with

$$\{S_{\rm F1} = S_{\rm F} + \Delta_{\rm F},\ S_{\rm D1} = S_{\rm D} + \Delta_{\rm D},\ S_{\rm F2} = S_{\rm F} - \Delta_{\rm F},\ D_{\rm D2} = S_{\rm D} - \Delta_{\rm D}\},$$

that is commonly used in FODO cells with 90° phase advance. Here the parameters $S_{\rm F}$ and $S_{\rm D}$ are determined from the first-order chromaticity correction, Since $\beta(s)$ and $D(s)$ are periodic functions of s in the repetitive FODO cells, the parameters $\Delta_{\rm F}, \Delta_{\rm D}$ will not affect the first-order chromaticity, which is proportional to the zeroth harmonic of the stopband integral. However, the chromatic stopband integrals due to the parameters $\Delta_{\rm F}$ and $\Delta_{\rm D}$ are given by

$$\Delta J_{p,\text{sext}} = \frac{\delta}{2\pi}\zeta_N\left(\frac{p\Phi}{2\nu\pi} - \frac{1}{2}\right)\left[\beta_{\rm F}\Delta_{\rm F}D_{\rm F} + \beta_{\rm D}\Delta_{\rm D}D_{\rm D}e^{-jp\Phi/4\nu}\right]e^{-j(N-1)[(p\Phi/2\nu\pi)-(1/2)]\pi}.$$

At $p \approx [2\nu]$ and $\Phi/2\pi \approx 1/4$ (90° phase advance), we have $\zeta_N \to N$, i.e. every FODO cell contributes additively to the chromatic stopband. The resulting stopband width is proportional to Δ_F and Δ_D parameters. By adjusting Δ_F and Δ_D parameters, the betabeat and the second-order chromaticity can be minimized. The scheme works best for a nearly 90° phase advance per cell with $N\Phi = \text{integer} \times \pi$, where the third-order resonance-driving term vanishes also for the four-family sextupole scheme. Fig. 2.42 shows an example of chromatic correction with four families of sextupoles in RHIC, where the second-order chromaticity and the betatron amplitude modulation can be simultaneously corrected.

Similarly, the six-family sextupole scheme works for 60° phase advance FODO cells, where the six-family scheme

$$\{S_{F1}, S_{D1}, S_{F2}, D_{D2}, S_{F3}, S_{D3}\} \tag{2.342}$$

has two additional parameters.