PHY 564
Advanced Accelerator Physics
Lecture 5

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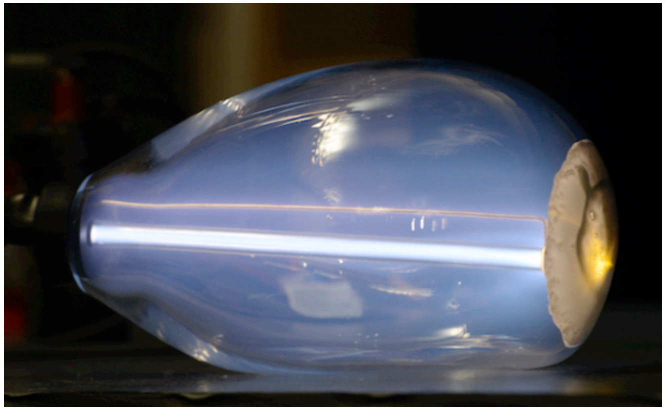
Accelerator Hamiltonian

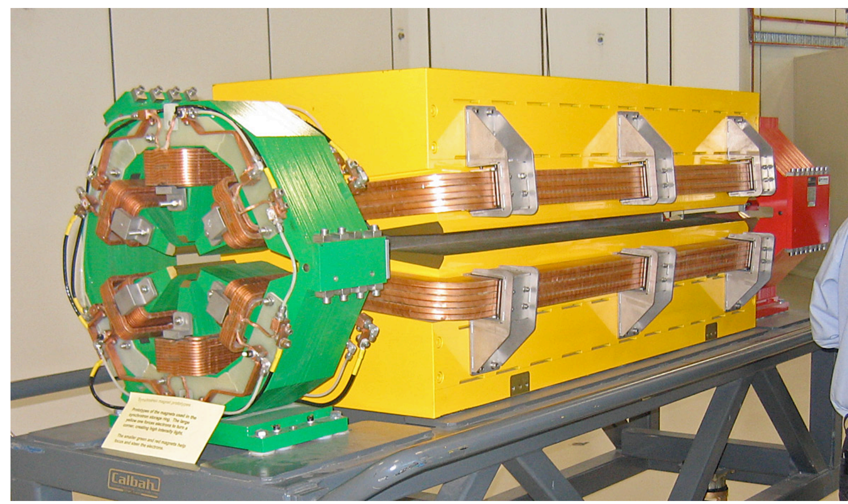
Intro: Before we are indulging ourselves on deriving accelerator Hamiltonian, let's discuss what accelerators are about. In practice they are about accelerating and circulating beams of relativistic charged particles. Beam – by a definition – is a group of particles which propagates along and around a common trajectory, which frequently called "orbit" or "reference trajectory". What is important that their motion is continuous, e.g. particles do not separate from the beam and go backwards. The later is very important, since the distance along the reference trajectory, s, will be used as an independent variable instead of the time. In addition, typical beams are confined transversely and usually propagate inside a vacuum chamber to avoid scattering. Exceptions are exceptions, and one can imagine an "accelerator" in which particles are completely disorganized and go everywhere in space in time – needless to say it most likely will be a useless device. Thus, let's focus on practical accelerators operating confined beams of charged particles.



A beam of particles in a cathode-ray tube

There is a number of very good reason for using s as independent variable: most of the accelerator elements are either DC (constant) or slowly varying in time, but always have a specific geometry – in other words all accelerators are bolted to the floor. Thus, arrival time of a particle into an accelerator element can vary, while element position, structure and duration along the reference trajectory is well defined. In circular accelerators (such as synchrotrons or storage rings), particles circulate for billions and billions of turns traversing the same magnetic structure (frequently called magnetic lattice!). This motion is nearly periodic in space along the trajectory.

LHC tunnel: a long periodic lattice of superconducting 80 kGs (8T) magnets



A magnet cell of an Australian light source: Green is a sextupole, yellow is a combined function dipole and red

2.1 Accelerator coordinate system.

In accelerator physics we usually study beams of particles, i.e. particles moving in approximately the same direction (a huge difference from detectors) with approximately the same momenta. It is traditional, and very useful to choose one particle in the beam as the reference particle and study its trajectory $\vec{r}_o(t)$ as natural reference. Furthermore, most accelerator equipment is bolted to the floor and, hence, can be better described by its position in space that its existence in time. This is the reason why accelerator physicists decided to use length along the reference trajectory, ***s***, as independent coordinate instead of time:

$$s(t) = \int_{t_i}^{t} \left| d\vec{r}_o(t) \right| = \int_{t_i}^{t} \left| \vec{\mathrm{v}}_o(t) \right| dt;$$

$$\vec{\mathrm{v}}_o(t) = \frac{d\vec{r}_o(t)}{dt}; \;\; \gamma(t) = 1/\sqrt{1 - \vec{\mathrm{v}}_o^{\,2}(t)/c^2}; \;\; \vec{p}_o(t) = \gamma(t) m \vec{\mathrm{v}}_o(t); \;\; E_o(t) = \gamma(t) mc^2 \tag{96}$$

It is important for independent variable to be a monotonous function (as is time), which requires that the reference particle never stops moving (except possibly at the beginning and the end of the reference trajectory).

Reference trajectories

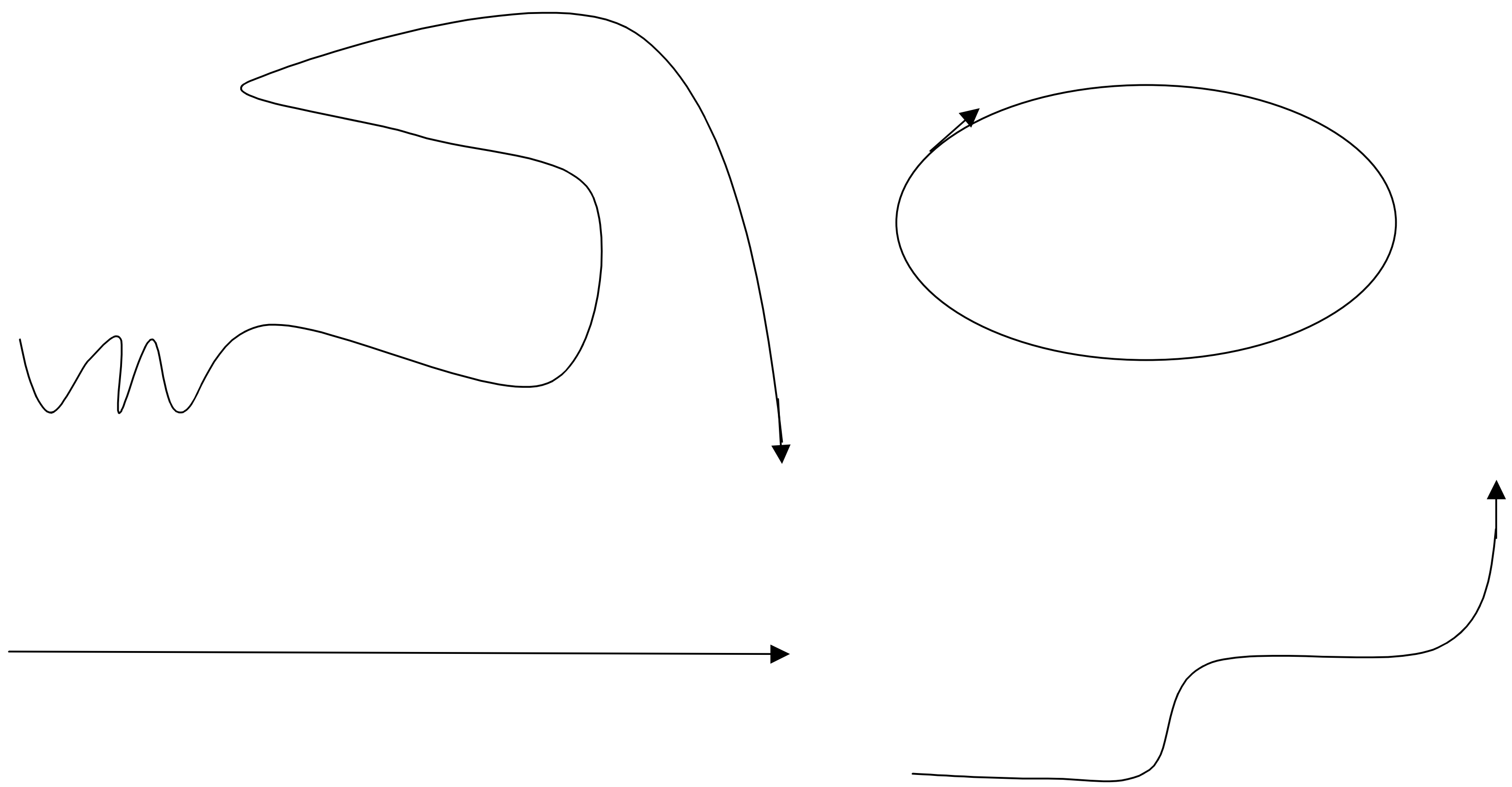


Fig. 1. Various possible reference trajectories, from a simple straight pass to a circular one, though all other possibilities.

The reference trajectory is determined by initial 4-momentum of the reference particle and the EM field along its trajectory. We should consider that trajectory is given (and from $\vec{r}_o(t)$ we also know the particle's 4-momentum in each point of trajectory) and so satisfy the equation of motion.

Usually EM fields are designed for the existence of such a trajectory (within constrains of Maxwell equation). Herein, the words reference trajectory and orbit are used interchangeably.

Inverting (96 we can write the 4D trajectory at the function of s:

$$\vec{r} = \vec{r}_o(s);\ \ t = t_o(s);\ \ \vec{p} = \vec{p}_o(s), E = E_o(s). \tag{97}$$

with the charge to the designer of accelerator to make it real trajectory:

$$\frac{d\vec{p}_o(s)}{ds} = \frac{dt_o(s)}{ds}\left(e\vec{E}\left(\vec{r}_o(s), t_o(s)\right) + \frac{e}{c}\left[\vec{\mathrm{v}}_o(s) \times \vec{B}\left(\vec{r}_o(s), t_o(s)\right)\right]\right) \tag{98}$$

Starting from this point, we use following conventions: Derivatives of any function with respect to the time will be shown by appropriate number of dots, while appropriate number of symbol ′ will be used to indicate derivatives with respect to ***s***:

$$f' = \frac{df}{ds};\;\; f'' = \frac{d^2 f}{ds^2} \ldots\ldots\;\; \dot{f} = \frac{df}{dt};\;\; \ddot{f} = \frac{d^2 f}{dt^2}. \tag{99}$$

There is infinite variety of possible reference trajectories. The most popular ones are flat, i.e. they lie in a plane. A typical example is the circular orbit of a storage ring with a horizontal trajectory. Many of reference orbits are piece-wise combinations of trajectories lying in various planes. Still, there are 3D reference orbits by design. As the matter of fact, all real reference orbits are 3D because of the field errors in magnets, and errors in aligning these magnets.

Hence, there is no good reason not to start this discussion from general 3D reference trajectory. Fortunately two French mathematicians, Jean Frédéric Frenet and Joseph Alfred Serret, in the mid-nineteenth century developed such a coordinate system, which is described by the Frenet-Serret formulas in classical differential geometry (O.Struik, Dirk J., Lectures on Classical Differential Geometry, Addison-Wesley, Reading, Mass, 1961). The Frenet-Serret coordinate system often is called the natural coordinate system. One important feature is that it has non-diagonal metrics. Hence, we have a bit of differential geometry to spice the mix.

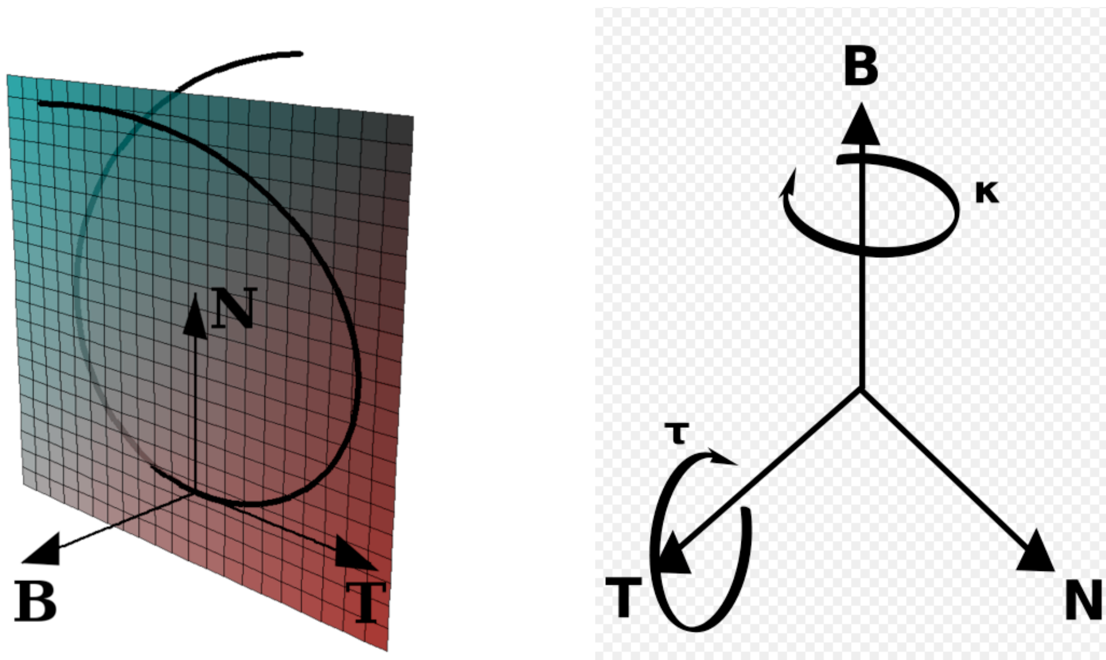
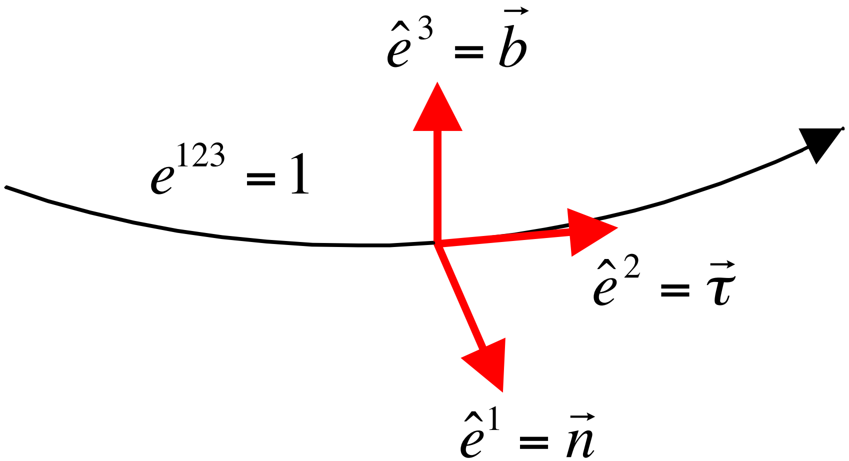


Fig. 2. Illustration of Frenet-Serret formulas and system from
http://en.wikipedia.org/wiki/Frenet-Serret

Figures 2 and 3 illustrate the Frenet-Serret coordinate system and define 3 orthogonal unit vectors: Normal $\hat{e}^1 = \vec{n}(s)$, tangent $\hat{e}^2 = \vec{\tau}(s)$, and normal and bi-normal $\hat{e}^3 = \vec{b}(s) = \left[\vec{n} \times \vec{\tau}\right]$:

$$\left(\vec{n} \cdot \vec{\tau}\right) = \left(\vec{b} \cdot \vec{n}\right) = \left(\vec{b} \cdot \vec{\tau}\right) = 0.$$



$$\vec{\tau} = \frac{d\vec{r}_o(s)}{ds} = \vec{r}_o{}'$$

$$\vec{n} = -\vec{r}_o{}'' \Big/ \left|\vec{r}_o{}''\right|$$

$$\vec{b} = \left[\vec{n} \times \vec{\tau}\right]$$

Fig. 3. Unit vectors in the Frenet-Serret coordinate system and their definitions

The reference trajectory must be smooth, with finite second derivatives, etc….etc… The position of any particle located in close proximityto the reference trajectory can uniquely expressed as

$$\vec{r} = \vec{r}_o(s) + x \cdot \vec{n}(s) + y \cdot \vec{b}(s). \qquad (100)$$

i.e., it is fully described by 3 contra-variant coordinates:

$$q^1 = x;\ q^2 = s,\ q^3 = y. \qquad (100\text{-}1)$$

The vectors $\left\{\vec{n},\vec{\tau},\vec{b}\right\}$ satisfy Frenet-Serret formulae:

$$\frac{d\vec{\tau}}{ds} - K(s)\cdot\vec{n};\ \frac{d\vec{n}}{ds} = K(s)\cdot\vec{\tau} - \kappa(s)\cdot\vec{b};\ \frac{d\vec{b}}{ds} = \kappa(s)\cdot\vec{n};. \qquad (101)$$

where

$$K(s) = 1/\rho(s) \qquad (101\text{-}1)$$

is the curvature of the trajectory, and $\kappa(s)$ is its torsion. If the torsion is equal to zero, the trajectory remains in one plane, as designed for majority of accelerators. Curvature of trajectory is more common – each dipole magnet makes trajectory to curve.

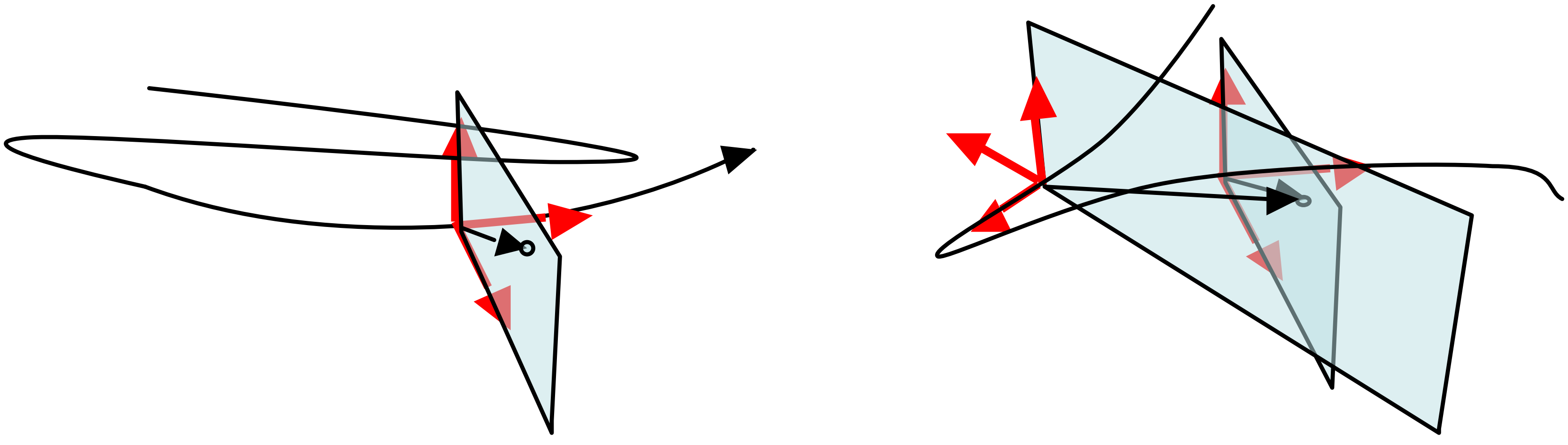
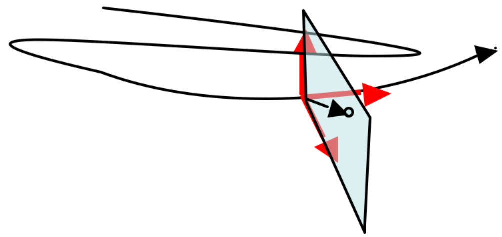


Fig. 4. Expansion of particle’s position in Frenet-Serret frame.

Proximity to the reference orbit is important for the uniqueness of the extension (100): As shown on the figure above, equation (101-2) may have multiple solutions if the requirement of proximity is not applied, i.e, the expansion (100) may have multiple branches and mathematically become too involved.



As shown in Fig.4, the transverse part of the position vector $\vec{r}_\perp = x \cdot \vec{n}(s) + y \cdot \vec{b}(s)$ lies in the plane defined by the normal and by-normal unit vectors $(\vec{n}(s),\ \vec{b}(s))$, while s is defined from equation:

$$\left(\vec{r} - \vec{r}_o(s)\right) \cdot \vec{\tau}(s) = 0. \qquad (101\text{-}2)$$

Now we expand the differential geometry:

$$d\vec{r} = \sum_{i=1}^{3} \vec{a}_i dq^i = \vec{n}dx + \vec{b}dy + \left\{\left(1 + Kx\right)\vec{\tau} + \kappa\left(\vec{n}y - \vec{b}x\right)\right\}ds \qquad (102)$$

with the co-variant basis of

$$\vec{a}_i = \frac{\partial \vec{r}}{\partial q^i}; \quad \vec{a}_1 = \vec{n}; \ \vec{a}_2 = \left(1 + Kx\right)\vec{\tau} + \kappa\left(\vec{n}y - \vec{b}x\right); \quad \vec{a}_3 = \vec{b}; \qquad (103)$$

A co-variant basis vector is readily derived from the orthogonal conditions:

$$\vec{a}_i \vec{a}^j = \delta_i^j;\; \vec{a}^1 = \vec{n} - \frac{\kappa y}{1+Kx}\vec{\tau};\, \vec{a}^2 = \frac{\vec{\tau}}{1+Kx};\; \vec{a}^3 = \vec{b} + \frac{\kappa x}{1+Kx}\vec{\tau}; \tag{104}$$

The components of the co- and contra-variant metric tensors are defined as follows:

$$g_{ik} = \vec{a}_i \cdot \vec{a}_k = \begin{bmatrix} 1 & \kappa y & 0 \\ \kappa y & (1+Kx)^2 + \kappa^2(x^2+y^2) & -\kappa x \\ 0 & -\kappa x & 1 \end{bmatrix}$$

$$g^{ik} = \vec{a}^i \cdot \vec{a}^k = \frac{1}{(1+Kx)^2} \cdot \begin{bmatrix} (1+Kx)^2 + \kappa^2 y^2 & -\kappa y & -\kappa^2 xy \\ -\kappa y & 1 & \kappa x \\ -\kappa^2 xy & \kappa x & (1+Kx)^2 + \kappa^2 x^2 \end{bmatrix} \tag{105}$$

$$g_o = \det\left[g_{ik}\right] = (1+Kx)^2$$

Any vector can be expanded about both co- and contra-variant bases, as well can $\left\{\vec{n},\vec{\tau},\vec{b}\right\}$:

$$\vec{R} \equiv R_x\vec{n} + R_s\vec{\tau} + R_y\vec{b} \equiv \sum_k R^k\vec{a}_k \equiv \sum_k R_k\vec{a}^k$$

$$R_k = \vec{R}\cdot\vec{a}_k;\; R_1 = R_x; R_2 = \left(1+Kx\right)R_s + \kappa\left(R_x y - R_y x\right);\; R_3 = R_y;$$

$$R^k = \vec{R}\cdot\vec{a}^k;\; R^1 = R_x - \frac{\kappa y}{1+Kx}R_s;\; R^2 = \frac{R_s}{1+Kx} + \kappa\left(R_x y - R_y x\right);\; R^3 = R_y + \frac{\kappa x}{1+Kx}R_s;$$

(106)

All this is trivial, and finally differential operators will look like:

$$\begin{gathered}\vec{\nabla}\varphi = \vec{a}^k\frac{\partial\varphi}{\partial q^k};\quad div\vec{A} = \left(\vec{\nabla}\cdot\vec{A}\right) = \frac{1}{\sqrt{g_o}}\frac{\partial}{\partial q^k}\left(\sqrt{g_o}A^k\right);\\ curl\vec{A} = \left[\vec{\nabla}\times\vec{A}\right] = \frac{e^{ikl}}{\sqrt{g_o}}\frac{\partial A_l}{\partial q^k}\vec{a}_i;\;\; \Delta\varphi = \vec{\nabla}^2\varphi = \frac{1}{\sqrt{g_o}}\frac{\partial}{\partial q^i}\left(\sqrt{g_o}g^{ik}\frac{\partial\varphi}{\partial q^k}\right)\end{gathered}. \tag{107}$$

As discussed before, the Hamiltonian of a charged particle in EM field in Cartesian coordinate system is

$$H(\vec{r},\vec{P},t) = c\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2} + e\varphi, \qquad \text{(from Lecture 3/4)}$$

where the canonical momentum is $\vec{P} = \vec{p} + \frac{e}{c}\vec{A}$. Let us explore how we can make the transformation to our "curved and twisted" coordinate system. The easiest way is to apply canonical transformation with generation function

$$F(\vec{P},q^i) = -\vec{P}\cdot\left(\vec{r}_o(s) + x\cdot\vec{n}(s) + y\cdot\vec{b}(s)\right). \tag{108}$$

to our new coordinates (101):

$$q^1 = x;\ q^2 = s,\ q^3 = y. \tag{109}$$

with new momenta obtained by simple differentiation

$$P_1 = P_x; P_2 = (1+Kx)P_s + \kappa\left(P_x y - P_y x\right);\ P_3 = P_y; \tag{110}$$

that alter the appearance of the Hamiltonian (L1.38)

$$H = c\sqrt{\begin{array}{l}(1+Kx)^{-2}\left(\left(P_2-\frac{e}{c}A_2\right)+\kappa x\left(P_3-\frac{e}{c}A_3\right)-\kappa y\left(P_1-\frac{e}{c}A_1\right)\right)^2 \\ +\left(P_1-\frac{e}{c}A_1\right)^2+\left(P_3-\frac{e}{c}A_3\right)^2+m^2c^2\end{array}}+e\varphi \qquad (111)$$

This is still the Hamiltonian with ***t*** as independent variable and three sets of canonical pairs $\{q^1, P_1\}, \{q^2, P_2\}, \{q^3, P_3\}$. Now, we change the independent variable to ***s*** by the easiest method, that, as always, is using the least-action principle: we consider the conjugate momentum to *s*, P_2, as a function of the remaining canonical variables: $\{q^1, P_1\}$, $\{q^3, P_3\}$, $\{-t, H\}$

$$S = \int_A^B P_1 dq^1 + P_2(....)ds + P_3 dq^3 - Hdt; \quad \delta S = 0; \tag{112}$$

Notably, the coordinates and time, the canonical momenta and the Hamiltonian appear in the 4-D scalar product form in the action integral.

$$P_i dx^i; \; x^i = \{ct, x, s, y\}; P_i = \{H/c, -P_1, -P_2, -P_3\}, i = 0,1,2,3.$$

This equivalency of the time and space is fundamental to the relativistic theory.

Let's use *s* a independent variable and *t* as one of the coordinates:

$$\delta S_{AB} = \delta\left(\int_A^B P_i dq^i - Hdt\right) = \int_A^B \left(\begin{array}{l} \sum_{i=1,3}\left(\delta P_i dq^i + P_i d\delta q^i + \dfrac{\partial P_2}{\partial q^i}\delta q^i ds + \dfrac{\partial P_2}{\partial P_i}\delta P_i ds\right) + \\ \left(-\delta H dt - H d\delta t + \dfrac{\partial P_2}{\partial t}\delta t ds + \dfrac{\partial P_2}{\partial H}\delta H ds\right) \end{array} \right) = 0$$

and integrating by parts $\left.\sum_{i=1,3} P_i \delta q^i - H\delta t\right|_A^B \equiv 0$, equations of motions as functions of *s*:

$$\delta S_{AB} = \int_A^B \left(\begin{array}{l} \sum_{i=1,3}\left(\delta P_i\left(\dfrac{\partial P_2}{\partial P_i}ds + dq^i\right) + \delta q^i\left(\dfrac{\partial P_2}{\partial q^i}ds - dP_i\right)\right) \\ +\left(\delta t\left(dH + \dfrac{\partial P_2}{\partial t}ds\right) + \delta H\left(\dfrac{\partial P_2}{\partial H}ds - dt\right)\right) \end{array} \right) = 0 \qquad (113)$$

$$\frac{dq^i}{ds} = -\frac{\partial P_2}{\partial P_i};\frac{dt}{ds} = \frac{\partial P_2}{\partial H}ds; \qquad \frac{dP_i}{ds} = +\frac{\partial P_2}{\partial q_i^i};\frac{dH}{ds} = -\frac{\partial P_2}{\partial t}$$

Or explicitly:

$$x' = \frac{dx}{ds} = \frac{\partial h^*}{\partial P_1}; \quad \frac{dP_1}{ds} = -\frac{\partial h^*}{\partial x}; \qquad y' = \frac{dy}{ds} = \frac{\partial h^*}{\partial P_3}; \quad \frac{dP_3}{ds} = -\frac{\partial h^*}{\partial y}$$
$$t' = \frac{dt}{ds} = \frac{\partial h^*}{\partial P_t} \equiv -\frac{\partial h^*}{\partial H}; \quad \frac{dP_t}{ds} = -\frac{\partial h^*}{\partial t} \rightarrow \frac{dH}{ds} = \frac{\partial h^*}{\partial t} \tag{114}$$

$$h^* = -(1+Kx)\sqrt{\frac{(H-e\varphi)^2}{c^2} - m^2c^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2}$$
$$+\frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right) \tag{115}$$

Thus, by choosing one of coordinates as independent variable, the new Hamiltonian is nothing but its conjugate canonical momentum with a minus sign.

The same result can be obtained indirectly (the way frequently used in text books) by using equivalency of the Canonical pairs:

$$H = H(x^i, P_1, P_2, P_3) \overbrace{\rightarrow}^{solve} P_2 = P_2(x^i, P_1, P_2, H)$$

$$rename \quad P_t = -H; h^* = -P_2(x^i, P_1, P_2, H)$$

$$S = \int P_1\, dx + P_3\, dy + P_z\, dz - H\, dt \equiv$$

$$\int P_1\, dx - P_3\, dy - h^*\, dz + P_t\, dt$$

While this gives the same result, it has an appearance of a trick, not direct derivation. Hence, we did it from the least action principle.

Applying a canonical transformation that exchanges the coordinate with momentum and then employs a new coordinate (old momentum) as the independent variable it would turn the old coordinate into the new Hamiltonian. In all cases, the Hamiltonian is the function of the remaining canonical variables. This capability of the Hamiltonian systems is unique and one we can take advantage of. An important restriction is the monotonous behavior of independent variable. Otherwise, some or all of the derivatives can be infinite in the point where the independent variable stumbles (i.e., where the new time stops).

The equations (114) and (115) are the general form of the single-particle Hamiltonian equation in an accelerator. It undoubtedly is nonlinear (the square root signifies relativistic mechanics), and cannot be solved analytically in general. Only few specific cases allow such solutions.

Most General Form of the Accelerator Hamiltonian

$$h^* = -(1+Kx)\sqrt{\frac{(H-e\varphi)^2}{c^2} - m^2c^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2}$$

$$+\frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right)$$

$$x' = \frac{dx}{ds} = \frac{\partial h^*}{\partial P_1}; \quad \frac{dP_1}{ds} = -\frac{\partial h^*}{\partial x}; \qquad y' = \frac{dy}{ds} = \frac{\partial h^*}{\partial P_3}; \quad \frac{dP_3}{ds} = -\frac{\partial h^*}{\partial y}$$

$$t' = \frac{dt}{ds} = \frac{\partial h^*}{\partial P_t} \equiv -\frac{\partial h^*}{\partial H}; \quad \frac{dP_t}{ds} = -\frac{\partial h^*}{\partial t} \rightarrow \frac{dH}{ds} = \frac{\partial h^*}{\partial t}$$

We always have a choice of the reference orbit (e.g. K and ϰ) as well as of the gauge of 4-potential. We can use this flexibility for our benefit!

Next class – we will use a specific gauge to express components of 4-potentail as explicit functions of electric and magnetic fields

Relations between units: corrected typo from class 4

Simple things useful in accelerator physics
SGS <-> SI <-> eV/TeV

Typo is corrected !

- 1 meter = 100 cm; 1kg=10^3 g; 1J = 10^7 erg; seconds are universal

Speed of the light	2.9979 x 10^{10} cm/sec	~ 3 x 10^{10} cm/sec
Electron charge, *e*	4.803 x 10^{-10} ESU	1.602 x 10^{-19} C
EM field, Gs	1 Gs = 299.79 (~300) V/cm	1 T =10^4 Gs
Energy	1 eV =1.602 x 10^{-12} erg	=1.602 x 10^{-19} J
Energy/rigidity (*pc*)	*e* x 1 Gs cm = 299.79 eV	~ 0.3 keV
$E=\sqrt{\vec{p}^2c^2+\left(mc^2\right)^2}$	*e* x 1 T m = 299.79 MeV	~ 0.3 GeV

We will introduce more "handy" formulae/relations in the future

I found one useful unit in old British – modern USA system:

1' = One foot ~ 30 cm ~ c10^{-9} sec*

This how I remember what is one foot.