PHY 564
Advanced Accelerator Physics
Lectures 28

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Lecture 28. Nonlinear beam dynamics. Part II

Nonlinear effects in particle's motion in accelerators (or in Hamiltonian mechanics in general) some time can be treated in perturbative manner – the way we learned in this course. But while giving analytical expressions for the results, it is limited by – usually – second order perturbation and not necessarily converging when brought to higher orders. Needless to say it becoming very cumbersome even in the second order… Resonant approach, while giving a nice intuitive understanding of the resonances, is limited to (a) a single resonance, (b) ignores non-resonant terms which definitely distort or even – at large amplitude - ruin the simple picture we looked at. Fortunately there is a very systematic and rigorous method for non-linear dynamics developed by Prof. Alex J. Dragt (UM) and his follower (many of them his former students). This fundamental work started in late 1970s and brought to a well-formulated theory in early 1980s. Naturally, the work did not stopped there and there are a lot of new addition to this method (frequently oriented to computing and analyzing non-linear maps), which are extension of the method. Method itself is uses a number of mathematical concepts and power of Lie algebraic tools. It exploits symmetries of Hamiltonian systems and is – at present - the most comprehensive approach to the non-linear beam dynamics. We can not follow each and every – some of them rather complex – derivations. Hence, we will deviate from tradition in our course to prove almost everything and will instead have a short introduction to this method. We may offer a dedicated course sometimes in near future.

Let's start from something we are well aware of: group (**G**) of 2n x 2n symplectic matrices (formally called **Sp(2n)**) satisfying simplicity conditions:

$$\mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S}; \qquad (28\text{-}1)$$

which satisfy group properties **G** :

1. It contains identity matrix **I**: since obvious $\mathbf{I}^T\mathbf{S}\mathbf{I} = \mathbf{S}$
2. If $\mathbf{M} \in \mathbf{G} \to \mathbf{M}^{-1} \in \mathbf{G}$ (contains inverse matrix) :

$$\mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S}; ® \mathbf{M}^{-1} = -\mathbf{S}\mathbf{M}^T\mathbf{S}; ® \mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S};$$

$$\mathbf{M}^{-1T}\mathbf{S}\mathbf{M}^{-1} = -\mathbf{S}\mathbf{M}\mathbf{S}\mathbf{M}^T\mathbf{S} = \mathbf{S}.$$

3. $\mathbf{M},\mathbf{N} \in \mathbf{G} \to \mathbf{M}\cdot\mathbf{N} \in \mathbf{G}$: since $(\mathbf{M}\mathbf{N})^T\,\mathbf{S}\cdot\mathbf{M}\mathbf{N} = \mathbf{N}^T\left(\mathbf{M}^T\mathbf{S}\mathbf{M}\right)\mathbf{N} = \mathbf{N}^T\mathbf{S}\mathbf{N} = \mathbf{S}$
4. $\mathbf{M}(\mathbf{N}\mathbf{L}) = (\mathbf{M}\mathbf{N})\mathbf{L}$, which is correct for any square matrices of the same order.

Thus, we proved that that symplectic matrices form symplectic group. Now we will focus on more formal definition of something we are familiar with, which is called lie algebraic properties. For any matrix **A** we defined exponential matrix function (heavily use in Lie algebras):

$$\exp(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}; \qquad (28\text{-}2)$$

which converge for any **A** . A bit trickier is inverse, i.e. natural logarithm function:

$$\ln(\mathbf{A}) = \ln\left(\mathbf{I} - (\mathbf{I} - \mathbf{A})\right) = -\sum_{n=1}^{\infty} \frac{(\mathbf{I} - \mathbf{A})^n}{n}.; \qquad (28\text{-}3)$$

uniqueness and convergence of which is much less trivial. It definitely converges when norm of $\mathbf{I} - \mathbf{A}$ is close to zero.

It definitely converges when norm of $\mathbf{I}-\mathbf{A}$ is close to zero. We know that for (real) matrix $\mathbf{A}$ with non-zero eigen values (e.g. non-zero determinant!) we can use Sylvester formula and find (a bit trickier to get it to be real) a solution of (28-3). We are all aware that *ln* of any number has branching at zero and is defined with accuracy of 2nπ. It means that $\mathbf{A}=\exp(\mathbf{B})$ has infinite number of solutions.

It is possible to show (a good exercise similar to proving exp(ln(x))=x) that if:

$$\mathbf{B}=\ln(\mathbf{A}) \rightarrow \mathbf{A}=\exp(\mathbf{B}). \tag{28-4}$$

If **M** is real and symplectic, than

$$\mathbf{D}=\ln(\mathbf{M}) \rightarrow \mathbf{D}^T\mathbf{S}+\mathbf{S}\mathbf{D}=0;. \tag{28-5}$$

or **D** is anti-commute with **S**. It is easy to prove:

$$\begin{aligned}
&\mathbf{D}=\ln(\mathbf{M}); -\mathbf{D}=\ln\left(\mathbf{M}^{-1}\right)=\ln\left(\mathbf{S}^{-1}\mathbf{M}^T\mathbf{S}\right)=\mathbf{S}^{-1}\ln\left(\mathbf{M}^T\right)\mathbf{S}=-\mathbf{S}\ln\left(\mathbf{M}^T\right)\mathbf{S};\\
&\mathbf{D}^T=\left(\mathbf{S}\ln\left(\mathbf{M}^T\right)\mathbf{S}\right)^T=\mathbf{S}\ln(\mathbf{M})\mathbf{S}=\mathbf{SDS}; \rightarrow \mathbf{D}^T-\mathbf{SDS}=-(\mathbf{D}^T\mathbf{S}+\mathbf{SD})\mathbf{S}=0;
\end{aligned}. \tag{28-6}$$

It means that (surprise-surprise) that **D=SH,** where **H** is symmetric matrix**:**

$$\mathbf{H} = -\mathbf{SD};\ \mathbf{D}^T=\mathbf{SDS} \rightarrow \mathbf{H}^T=\mathbf{D}^T\mathbf{S}=\mathbf{SDS}^2=-\mathbf{SD}=\mathbf{H}. \tag{28-7}$$

We already proved many times that for $\mathbf{H}^{\mathbf{T}}=\mathbf{H}$,

$$\mathbf{M}=\exp(\mathbf{SH}) \rightarrow \mathbf{M}^T\mathbf{SM}=\mathbf{S}, \tag{28-8}$$

which is a two-liner:

$$\mathbf{M}^T=\exp\left((\mathbf{SH})^T\right)=\exp(-\mathbf{HS})=\exp\left(-\mathbf{S}^{-1}\mathbf{SHS}\right)=\mathbf{S}^{-1}\exp(-\mathbf{SH})\mathbf{S}=\mathbf{S}\exp(-\mathbf{SH})\mathbf{S}$$

$$\mathbf{S}^{-1}=-\mathbf{S};\ \mathbf{M}^T\mathbf{SM}=-\mathbf{S}\exp(-\mathbf{SH})\mathbf{S}^2\exp(\mathbf{SH})=\mathbf{S}\exp(\mathbf{SH}-\mathbf{SH})=\mathbf{S}.$$

What we shown is that symplectic matrix can be written on form

$$\mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S} \to \mathbf{M} = \exp(\mathbf{S}\mathbf{H}),\ \mathbf{H}^T = \mathbf{H}. \tag{28-9}$$

Now we are ready to define Lie algebra for matrices: A set of matrices forms **Lie algebra** if:

1. If matrix **A** is in the Lie algebra, than so any product with a scalar **a**, **aA** ;
2. If matrices **A** and **B** is in the Lie algebra, then so their sum **A+B**.
3. If matrices **A** and **B** is in the Lie algebra, , then so their commutator **[A,B]**, defined as

$$[\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA}, \tag{28-10}$$

which is something new we did not touched yet in our course, but something having a very fundamental relation with Poisson brackets in Hamiltonian mechanics. The next is to show that our **D=SH** , $\mathbf{H}^{\mathbf{T}}$**=H** set of matrices **D** form an Lie algebra. From observing that **H=-SD**, two first conditions are trivial adding symmetric matrices and multiplying them by a scalar keeps them symmetric. Third condition is a new and can be easily proved:

$$\mathbf{A} = \mathbf{SH}_1, \mathbf{B} = \mathbf{SH}_2; [\mathbf{A},\mathbf{B}] = \mathbf{SH}$$

$$\mathbf{H} = -\mathbf{S}[\mathbf{A},\mathbf{B}] = \mathbf{SBA} - \mathbf{SAB} = \mathbf{H}_1\mathbf{SH}_2 - \mathbf{H}_2\mathbf{SH}_1; \tag{28-11}$$

$$\mathbf{H}^T = \left(\mathbf{H}_1\mathbf{SH}_2 - \mathbf{H}_2\mathbf{SH}_1\right) = \left(\mathbf{H}_2{}^T\mathbf{S}^T\mathbf{H}_1{}^T - \mathbf{H}_1{}^T\mathbf{S}^T\mathbf{H}_2{}^T\right) = \mathbf{H}_1\mathbf{SH}_2 - \mathbf{H}_2\mathbf{SH}_1 = \mathbf{H};$$

which proves that $[\mathbf{A},\mathbf{B}] = \mathbf{SH}$ with $\mathbf{H}^T = \mathbf{H}$.

Further, is possible to prove that symplectic matrix can be presented in form of the product of exponents

$$\mathbf{M} = \exp\left(\mathbf{SH}_a\right)\exp\left(\mathbf{SH}_s\right), \mathbf{SH}_a = -\mathbf{H}_a\mathbf{S}; \mathbf{SH}_c = \mathbf{H}_c\mathbf{S}; \tag{28-12}$$

with commuting and anti-commuting generating matrices $\mathbf{H}_a, \mathbf{H}_c$. This can be proven using the fact that an arbitrary real non-singular matrix can be decomposed as product of real positive definite symmetric matrix **P** and orthogonal matrix **O** (we use it without prove!):

$$\mathbf{M} = \mathbf{PO}; \mathbf{P}^T = \mathbf{P}; \ \mathbf{O}^T = \mathbf{O}^{-1}; \tag{28-13}$$

For symplectic matrix we have

$$\mathbf{M} = \mathbf{S}^{-1}\left(\mathbf{M}^{-1}\right)^{\mathbf{T}}\mathbf{S} \rightarrow \mathbf{PO} = \left(\mathbf{S}^{-1}\mathbf{P}^{-1}\mathbf{S}\right)\left(\mathbf{S}^{-1}\mathbf{OS}\right)$$

where we used $\mathbf{P}^T = \mathbf{P}; \ \mathbf{O}^T = \mathbf{O}^{-1}$ and with $\mathbf{S}^{-1}\mathbf{P}^{-1}\mathbf{S}$ being real, symmetric and positive definite and $\mathbf{S}^{-1}\mathbf{O}^{-1}\mathbf{S}$ real and orthogonal. Since polar decomposition is unique (we use it without prove!) than

$$\mathbf{P} = \mathbf{S}^{-1}\mathbf{P}^{-1}\mathbf{S}; \mathbf{O} = \mathbf{S}^{-1}\mathbf{OS}; \rightarrow \mathbf{P} = -\mathbf{S}\left(\mathbf{P}^{-1}\right)^T\mathbf{S}; \mathbf{O} = -\mathbf{S}\left(\mathbf{O}^{-1}\right)^T\mathbf{S}.;$$

$$\mathbf{P}^T\mathbf{SP} = \mathbf{P}^T\left(\mathbf{P}^{-1}\right)^T\mathbf{S} = \mathbf{S}; \qquad \mathbf{O}^T\mathbf{SO} = \mathbf{O}^T\left(\mathbf{O}^{-1}\right)^T\mathbf{S} = \mathbf{S}\,\#$$

e.g. both of these matrices are symplectic. A bit more of exercises is needed to prove that $\mathbf{A} = \ln\mathbf{O}$ is asymmetric matrix $\mathbf{A}^T = -\mathbf{A}$ and $\mathbf{B} = \ln\mathbf{P}$ is symmetric matrix $\mathbf{B}^T = \mathbf{B}$: $-\mathbf{A} = \ln\mathbf{O}^{-1} = \ln\mathbf{O}^T = \mathbf{A}^T; \mathbf{B}^T = \ln\mathbf{P}^T = \ln\mathbf{P} = \mathbf{B}$. As we found that for any logarithm of symplectic matrix condition (28-5) applies $\mathbf{D} = \ln\left(\mathbf{M}\right) \rightarrow \mathbf{D}^T\mathbf{S} + \mathbf{SD} = 0;$ requiring:

$$\begin{gathered}\mathbf{A}^T\mathbf{S} + \mathbf{SA} = 0 \rightarrow \mathbf{AS} = \mathbf{SA}; \mathbf{A} = \mathbf{SH}_c \rightarrow \mathbf{H}_c\mathbf{S} = \mathbf{SH}_c \\ \mathbf{B}^T\mathbf{S} + \mathbf{SB} = 0 \rightarrow \mathbf{BS} = -\mathbf{SB}; \mathbf{B} = \mathbf{SH}_a \rightarrow \mathbf{H}_a\mathbf{S} = -\mathbf{SH}_a\,\#\end{gathered} \tag{28-14}$$

This proves (relying on a couple of theorem from linear algebra we took for granted) that (28-12) is correct. Since, $\mathbf{S}^2 = -\mathbf{I} = (i\mathbf{I})^2$ and generating matrices either commute or anti-commute with **S**, one can find real $\mathbf{H}_a, \mathbf{H}_c \ldots$ again without proof.

Now we are ready to connect our – so far an abstract exercise – to Poisson brackets, which are defined fro two functions of coordinates and momenta as

$$X = \{x_i, i = 1, 2n\} = \{\{q_k, P^k\} k = 1, n\};$$

$$f = f(X, s) \equiv f(q_k, P^k, s); g = g(X, s) \equiv g(q_k, P^k, s);$$

$$[f, g]_{def} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial P^k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial P^k} \right) = \sum_{i,j=1}^{2n} \left(\frac{\partial f}{\partial x_i} S_{ij} \frac{\partial g}{\partial x_j} \right) = \tag{28-15}$$

$$(\partial_X f, \mathbf{S} \cdot \partial_X g) = (\partial_X f)^T \mathbf{S} \cdot (\partial_X g).$$

From Hamiltonian mechanics we know that

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H] = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X} S \frac{\partial H}{\partial X} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_k} \frac{\partial H}{\partial P^k} - \frac{\partial f}{\partial P^k} \frac{\partial H}{\partial q_k};$$

and time-independent function “commuting” with Hamiltonian are invariants of motion.

Let's now introduce one more object, **Lie operator** *:f:* defined as :

$$:f:g=[f,g];$$
$$:f:^0=g;\ :f:^2 g=[f[f,g]];\ \ :f:^{n+1} g=[f,:f:^n g]. \qquad (28\text{-}16)$$

together with its powers. Obviously the Lie operator and its power are linear operators

$$:f:^n(a\cdot g+b\cdot h)=(a\cdot :f:^n g+b\cdot :f:^n h) \qquad (28\text{-}17)$$

since functions the operator acts upon appearing linearly. Similarly, since $:f:$ is a differential operator, the following rule

$$:f:(g\cdot h)=(:f:g)\cdot h+g\cdot(:f:h) \qquad (28\text{-}18)$$

is trivial to prove. Furthermore, similarly to the ordinary differentiation $:f:^n$ obeys Leibnitz rule

$$:f:^n(g\cdot h)=\sum_{m=0}^{n} C_m^n(:f:^m g)(:f:^{n-m} h);\ C_m^n=\frac{n!}{m!(n-m)!}. \qquad (28\text{-}19)$$

Finally, the Jacoby identity

$$[f,[g,h]]=[[f,g],h]+[g,[f,h]] \qquad (28\text{-}20)$$

which I recommend you to prove as an exercise (not a home work!) is equivalent to identity for Lie operators

$$:f:[g,h]=[:f:g,h]+[g,:f:h] \qquad (28\text{-}21)$$

Now we will convert linear Lie operators into a linear algebra by defining their product (algebraic, not simple multiplication) of Lie operators:

$$\{:f:,:g:\}=:f::g:-:g::f: \qquad (28\text{-}22)$$

or using Jacoby identity

$$\{:f:,:g:\}h=(:f::g:-:g::f:)h=[:f:,:g:]h=:[f,g]:h \qquad (28\text{-}23)$$

with $:[f,g]:$ being a compact form of the product of two operators.

Hence we established homomorphism between the Lie algebra of function (Poisson brackets) and Lie operators. Naturally (28-22) turns the set Lie operator into a Lie algebra.

We are non done yet with definitions: we define Lie transform as an exponent of the Lie operators:

$$\exp(:f:) = \sum_{n=0}^{\infty} \frac{:f:^n}{n!} \tag{28-24}$$

which have unbelievably beautiful properties:

$$\exp(:f:)(gh) = \left(\exp(:f:)g\right)\left(\exp(:f:)h\right) \tag{28-25}$$

which can be prove using Leibnitz rule in manner similar to prove of $\exp(x+y) = \exp(x)\exp(y)$ in mathematical analysis.

Applying it to

$$\exp(:f:)x^n = \left(\exp(:f:)x\right)^n;$$

$$g(X) = \sum_{n=0}^{\infty} g_n X^n \to \exp(:f:)g(X) = \sum_{n=0}^{\infty} g_n \left(\exp(:f:)X\right)^n = g\left(\exp(:f:)X\right). \tag{28-26}$$

with the later being the most remarkable quality: Lie transformation of a function is a function of Lie transformation of its argument! (We cheated a bit here – we really needed to expand function of multiple variables as $\sum_{k=0}^{\infty} g_{k_1 \ldots k_{2n}} x_1^{k_1} \cdot\cdot x_{2n}^{k_{2n}}$ with the same final result). Lastly, the important (and elegant) property we will use later:

$$\exp(:f:)[g,h] = \left[\exp(:f:)g, \exp(:f:)h\right] \tag{28-27}$$

It worth noting that all relations mentioned above without checking are relatively straight forward to prove, but proved are necessarily compact.

Let's now switch to symplectic maps denoted as:

$$\mathtt{M}: x \to \bar{x}(x,s); \mathtt{M}: X \to \bar{X}(x,s); \tag{28-28}$$

which generate local symplectic matrices

$$\mathbf{M}(s,X) = \left[\frac{\partial \bar{x}_i}{\partial x_j}\right] = \frac{\partial \bar{X}}{\partial X}; \quad \mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S}. \tag{28-29}$$

We discussed the invariants and result of these important features of symplectic maps such as Poincare invariants and will not repeat it. Instead we will focus on connection between Lie algebras and symplectic maps. First, let's show that Lie transformation is symplectic, lets consider

$$\bar{x} = \mathtt{M}x; \mathtt{M} = \exp(:f:); \tag{28-30}$$

than we have

$$\left[\bar{x}_i, \bar{x}_j\right] = \left[\left(\exp(:f:)x\right)_i, \left(\exp(:f:)x\right)_j\right] = \exp(:f:)\left[x_i, x_j\right] = S_{ij} \tag{28-31}$$

which proves the symplecticity of local transformation and the map as a whole.

As we discussed in our class, accelerator physics is interested in particles motion around the reference orbit, e.g. in maps which map origin X=0 into itself. It easy very easy to show that

$$\mathtt{M} = \exp(:f:);\ f = \sum_{k=1}^{2n} a_i x_i; \tag{28-32}$$

generate a displacement of the origin. For example $f = ax$ generates

$$f = ax, a[x,p] = \frac{\partial x}{\partial x}\frac{\partial p}{\partial p} = a;\ \bar{p} = a;\ \ :x:^n [x,p] = 0, n > 0 \tag{28-33}$$

First, we are not interest in such trivial shifts. Second, in general case, we always eliminate shift of the origin by choosing appropriately coefficients in (28-32).

Let's, for a moment, consider a Lie transformation with quadratic terms

$$f_2 = -\frac{1}{2}X^T \mathbf{H} X = -\frac{1}{2}\sum_{i,j=1}^{2n} h_{ij} x_i x_j; \mathbf{H}^T = \mathbf{H}. \tag{28-34}$$

Let's calculate action of $:f_2:$ on x_k:

$$\begin{gathered} :f_2:x_k = -\frac{1}{2}\sum_{i,j=1}^{2n} h_{ij}\left[x_i x_j, x_k\right]; \\ \left[x_i x_j, x_k\right] = \left[x_i, x_k\right] x_j + x_i\left[x_j, x_k\right] = S_{ik}x_j + S_{jk}x_i \\ -\frac{1}{2}\sum_{i,j=1}^{2n} h_{ij}\left(S_{ik}x_j + S_{jk}x_i\right) = \sum_{i}^{2n}(\mathbf{SH})_{ki} x_i \\ :f_2:x_k = (\mathbf{SH})_{ki}\, x_i \rightarrow :f_2:X = \mathbf{SH}X \end{gathered} \tag{28-35}$$

to see that it generates a linear matrix transformation.

Then we prove that Lie transformation with second order Hamiltonian polynomial as a generation function

$$:f_2:X=(\mathbf{SH})X;\ :f_2:^n X=(\mathbf{SH})^n X;$$
$$\exp(:f_2:)=\exp(\mathbf{SH}). \tag{28-36}$$

generates linear transformation. Which is equivalent to that generated by s-independent Hamiltonian of linear motion. As we discussed, linear motion is a trivial (when stable!) and is reduced to **n** independent oscillators with their amplifies (actions) and phases.

So far we had shown that Lie transforms are symplectic maps, that linear Lie map generated by second order Hamiltonian generate linear symplectic matrix and, vice versa, we can find such Lie transform for any symplectic matrix (for example using Sylvester formula for ln**M**). The remaining and very potent question remains: if a any analytical symplectic map can be presented in exponential form of a Lie operator? The answer is given by the **factorization theorem**: the keystone for application of the Lie transformation to non-linear Hamiltonian maps.

Factorization theorem: For an analytical symplectic map $\mathtt{M}$ (which transfers the origin in itself) and relation are assumed to be expandable into as power series:

$$\bar{X}=\mathtt{M}X;\ \bar{x}_i=M_{ik}x_k+\sum_{\sum_{i=1}^{2n}p_i=2}^{\infty}a_{1\ldots 2n}x_1^{p_1}\cdots x_{2n}^{p_{2n}}; \tag{28-37}$$

the map can be written in from of

$$\mathtt{M}=\exp(:f_2:)\exp(:f_3:)\exp(:f_4:)\exp(:f_5:)\ldots \tag{28-38}$$

where f_m are homogeneous polynomials of power m of $\{x_i\}, i=1,2n$.

Sketch of a proof – which is long- in based on the observation that if f_m and g_k are homogenies polynomials of order m and k, than their Poisson bracket

$$\left[f_m, g_k\right] = p_{m+k-2}$$

is also a homogeneous polynomial of order m+k-2. This is why f_2 generates linear map with linear polynomial X. Hence, f_3 will generate second order term and its exponential will generate all higher orders as well.

Let's apply using the linear map at the origin (X=0) the inverse transformation:

$$\exp(-:f_2:) = \exp(-\mathbf{SH}) \tag{28-39}$$

to both sides of (28-37)

$$\exp(-:f_2:)\bar{X} = \exp(-:f_2:)\mathtt{M}X =$$

$$X + \exp(-:f_2:)(\sum_{2+}^{\infty} a_{1\ldots 2n} x_1^{p_1} \cdots x_{2n}^{p_{2n}}, \text{higher orders}) \tag{28-40}$$

$$\exp(-:f_2:)\bar{x}_i = x_k + \exp(-:f_2:)\sum_{2+}^{\infty} a_{1\ldots 2n} x_1^{p_1} \cdots x_{2n}^{p_{2n}};$$

Suppose that f_3 is some cubic polynomial

$$\exp(-:f_3:)\exp(-:f_2:)\bar{X} = X + -:f_3:X + (\text{higher orders}); \tag{28-41}$$

Than (hopefully) we can select coefficients of f_3 to leave only cubic and higher order terms. Than we repeat the procedure for $f_4, f_5 \ldots\ldots$

$$\ldots\ldots\exp(-:f_5:)\exp(-:f_5:)\exp(-:f_3:)\exp(-:f_2:)\bar{X} \to X \tag{28-42}$$

with natural conclusion that multiplying (28-42) by (28-38) we get:

$$\bar{X} = \mathtt{M}X. \tag{28-43}$$

While logically straight forward, the process (especially for 3D case) become cumbersome right away and in real situation (with few exceptions which prove the rule) computers do it much better.

Thus, we concluded that any analytical symplectic map can be presented as a product of linear (Gaussian optics) Lie transformation and product of Lie transformations comprising homogeneous polynomials of increasing power:

$$\mathtt{M} = \overbrace{\exp(:f_2:)}^{Gaussian\,optics} \bullet \overbrace{\exp(:f_3:)\exp(:f_4:)\exp(:f_5:)\ldots}^{Abberations,\,Nonlinear\ effects} \qquad (28\text{-}44)$$

While looking as a final result, the remaining question is – how we can use it?

While there are hundreds of very important Lie algebraic relations and many-many tricks, one is important for interpretation (normalization) of the non-linear symplectic maps. In linear case we have set the action and angle canonical pairs describing each oscillator:

$$\{\varphi_k, I_k\} \Leftrightarrow \tilde{x}_k = \sqrt{2I_k}\cos(\psi+\varphi_k); \tilde{p}_k = -\sqrt{2I_k}\sin(\psi+\varphi_k); I_k = \frac{\tilde{x}_k{}^2 + \tilde{p}_k{}^2}{2}; \qquad (28\text{-}45)$$

where $\{\tilde{x}_k, \tilde{p}_k\}$ are also canonical pairs. We could bring our linear map (matrix) to an oscillator turn using

$$U = [\ldots, \mathrm{Re}\, Y_k; \mathrm{Im}\, Y_k \ldots]; MY_k = e^{i\mu_k} Y_k \rightarrow M \cdot U = UR; k = 1,..,n$$

$$R = \begin{bmatrix} \ldots & 0 & 0 \\ 0 & R_k & 0 \\ 0 & 0 & \ldots \end{bmatrix}; R_k = \begin{bmatrix} \cos\mu_k & -\sin\mu_k \\ \sin\mu_k & \cos\mu_k \end{bmatrix}; \; U^{-1} \cdot M \cdot U = R = \exp(:\vec{\mu}\cdot\vec{I}:).$$

(28-46)

In linear approximation trajectories in $\{\tilde{x}_k, \tilde{p}_k\}$ planes are boring circles with radius $\sqrt{2I_k}$. This representation is called normal form of representation for linear symplectic map.

While considering nonlinear effect, we noticed that tune can depend on particle's actions

$$\mu_k = \mu_k\left(I_1,...I_n\right) \leftrightarrow \vec{\mu} = \vec{\mu}\left(\vec{I}\right)$$

and this is the reason for looking for a nonlinear transformation which brings a non-linear map to a specific form:

$$\mathtt{M} = \mathtt{A}^{-1}\mathtt{RA};\ \ \mathtt{R} = \exp\left(:f:\right);\ \ f = f\left(\vec{I}\right). \qquad (28\text{-}47)$$

The idea of such transformation (rotation with angle depending on amplitude!) is to capture tune dependence of the amplitudes and to separate the geometrical aberrations (phase dependent) and resonance terms from it. Let make transformation into the normalized phase space where

$$\mathtt{M} = R\cdot\exp\left(:f_3:\right)\exp\left(:f_4:\right)\exp\left(:f_5:\right) \qquad (28\text{-}48)$$

There is a step-by step process on how to separate contribution to the rotation from various orders:

$$\mathtt{R}_3 = e^{:F_3:}Me^{:-F_3:} = e^{:F_3:}Re^{:f_3:}e^{:f_4:}....e^{:-F_3:} = RR^{-1}e^{:F_3:}Re^{:f_3:}e^{:-F_3:}e^{:F_3:}e^{:f_4:}...e^{:-F_3:} \qquad (28\text{-}49)$$

where we have to use one more magic feature of Lie transformations:

$$\exp\left(:h:\right)\exp\left(:g:\right)\exp\left(-:h:\right) = \exp\left(\exp\left(:h:\right):g:\right)$$

This important ratio can be derived by introducing adjoin Lie operator

$$def: \quad \#f\#:g \coloneqq \{:f:,:g:\}; \;\; \#f\#^2:g \coloneqq \{:f:,\{:f:,:g:\}\};$$

$$\#f\#^0:g \coloneqq :g:; \#f\#^{n+1}:g \coloneqq \{:f:,\#f\#^n:g:\};$$

$$\{\#f\#,\#g\#\} = \#f\#\#g\# - \#g\#\#f\#;$$

$$\{\#f\#,\#g\#\} = \#\{f,g\}\# = \#\{f,g\}\#;$$

$$\#f\#^n:g \coloneqq :\left(:f:^n g\right): \;\; ;\#f\#\left(:g::h:\right) = \left(\#f\#:g:\right):h: + :g:\left(\#f\#:h:\right); \quad (28\text{-}50)$$

$$\exp(:f:):g:\exp(-:f:) = \exp(\#:f:\#):g:;$$

$$\exp(\#:f:\#) = \sum_{n=0} \frac{\#:f:\#^n}{n!};$$

$$\exp(:f:):g:^n \exp(-:f:) = \exp(\#:f:\#):g:^n;$$

$$\exp(:f:)\exp(:g:)\exp(-:f:) = \exp\left(\exp(:f:):g:\right);$$

which can be also proved with some modest (but not a two-liner effort – see prove in [1]).

Using (28-50) we get

$$\mathrm{R}_3 = Re^{:R^{-1}F_3:}e^{:f_3:}e^{:-F_3:}e^{:e^{:F_3:}f_4:}$$

Next step requires using one of key formula named after Baker-Campbell-Hausdorff who proved it:

$$\exp(:A:)\exp(:B:) = \exp(:C:), \quad \to \quad C = A + B + \frac{1}{2}[A,B] + higher\ oders\cdots (28\text{-}51)$$

$$\mathrm{R}_3 = Re^{:R^{-1}F_3+f_3-F_3+O(4):}e^{:e^{:F_3:}f_4:} \to Re^{:f_3^{(1)}{}_3:}e^{:f_4^{(1)}:}; \quad f_3^{(1)} = R^{-1}F_3 + f_3 - F_3$$

with solution being

$$F_3 = \frac{f_3 - f_3^{(1)}}{I - R^{-1}}; \quad f_3 = \sum_m \bar{f}_{3,m}(I_x)e^{im\phi}$$

$$f^{(1)}{}_3 = \bar{f}_{3,0}(I_x); F_3 = \sum_{m\neq 0} \frac{\bar{f}_{3,m}(I_x)e^{im\phi}}{1-e^{-im\mu}} \qquad (28\text{-}52)$$

Similarly the higher orders can be treated. If one takes a map of a octupole:

$$\mathrm{M} = Re^{:f_4:}; \ f_4 = -\frac{1}{24}k_3 l x^4 \to \mathrm{M}_4 = Re^{:f_{4,0}:} = e^{:-\mu J - \frac{1}{16}k_3 l\beta^2 I^2:};$$

$$f_4 = -\frac{1}{6}k_3 l\beta^2 I^2\cos^4\varphi = -\frac{1}{48}k_3 l\beta^2 I^2(3 + 4\cos 2\varphi + \cos 4\varphi); \qquad (29\text{-}53)$$

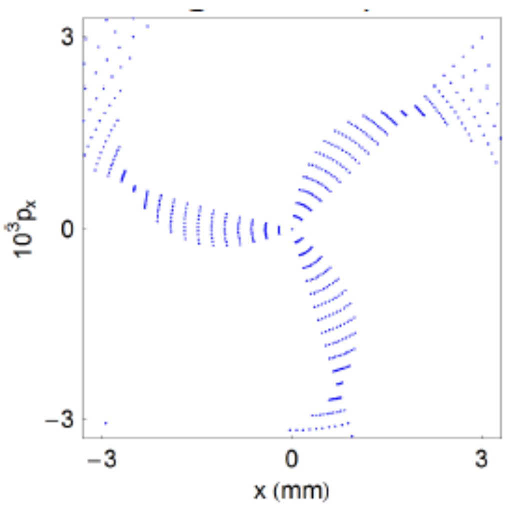
$$F_4 = \sum_{m\neq 0}\frac{f_{4,m}(J)e^{im\phi}}{1-e^{-im\mu}}$$

e.g. familiar result we get by averaging the nonlinear part of Hamiltonian.

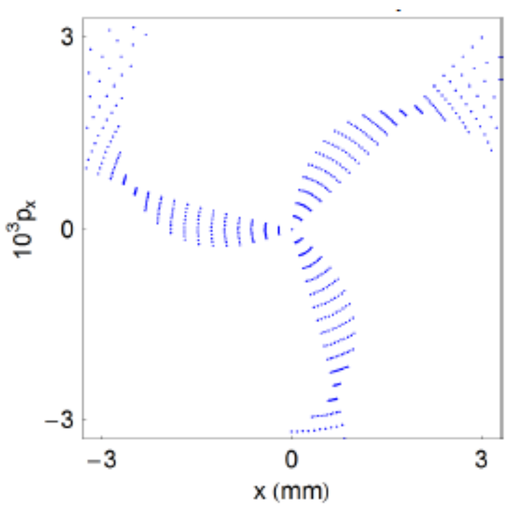
Normal form treatment

The normalized map now contains only action variable (easy to integrate) while all the phase information has been pushed to higher order.

From the generator F_4, we see the octupole drives half integer and quarter integer resonances. We can track the Poincare map using exact map and the normalized map respectively (assum $k_3 l = 4800\ m^{-3}$ and $\beta = 1$ m). Assuming the tune μ is $0.33 \times 2\pi$ far from resonances



exact map



normalized map

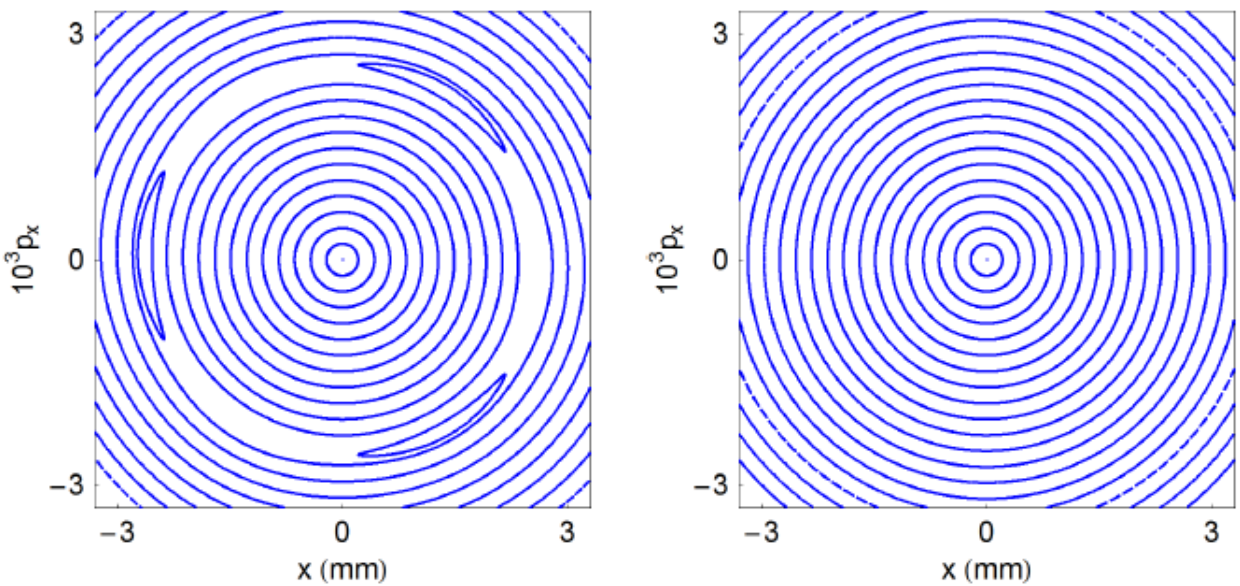
30 turns

Tune shift with amplitude!!

Y. Jing

Normal form treatment

Tracking for longer turns results in different feature where we pay the price of the simplified (normalized) map. Some of the phase information (3rd order resonance island) is lost during this process.



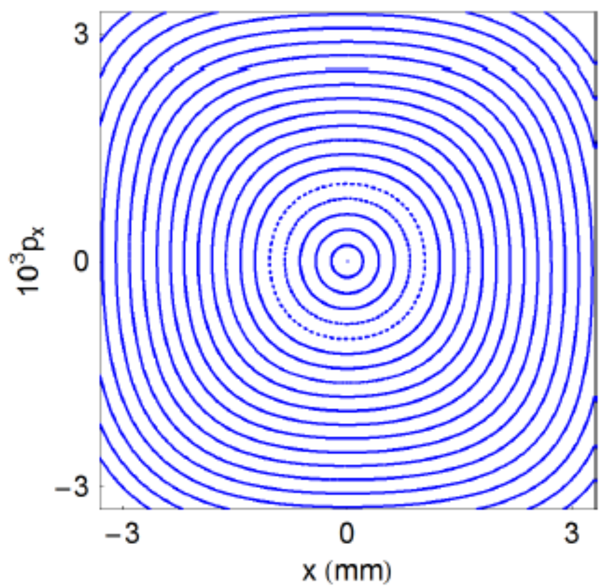
2500 turns

exact map
normalized map

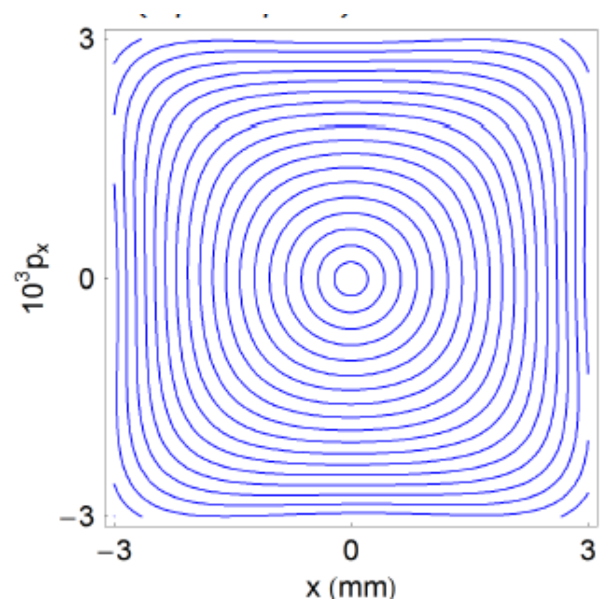
Y. Jing

Normal form treatment

Tracking for tunes near 4th order resonance is a bit tricky. Since the k_3l is positive, the tune shift with amplitude drives the tune up. Thus if the tune μ is $0.252 \times 2\pi$, we barely see resonances. The two tracking results resemble



exact map

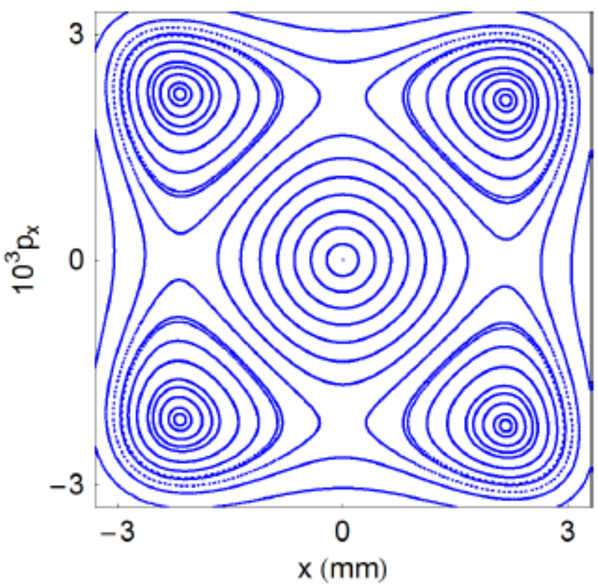


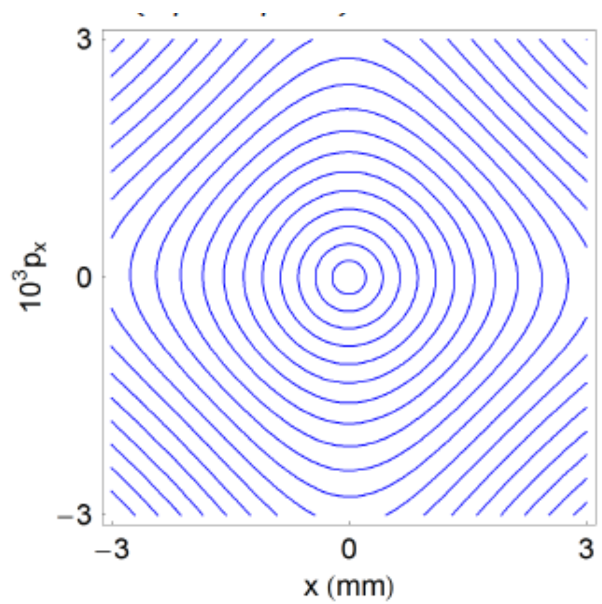
normalized map

2500 turns

Y. Jing

Normal form treatment





2500 turns

exact map
normalized map

Normal form of a one turn map preserves the information on tune amplitude dependence while loses the key phase information (when close to resonances). Need to retain higher order terms!

Y. Jing

Much more in:

[1] A. Dragt, Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics

[2] E. Forest, Beam dynamics, Harwood Academic Publishers

End of classes