USPAS “Hadron Beam Cooling in Particle Accelerators”

HW1 – Monday, January 30, 2023

Problem 1: Reference particle and reference orbit. 6points

Using accelerator Hamiltonian (M1.19), corresponding differential equations (M1.20), expansion of the vector and scalar potentials (M1.21), show that for a reference particle that is following a reference “trajectory”:

$$\vec{r} = \vec{r}_o(s);\ \ t = t_o(s);\ \ H = H_o(s) = E_o(s) + \varphi_o(s, t_o(s)),$$

with $x \equiv 0;\ y \equiv 0;\ p_x \equiv 0; p_y \equiv 0$ and $h^*\big|_{ref} = -p_o(s)$ result in the following conditions:

$$K(s) \equiv \frac{1}{\rho(s)} = -\frac{e}{p_o c}\left(B_y\big|_{ref} + \frac{E_o}{p_o c} E_x\big|_{ref} \right); \tag{1}$$

$$B_x\big|_{ref} = \frac{E_o}{p_o c} E_y\big|_{ref}; \tag{2}$$

$$\frac{dt_o(s)}{ds} = \frac{1}{\mathrm{v}_o(s)} \tag{3}$$

$$\frac{dE_o(s)}{ds} = -e\frac{\partial \varphi}{\partial s}\bigg|_{ref} \equiv eE_2(s, t_o(s)). \tag{4}$$

Hints:

1. Use condition $\vec{A}\big|_{ref} = 0$ with

$$x\big|_{ref} = 0;\ y\big|_{ref} = 0;\ P_1\big|_{ref} = p_x\big|_{ref} + \frac{e}{c}A_1\big|_{ref} \equiv 0;\ \ P_3\big|_{ref} = p_y\big|_{ref} + \frac{e}{c}A_3\big|_{ref} \equiv 0;$$

or in the differential form

$$\frac{dx}{ds}\bigg|_{ref} = \frac{\partial h^*}{d P_1}\bigg|_{ref} = 0;\ \ \frac{dy}{ds}\bigg|_{ref} = \frac{\partial h^*}{dP_3}\bigg|_{ref} = 0;$$
$$\frac{dP_1}{ds}\bigg|_{ref} = -\frac{\partial h^*}{dx}\bigg|_{ref} = 0;\ \frac{dP_3}{ds}\bigg|_{ref} = -\frac{\partial h^*}{dy}\bigg|_{ref} = 0;$$

2. Keep only necessary (i.e. relatively low order) terms in expansion of vector potentials.

Problem 2: Trace and determinant. 4 points

Solution of any linear n-dimensional differential equation

$$\frac{dX}{ds} = \mathbf{D}(s)X$$

can be expressed in a form of transport matrix

$$X(s) = \mathbf{M}(s)X_o;\ X_o = X(s=0)$$

with

$$\frac{d\mathbf{M}(s)}{ds} = \mathbf{D}(s)\mathbf{M}(s);\ \mathbf{M}(s=0) = \mathbf{I}; \qquad (1)$$

where **I** is unit *nxn* matrix. Prove that

$$\det\left(\mathbf{M}(s)\right) = \exp\left(\int_0^s Trace\left(\mathbf{D}(\zeta)\right)d\zeta\right).$$

Hints:

1. Prove first that
$$\frac{d}{ds}\det\mathbf{M} = Trace\left(\mathbf{D}\right)\cdot\det\mathbf{M}$$
2. Use infinitesimally small step in eq. (1) to conclude that
$$d\mathbf{M}(s) = \mathbf{D}(s)\mathbf{M}(s)ds + O\left(ds^2\right) \Rightarrow \mathbf{M}(s+ds) = \left(\mathbf{I} + \mathbf{D}(s)ds\right)\cdot\mathbf{M}(s) + O\left(ds^2\right);$$
$$\det\mathbf{M}(s+ds) = \det\left(\mathbf{I} + \mathbf{D}(s)ds\right)\cdot\det\mathbf{M}(s) + O\left(ds^2\right) \rightarrow \qquad (1)$$
$$\frac{1}{\det\mathbf{M}}\frac{d\left(\det\mathbf{M}\right)}{ds} = \frac{\det\left(\mathbf{I} + \mathbf{D}(s)ds\right) - 1}{ds};$$
3. What remained is to prove us that
$$\det\left(\mathbf{I} + \varepsilon\mathbf{D}\right) = 1 + \varepsilon\cdot Trace[\mathbf{D}] + O\left(\varepsilon^2\right)$$

where ε is infinitesimally small real number and term $O\left(\varepsilon^2\right)$ contains second and higher orders of ε.

4. First, fist look on the product of diagonal elements $\prod_{m=1}^{n}\left(1 + \varepsilon a_{mm}\right)$ in $\det\left[I + \varepsilon A\right]$in the first order of ε. Then prove that contributions to determinant from non-diagonal terms $a_{km}; k \neq m$ is $O\left(\varepsilon^2\right)$ or higher order of ε. It is possible to do it directly for an arbitrary *nxn* matrix, or start from *n=1* and use induction from *n* to *n+1*.

By doing this you also prove the sum of decrements theorem!

P.S. Any elegant and unexpected solution will have result in quadrupled points.

Problem 3: Trace and determinant. 15 points

Part 1. **5 points**. Prove that for uncoupled vertical oscillations

$$\frac{dy}{ds} = y'; \ \frac{dy'}{ds} \equiv y'' = -K_1(s)\,y; \tag{1}$$

the phase space distribution

$$F(y,y',s) = f\left(\zeta(y,y',s)\right); \ \zeta(y,y',s) = \left(\mathrm{w}(s)\,y' - \mathrm{w}'(s)\,y\right)^2 + \left(\frac{y}{\mathrm{w}(s)}\right)^2 \tag{2}$$

with an arbitrary differentiable $f(\zeta)$ and beam envelope

$$\mathrm{w}''(s) + K_1(s)\,\mathrm{w}(s) = \frac{1}{\mathrm{w}(s)^3} \tag{3}$$

satisfied Vlasov equation:

$$\frac{\partial F}{\partial s} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' = 0. \tag{4}$$

Hint: Use well-known $\dfrac{\partial_{y,y',s} f(\zeta)}{\partial y, y', s} = \dfrac{df(\zeta)}{d\zeta}\cdot\dfrac{\partial_{y,y',s}\zeta}{\partial y, y', s}$ and equations (1) and (3) to prove (4)

Part 2. **10 points**. Prove that phase space distribution

$$F(y,y',s) = f(\zeta) = c\cdot\exp\left(-\frac{\zeta}{2\varepsilon}\right); \tag{5}$$

satisfies **phase-averaged** Fokker Plank equation:

$$\frac{\partial F}{\partial s} + \frac{\partial F}{\partial y}y' - \frac{\partial}{\partial y'}F\left(K_1 y - \xi y'\right) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(F\cdot D_{yy}\right) + \frac{1}{2}\frac{\partial^2}{\partial y\,\partial y'}\left(F\cdot D_{yy'}\right) + \frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(F\cdot D_{y'y'}\right) = 0 \tag{6}$$

for uncoupled vertical oscillations with additional damping terms and random noise (diffusion)

$$\frac{dy}{ds} = y'; \ \frac{dy'}{ds} \equiv y'' = -K_1(s)\,y - \xi(s)\,y' + \upsilon(s)\cdot\sum_{i=1}^{N} rnd_i\cdot\delta(s-s_i); s_i\in(0,C)$$
$$\langle rnd\rangle = 0; \langle rnd^2\rangle = 1 \tag{7}$$

with constant emittance $\varepsilon = \dfrac{\langle D_{y'y'}\mathrm{w}^2\rangle}{2\langle\xi\rangle}$.

Step 1: First, eliminate fast oscillating terms using eq. (4): $\dfrac{\partial F}{\partial s} = -\dfrac{\partial F}{\partial y}y' - \dfrac{\partial F}{\partial y'}y''$.

Step 2: Evaluate three diffusion coefficients

$$D_{uv} = \lim_{\tau\to 0}\frac{1}{\tau}\left(u(s+\tau)-u(s)\right)\left(v(s+\tau)-v(s)\right);$$

Show that $D_{yy}=0$ by finding that $\left(y(s+\tau)-y(s)\right)^2 \sim \tau^2$, and that $\langle D_{yy'}\rangle = 0$, when averaging is taken of the random kicks with $\langle g(y,y')\cdot rnd\rangle = g(y,y')\cdot\langle rnd\rangle = 0$. Finally, calculate $\langle D_{y'y'}\rangle$ using following manipulations:

$$y'(s+\tau) = y'(s) + K(s^*)y(s^*) + \sum_{s_i\in\{s,s+\tau\}} \upsilon(s_i)\cdot rnd_i;\ s^*\in\{s,s+\tau\}$$

Show that after averaging over random kick strength, the only non-zero term originates only from square of the random kicks $\left\langle\left(\sum_{s_i\in\{s,s+\tau\}}\upsilon(s_i)\cdot rnd_i\right)^2\right\rangle \to \sum_{s_i\in\{s,s+\tau\}}\upsilon^2(s_i)\cdot\langle rnd_i^{\,2}\rangle$

Here you need to use the fact stand random kicks are not correlated:

$$\langle rnd_i\cdot rnd_{j\neq i}\rangle = 0$$

to arrive to $\langle D_{y'y'}\rangle$ independent on y and y', which allows you to take it out of $\frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(F\cdot D_{y'y'}\right)$.

Step 3: after completing all differentiations, use expression for y and y'

$$y = a\mathrm{w}(s)\cdot\cos\varphi;\ y' = a\left(\mathrm{w}'(s)\cdot\cos\varphi - \frac{\sin\varphi}{\mathrm{w}(s)}\right)$$

and average over betatron phases φ arrive to equation in form of $F(y,y',s)\cdot g\left(\xi(s),\mathrm{w}(s)D_{y'y'}(s),a^2,\varepsilon\right)=0$, which means that g=0.

Step 3: Assuming that a^2,ε (i.e. practically are constants!) are slow function compared with $\xi(s),\mathrm{w}(s)D_{y'y'}(s)$, average over the ring circumference to arrive to conclusion that $\varepsilon = \frac{\langle D_{y'y'}\mathrm{w}^2\rangle_C}{2\langle\xi\rangle_C}$ satisfies the Fokker-Plank equation.