PHY 554
Lecture 8
Quadrupole field errors, chromaticity


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What you learned in last class:

Distributed dipole field errors & integer resonances

$$X_{co}(s) = \sqrt{\beta(s)} \sum_{k=-\infty}^{\infty} \frac{\nu^2 f_k}{\nu^2 - k^2} e^{jk\phi(s)}$$

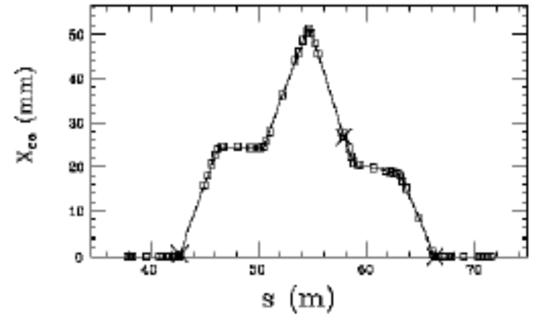
Where the field error is expanded in Fourier series $\left[\beta^{3/2}(\phi)\frac{\Delta B(\phi)}{B\rho}\right] = \sum_{k=-\infty}^{\infty} f_k e^{jk\phi}$

$$f_k = \frac{1}{2\pi}\oint\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}d\varphi = \frac{1}{2\pi\nu}\oint\left[\beta^{1/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}ds$$

$$\text{Sensitivity factor} \equiv \frac{\left\langle \left(X_{co}(s)\right)^2\right\rangle^{1/2}}{\theta_{rms}} \propto \sqrt{\beta(s)}$$

closed orbit bump: $X_{co}(s_f) = 0,\ X'_{co}(s_f) = 0$

$$\Delta x_{co}(s) = \left\{\sqrt{\beta_x(s_k)\beta_x(s)}\,\sin(\Delta\psi_x(s))\right\}\theta_k$$



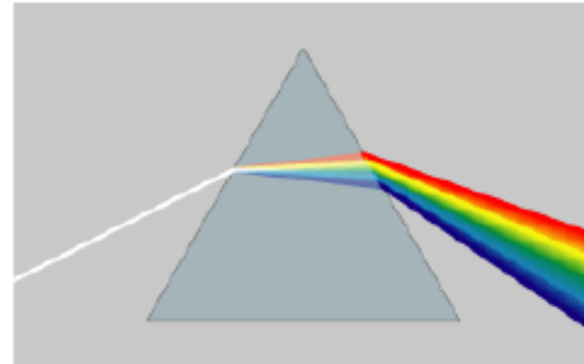
Orbit length change:

$$\Delta C = C - C_0 = \theta_0 \oint \frac{G_x(s,s_0)}{\rho} ds = D(s_0)\theta_0$$

$$\Delta C = \oint D(s_0)\frac{\Delta B_y(s_0)}{B\rho} ds_0$$

Off-momentum and dispersion

For different particle energy $\delta = \dfrac{p - p_0}{p_0}$



$$x = x_\beta + D\delta \qquad x' = x'_\beta + D'\delta$$

$$x''_\beta + K_x(s)x_\beta = 0, \qquad K_x(s) = \frac{1}{\rho^2} - K(s)$$

$$D'' + K_x(s)D = \frac{1}{\rho}$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1)\begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

Extend the matrix representation to 3 by 3

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

For a pure dipole (K=0):

$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \mathrm{L} & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

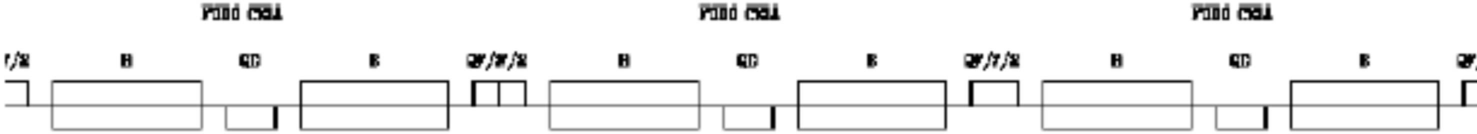
$$\theta << 1 \quad i.e. \quad L << \rho$$

For quadrupoles:

$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell & 0 \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Defocusing
change K -> -K

FODO cell



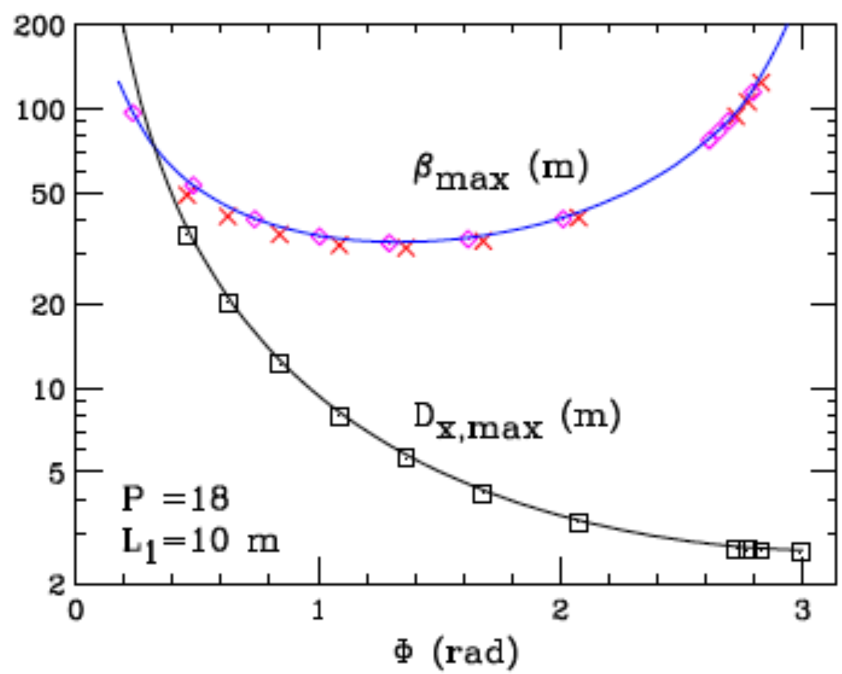
$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Closed orbit condition:

$$\begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\dfrac{L^2}{2f^2} & 2L(1+\dfrac{L}{2f}) & 2L\theta(1+\dfrac{L}{4f}) \\ -\dfrac{L}{2f^2}+\dfrac{L^2}{4f^3} & 1-\dfrac{L^2}{2f^2} & 2\theta(1-\dfrac{L}{4f}-\dfrac{L^2}{8f^2}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix}$$

$$D_F = \frac{L\theta(1+\frac{1}{2}\sin\frac{\Phi}{2})}{\sin^2\frac{\Phi}{2}}, \quad D'_F = 0$$

$$\beta_{\max} = \frac{2L_1(1+\frac{L_1}{2f})}{\sin\Phi} = \frac{2L_1(1+\sin\frac{\Phi}{2})}{\sin\Phi}$$



Connection between orbit distortions and dispersion function

- Equation for orbit distortions and dispersion function differ only by expression on the right-hand side
- Hence – they have the same analytical form of expression

$$x'' + K_x(s)x = \frac{e\delta B_y(s)}{pc} \Leftrightarrow D'' + K_x(s)D = K_0(s) \equiv \frac{1}{\rho(s)}$$

$$x(s) = \frac{w_x(s)}{\sin\frac{\mu_x}{2}} \oint w_x(s') \frac{e\delta B_y(s')}{pc} \cos\left(\frac{\mu_x}{2} - \left|\psi_x(s) - \psi_x(s')\right|\right);$$

$$y(s) = -\frac{w_y(s)}{\sin\frac{\mu_y}{2}} \oint w_y(s') \frac{e\delta B_x(s')}{pc} \cos\left(\frac{\mu_y}{2} - \left|\psi_y(s) - \psi_x(s')\right|\right);$$

$$w_{x,y}{}^2 \equiv \beta_{x,y};$$

$$\mu_{x,y} \equiv 2\pi\nu_{x,y};$$

$$D = \frac{w_x(s)}{\sin\frac{\mu_x}{2}} \oint w_x(s') K_0(s) \cos\left(\frac{\mu_x}{2} - \left|\psi_x(s) - \psi_x(s')\right|\right);$$

- Integer resonances - instable orbits: $\nu_{x,y} = \text{integer}$
- Note: $Q_{x,y}$ is frequently used in accelerator literature instead of $\nu_{x,y}$

Today we will focus on

- Effects of quadrupole field errors
- And related effects:
  - β-beat
  - Chromaticity (tuned dependence on momentum)
  - Parametric resonance
- Hill's equation for particle moving in modified focusing:

$$\boxed{\begin{gathered} x'' + K_o(s)x = 0 \Rightarrow x(s) = a\sqrt{\beta(s)}\cos(\psi(s)+\varphi); \\ x'' + \left(K_o(s) + k(s)\right)x = 0 \end{gathered}}$$

where change in focusing can be caused by quadrupole strength errors or a deviation of momentum from the ideal, or orbit deviation in nonlinear elements (sextopoles, quadrupoles, etc.)

Perturbation by a infinitesimally short quadrupole

$$\boxed{\delta(s-s')k(s')ds'}$$

Matrix of short quad

$$x \to x; x' \to x' - x\cdot\delta k(s); \delta k = k(s)ds;$$

$$M_\delta(s, s+ds) = \begin{bmatrix} 1 & 0 \\ -\delta k & 1 \end{bmatrix} + O(ds^2);$$

will modify one-turn matrix M_o

$$M = M_\delta \cdot M_o = \begin{bmatrix} 1 & 0 \\ -d\delta k & 1 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} - d\delta k\cdot m_{11} & m_{22} - d\delta k\cdot m_{12} \end{bmatrix};$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \equiv \begin{bmatrix} \cos\mu_o + \alpha\sin\mu_o & \beta\sin\mu_o \\ -\gamma\sin\mu_o & \cos\mu_o + \alpha\sin\mu_o \end{bmatrix};$$

$$\cos\mu \equiv \cos(\mu_o + d\delta\mu) = \frac{TraceM}{2} = \cos\mu_o - \frac{\delta k\cdot\beta\sin\mu_o}{2};$$

$$\cos(\mu_o + d\delta\mu) = \cos\mu_o\cos d\delta\mu - \sin\mu_o\sin d\delta\mu \cong \cos\mu_o - d\delta\mu\sin\mu_o$$

give us tune shift

$$d\delta\mu = \frac{\delta k\cdot\beta}{2} = \frac{\beta(s)k(s)ds}{2}; d\delta\nu = \frac{\beta(s)k(s)ds}{4\pi}$$

$$\boxed{k(s) \equiv \oint_C k(s')\delta(s-s')ds' \Rightarrow \delta\nu = \frac{\delta\mu}{2\pi} = \frac{1}{4\pi}\oint_C \beta(s)k(s)ds}$$

There is also associated changes in β-function

The β-function can be obtained by a one-turn map, i.e.

$$M_\delta(s_1) = \begin{bmatrix} 1 & 0 \\ -k(s_1)ds & 1 \end{bmatrix}; \beta_i \equiv \beta_o(s_i); \psi_i \equiv \psi_o(s_i) = \nu\phi_o(s_i);$$

$$M(s_2|s_2+C) = M_o(s_1|s_2+C)M_\delta(s_1)M_o(s_2|s_1);$$

HW6
Problem 1

$$\delta M_{12}(s_2|s_2+C) = -\beta_1\beta_2 k(s_1)ds \cdot \sin(\psi_1-\psi_2)\cdot\sin(\mu_o-\psi_1+\psi_2)$$

$$= \frac{1}{2}\beta_1\beta_2 k(s_1)ds\cdot\left[\cos\mu_o - \cos(\mu_o - 2(\psi_1-\psi_2))\right]$$

$$\delta M_{12}(s_2|s_2+C) \equiv \delta(\beta_2\sin\mu) = \delta\beta_2\sin\mu_o + \delta\mu\cdot\beta_2\cos\mu_o; \delta\mu = \frac{1}{2}\beta_1 k(s_1)ds;$$

$$\frac{\delta\beta_2}{\beta_2} = -\frac{\beta_1}{2\sin\mu_o}k(s_1)ds\cdot\cos(\mu_o+2(\psi_2-\psi_1))$$

β-beat occurs with double of the betatron phase advance
and for distributed errors is expressed as an integral

$$\boxed{k(s) \equiv \oint_C k(s')\delta(s-s')ds' \Rightarrow \frac{\delta\beta(s)}{\beta_o(s)} = -\frac{1}{2\sin\mu_o}\int_s^{s+C}\beta_o(z)k(z)dz\cdot\cos(\mu_o+2(\psi(s)-\psi_o(z)))}$$

β-beat and parametric resonances : ν=half integer

We can rewrite the expression for β-beat with clear indication of double betatron frequency oscillation of relative value of β-function:

HW6 Problem 2

$$f(s)=\frac{\delta\beta(s)}{\beta_o(s)}=-\frac{1}{2\sin\mu_o}\int_{\psi(s)}^{\psi(s)+\mu}\beta_o^{\,2}(z)k(z)\cdot\cos\left(\mu_o+2(\psi-\varphi)\right)d\varphi;d\varphi=\frac{ds}{\beta_o};$$

$$\frac{d^2}{d\psi^2}f(s)+4f(s)=-2\beta_o^{\,2}(s)k(s).$$

Parametric resonances or stop-bands

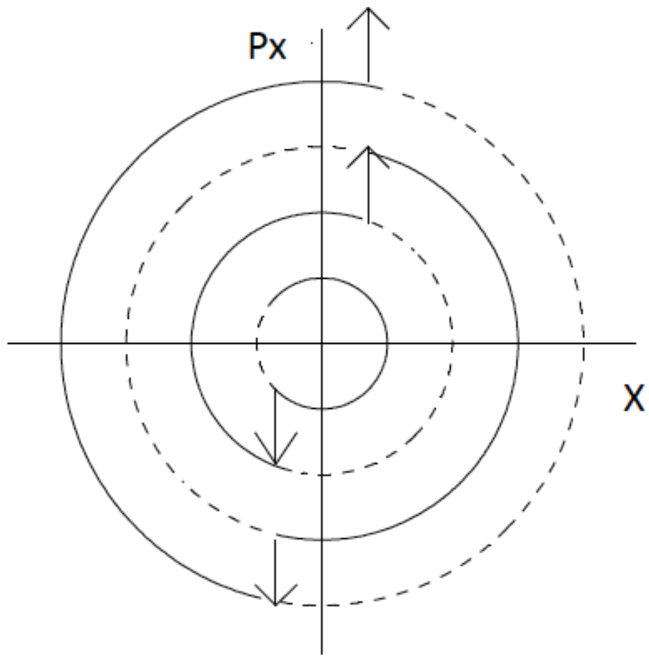
While it is obvious that β-function become infinite when tune is a half-integer and $\sin\mu_o=0$. Fourier expansion of the term under integral just makes it obvious with $2\nu_o\pm n$ appearing in the denominator

$$\beta_o^{\,2}(z)k(z)=\sum_{n=-\infty}^{\infty}A_n e^{2\pi in\frac{\psi(z)}{\mu_o}};A_n=\oint\beta_o(z)k(z)e^{2\pi in\frac{\psi(z)}{\mu_o}}ds;\Delta\psi(C)=\mu_o=2\pi\nu_o;$$

$$\frac{\Delta\beta(s)}{\beta(s)}=-2\nu_o\sum_{n=-\infty}^{\infty}\frac{A_n}{(2\nu_o)^2-n^2}e^{i\frac{n\psi(s)}{\nu}}=-2\nu_o\sum_{n=-\infty}^{\infty}\frac{A_n}{(2\nu_o-n)(2\nu_o+n)}e^{i\frac{n\psi(s)}{\nu}}$$

Parametric resonances : ν=half integer

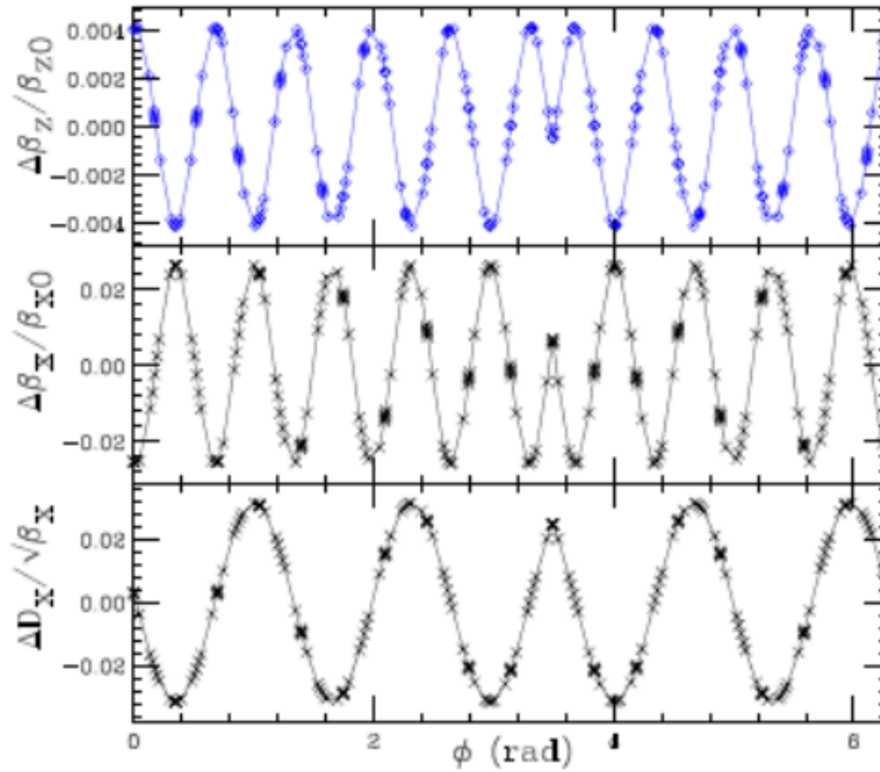
In fact there is are of unstable betatron motion around each half-integer tune resonance. It takes a bit more math to prove it, but this picture tell the story vary well that the amplitude of oscillation will grow exponentially at parametric resonance



Schematic plot of a particle trajectory at a half-integer betatron tune resulting from an error quadrupole kick $p_x = \beta_x \Delta X' = -\beta_x X/f$, where f is the focal length, X is the displacement from the quadrupole center, and β_x is the betatron amplitude function at the quadrupole. The quadrupole kick is proportional to the displacement X. At a half-integer betatron tune, the betatron coordinate changes sign in each consecutive revolution and the kick angles coherently add in each revolution to produce unstable particle motion.

Example of one quadrupole error in FODO cell lattice

Consider a simple accelerator lattice made of 18 FODO cells with half cell length 10-m, and dipole length 8 m bending angle 10^o. The betatron tunes are set at ν_x=4.79302 and ν_z=4.78298 by quadrupoles. Now, consider an 1% decrease in focusing quadrupole strength at the end of the 10th cell.



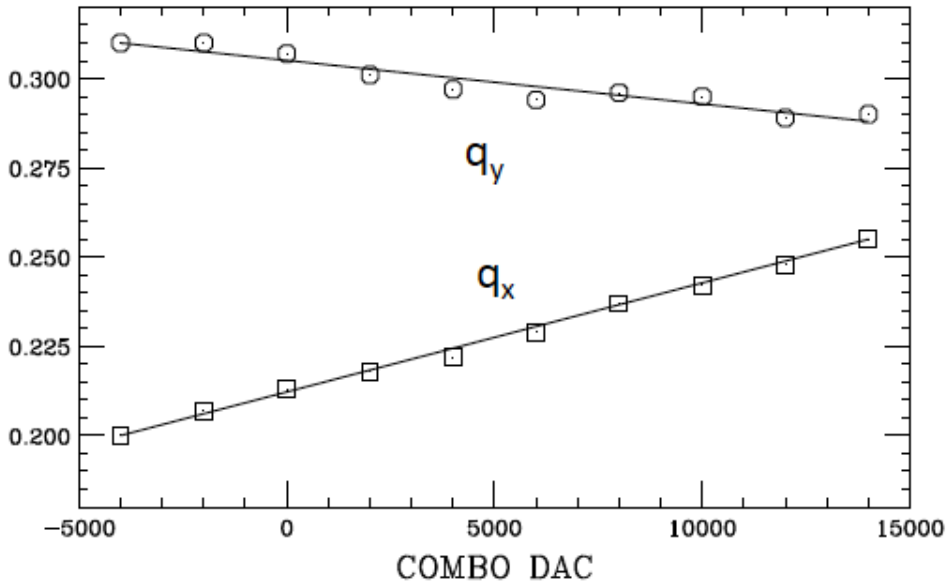
Perturbation of betatron amplitude functions vs ϕ (either ϕ_x or ϕ_y) resulting from 1% decrease in gradient strength of the 10th focusing quadrupole. The betatron amplitude function perturbation is dominated by harmonics nearest $[2\nu_x]$ and $[2\nu_y]$. Since β_x/β_y~6.37 at the focusing quadrupole location, the resulting error $\Delta\beta x/\beta x$ is about $6.37\Delta\beta y/\beta y$. A single kick at the error quadrupole location can be identified in the top 2 plots. The bottom plot shows the effect of quadrupole error on dispersion function shown as $\Delta D_x/\sqrt{\beta_x}$ vs $\phi=\phi_x$. A single kick at the error quadrupole location is visible to the dispersion closed orbit.

Applications of quadrupole error

$$\Delta\nu \approx \frac{1}{4\pi}\oint \beta_1 k(s_1)ds_1$$

1. Betatron amplitude function measurement

$$\left\langle \beta_{x,y} \right\rangle = 4\pi \frac{\Delta\nu_{x,y}}{\Delta Kl}$$



The horizontal and vertical tunes, determined by the FFT spectrum of the betatron oscillations, vs quadrupole field strength. The slope can be used to determine the **average** betatron amplitude function in a quadrupole.

The fractional parts of betatron tunes were $q_x=4-\nu_x$ and $q_y=5-\nu_y$. The experimental result of fractional horizontal tune appeared to "increase" with the strength of the quadrupole.

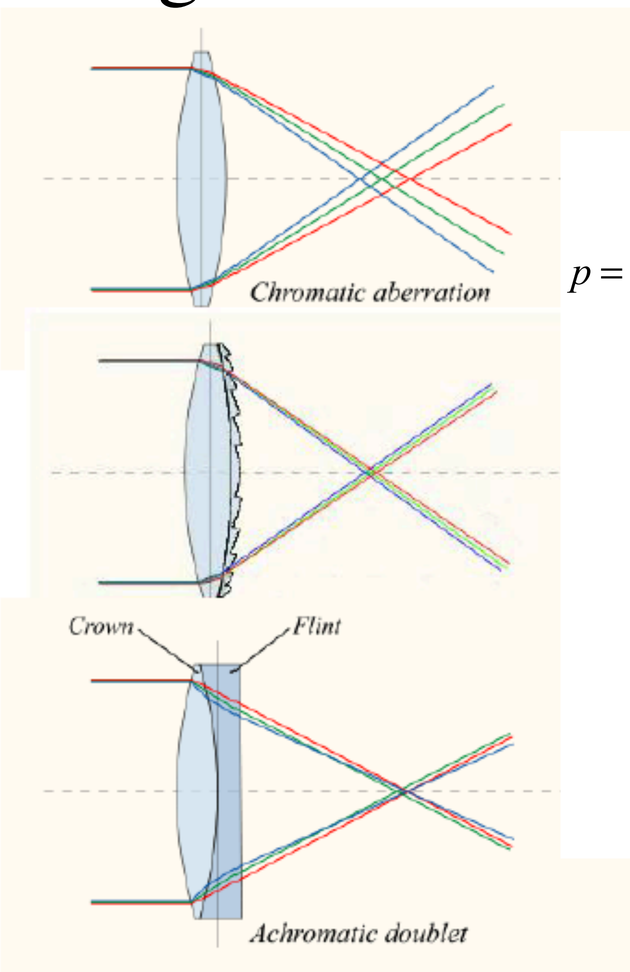
Q: Is the quadrupole focusing or defocusing? At this location, what can you say about the betatron amplitude functions?

2. Tune jump

$$\Delta\nu = \frac{1}{4\pi}\oint \beta_1 \frac{\Delta B_1}{B\rho} ds_1$$

Chromatism: betatron tune dependence on particle’s momentum

Origin of term



For off-momentum particle

$$p=p_o;\left\{\begin{array}{c} x''+K_x(s)x=0\\ y''+K_y(s)y=0\end{array}\right\};\left\{\begin{array}{c}K_x(s)=K_o^{\,2}(s)-K_y(s)\\ K_y(s)=-\dfrac{eG}{p_oc}\end{array}\right\};\left\{\begin{array}{c}G=\dfrac{\partial B_y}{\partial x}=\dfrac{\partial B_x}{\partial y}\\ \rho_o(s)=\dfrac{p_oc}{eB_{yo}(s)}\end{array}\right\};$$

$$p=p_o(1+\delta);D''+K_x(s)D=K_o(s)\equiv\frac{1}{\rho_o(s)}\Rightarrow x=D\delta+x_\beta+O(\delta^2);$$

$$Expand: x''-K_o(1+K_ox)=-\frac{e}{p_oc(1+\delta)}(1+K_ox)^2$$

$$\left\{\begin{array}{c}x_\beta''+\left(K_x(s)+\delta\cdot k_x(s)\right)x_\beta=O(\delta^2)\\ y''+\left(K_y(s)+\delta\cdot k_y(s)\right)y=O(\delta^2)\end{array}\right\};\left\{\begin{array}{c}k_x(s)=-2K_o^{\,2}(s)-k_y(s)\\ k_y(s)=-K_y(s)\end{array}\right\};$$

Definition of chromaticity

$$\boxed{\delta\nu_{x,y} = \frac{\delta\mu_{x,y}}{2\pi} = \frac{\delta}{4\pi}\oint_C \beta_{x,y}(s)k_{x,y}(s)\,ds \overset{def}{\Rightarrow} C_{x,y} \equiv \frac{d\nu_{x,y}}{d\delta} = \frac{1}{4\pi}\oint_C \beta_{x,y}(s)k_{x,y}(s)\,ds}$$

Strong focusing case

$$\left|\frac{1}{\rho_o^{\,2}}\left(1+K\cdot\frac{D}{\rho_o}\right)\right| << \left|K_{x,y}\right|$$

$$\left\{\begin{array}{c} x_\beta'' + \left(K_x(s)+\delta k_x(s)\right)x_\beta = O\left(\delta^2\right) \\ y'' + \left(K_y(s)+\delta k_y(s)\right)y = O\left(\delta^2\right) \end{array}\right\}; \left\{\begin{array}{c} \delta k_x(s) \approx -\delta\cdot K_x(s) \\ \delta k_y(s) = -\delta\cdot K_y(s) \end{array}\right\};$$

$$\boxed{\delta\nu_{x,y} = \frac{\delta\mu_{x,y}}{2\pi} = -\frac{\delta}{4\pi}\oint_C \beta_{x,y}(s)K_{x,y}(s)\,ds \overset{def}{\Rightarrow} C_{x,y} \equiv \frac{d\nu_{x,y}}{d\delta} = -\frac{1}{4\pi}\oint_C \beta_{x,y}(s)K_{x,y}(s)\,ds}$$

The chromaticity induced by focusing element of the ring is called natural chromaticity. It is obviously negative for weak focusing lattice. With β-functions having maxima where K is positive, it is negative in general. Even though, it is not a mathematically rigorous statement….

Specific chromaticity

$$\boxed{\overset{def}{\Rightarrow} \xi_{x,y} = \frac{C_{x,y}}{\nu_{x,y}}}$$

Simple FODO cell

$$\boxed{C_{x,y} \equiv -\frac{1}{4\pi}\oint_C \beta_{x,y}(s) K_{x,y}(s)\, ds \cong \frac{1}{4\pi}\sum_{lenses} \frac{\beta_{x,y}}{f} = -\frac{1}{4\pi}\left(\frac{\beta_{\max}}{f} - \frac{\beta_{in}}{f}\right)}$$

Using available expression for FODO cell we can estimate the specific chromaticity to be ~ 1

$$\sin\frac{\Phi}{2} = \frac{L_1}{2f} \qquad \beta_{\max} = \frac{2L_1(1+\sin(\Phi/2))}{\sin\Phi}, \quad \beta_{\min} = \frac{2L_1(1-\sin(\Phi/2))}{\sin\Phi}$$

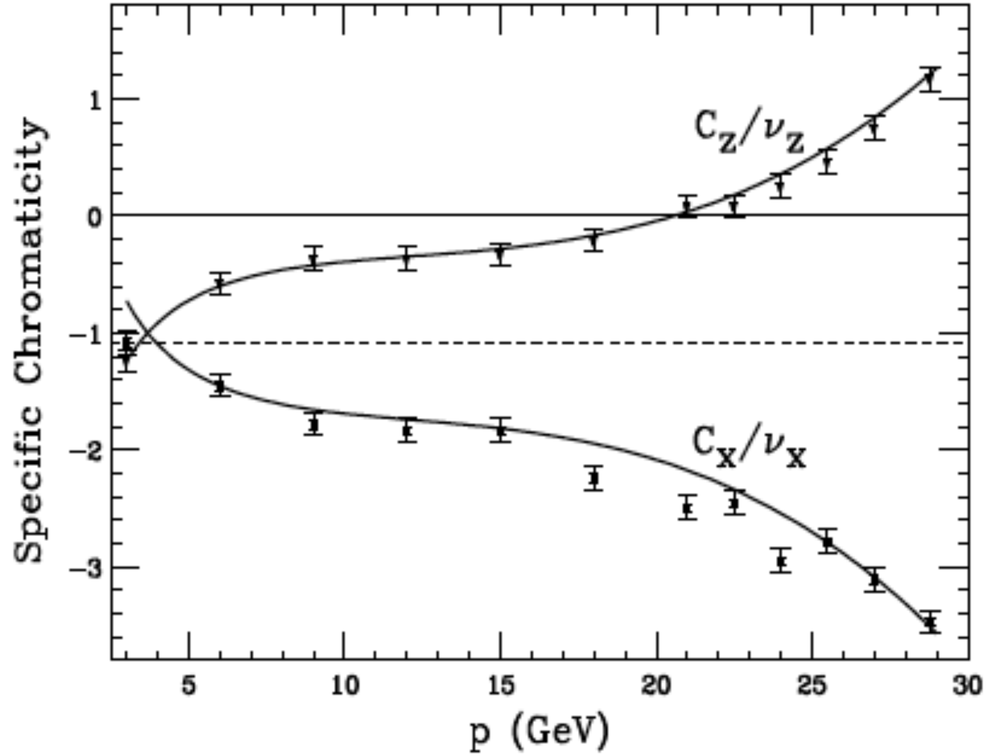
$$\boxed{C_{FODO\ x,y} \cong= -\frac{\tan\Delta\mu_{cel}}{\Delta\mu_{cel}} \nu_{x,y} \propto \nu_{x,y} \Rightarrow \xi_{FODO} \propto 1}$$

but for high luminosity colliders and high brightness light sources it can be significantly large than one- typically 2 to 4.

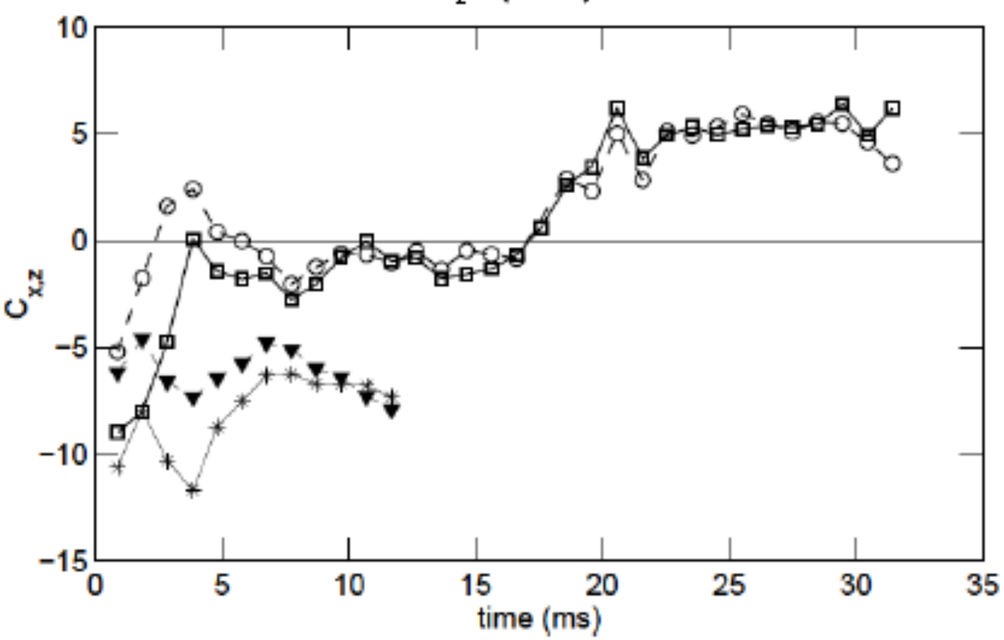
Examples:

BNL AGS (E. Blesser 1987):
Chromaticities measured at the AGS.

$$C_{X,\mathrm{nat}}^{\mathrm{FODO}} = -\frac{\tan(\Phi/2)}{\Phi/2}\nu_X \approx -\nu_X$$



Fermilab Booster (X. Huang, Ph.D. thesis, IU 2005): The measured horizontal chromaticity C_x when SEXTS is on (triangles) or off (stars), and the measured vertical chromaticity C_y when SEXTS is on (dash, circles) or off (squares). The error bar is estimated to be 0.5. The natural chromaticities are $C_{nat,y}$=−7.1 and $C_{nat,x}$=−9.2 for the entire cycle. The betatron tunes are 6.7(x) and 6.8(y) respectively.



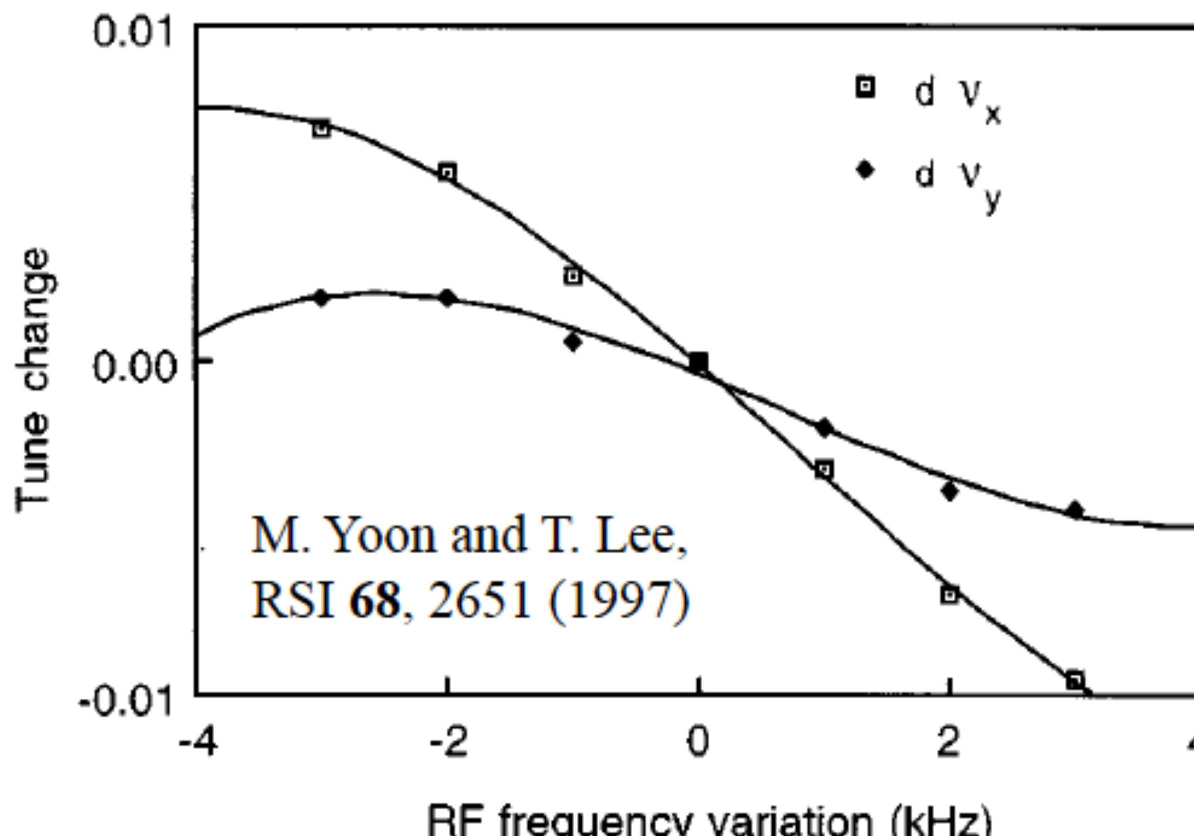
Chromaticity measurement:

The chromaticity can be measured by measuring the betatron tunes vs the rf frequency f, i.e.

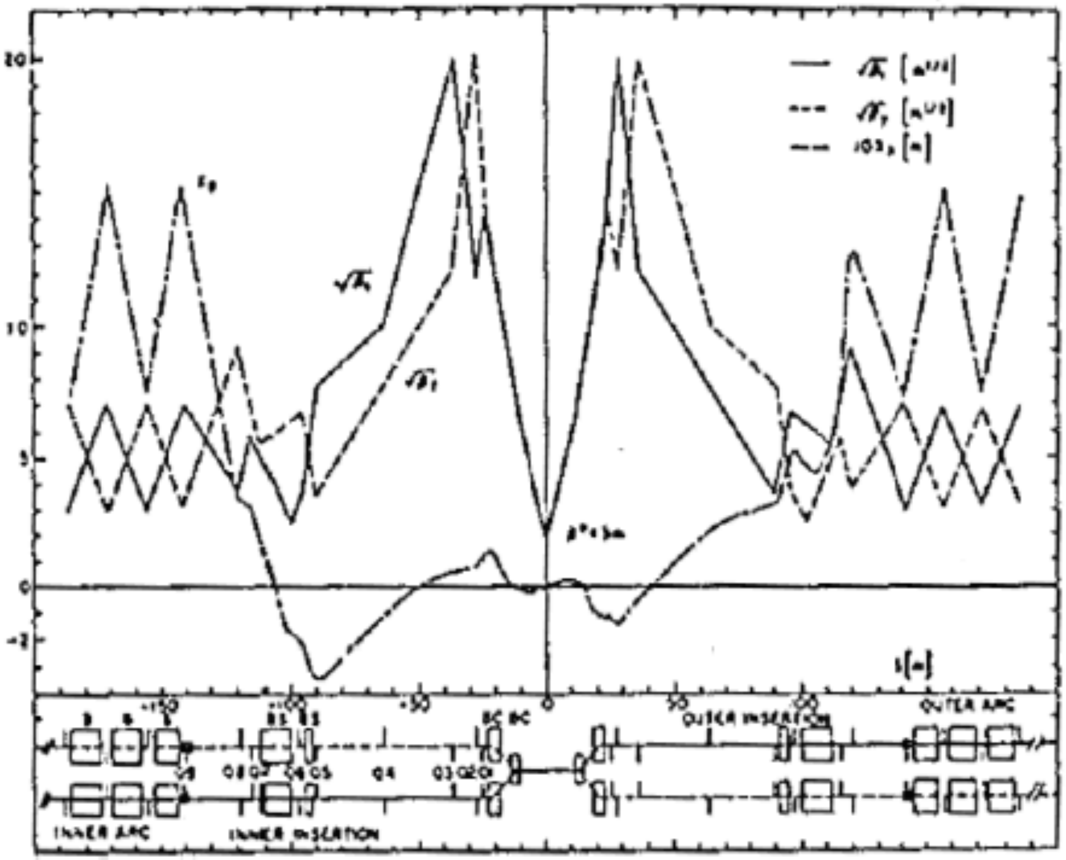
$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = (\alpha_c - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} = \eta\delta,$$

$$\Delta f/f_0 = -\eta\delta,$$

$$C = \frac{d\nu}{dp/p} = -\eta f_{rf}\frac{d\nu}{df_{rf}}$$



The chromaticites are C_x=+2.9, C_y=+1.4.

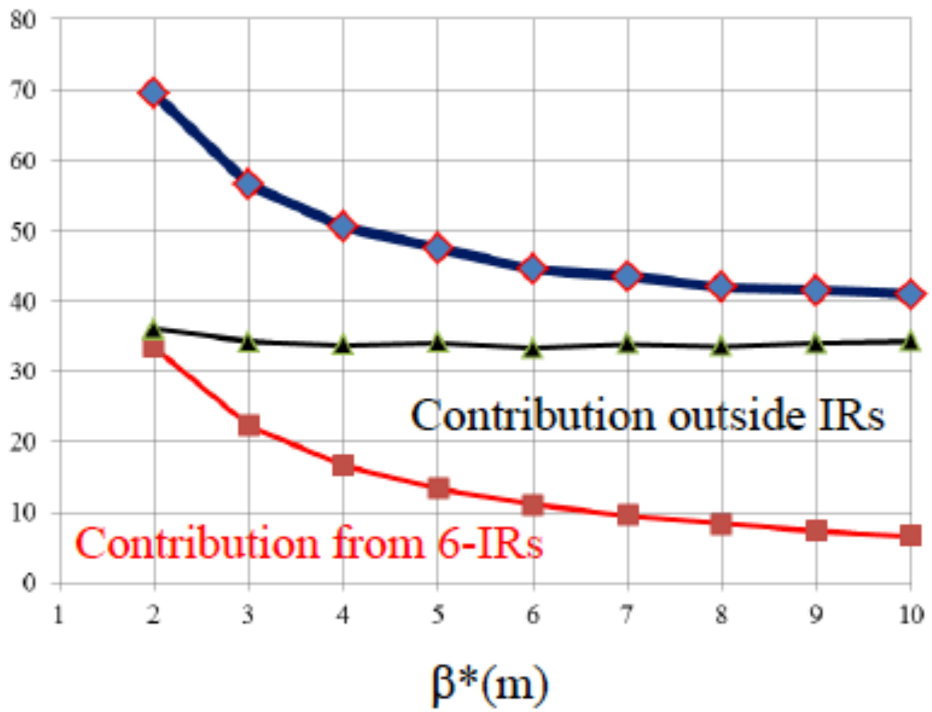


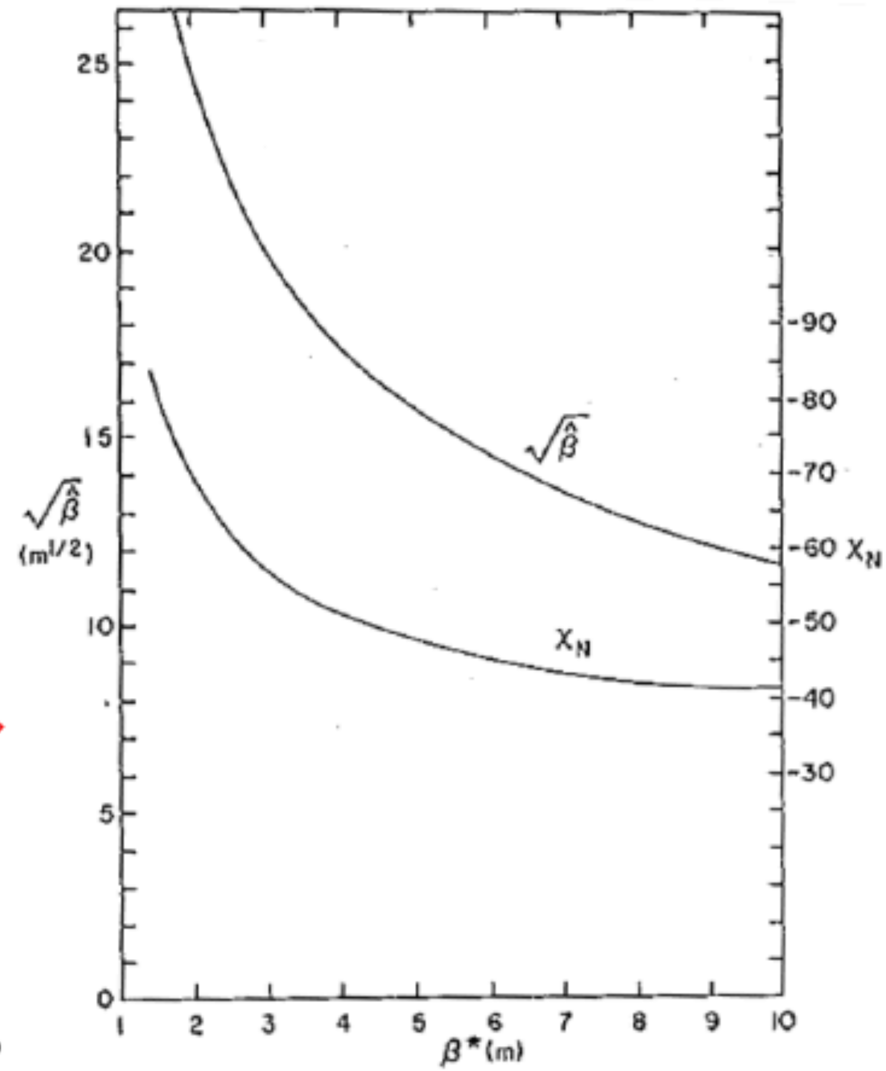
Contribution of low β triplets in an IR to the natural chromaticity is

$$C_{total} = N_{IR} C_{IR} + C_{bare\,machine}$$

$$\mathrm{C}_{\mathrm{IR}} = -\frac{2\Delta s}{4\pi\beta^*} \approx -\frac{1}{2\pi}\sqrt{\frac{\beta_{\max}}{\beta^*}}$$

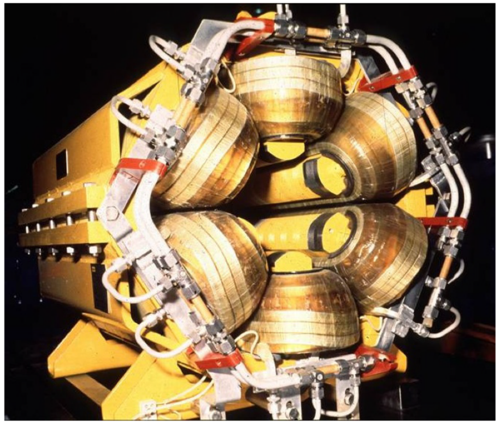
The total chromaticity is composed of contributions from the low β-quads and the rest of accelerators that is made of FODO cells. The decomposition to fit the data is **Δs≈35** m in RHIC.





Why do we care about chromaticity

- It was discovered early in operating storage rings that negative values of chromaticity cause violent collective "head-tail" transverse instability (to be exact – for ring operating above transition energy, which are normal for electron storage ring) – you will learn about it later in the course
- This instability occurs at very low beam current and has to be suppressed
- The only known way is to have slightly positive chromaticity for both vertical and horizontal planes – this is called chromaticity compensation
- It possible to do for strong focusing lattice using nonlinear element called sextupoles.
- *In your home work you are asked to prove that using sextupoles in weak focusing ring does not allow to compensate chromaticity*
- Sextupoles, as nonlinear elements, introduce nonlinear high order resonance – you will study them late in the course



$$B_y + iB_x = S(x+iy)^2;$$

$$B_y(s) = S\cdot\left(x^2 - y^2\right); B_x(s) = 2s\cdot xy$$

How it works? Particles with momentum deviation experience difference focusing:

$$B_y(s) = S\cdot\left(x^2 - y^2\right); \quad x = D(s)\delta + x_\beta \Rightarrow \delta G(s) = \left.\frac{\partial B_y}{\partial x}\right|_{x=D\delta} = 2\delta\cdot D(s)\cdot S(s)$$

$$k_{Sx}(s) = D(s)\cdot K_2(s); K_2(s) \equiv 2\frac{eS(s)}{pc}; k_{Sy}(s) = -k_{Sx}(s);$$

$$\Delta C_{Sx,y} \equiv \pm\frac{1}{4\pi}\oint_C D(s)\beta_{x,y}(s)K_2(s)\,ds$$

Alternating sign of sextupole field – positive where D β_x is large and defocusing where D β_y is large.
For strong focusing lattice we have a combination to bring to zero

$$C_x \equiv \frac{1}{4\pi}\oint_C \beta_x(s)\left\{D(s)K_2(s) - K_x(s)\right\}ds;$$

$$C_y \equiv -\frac{1}{4\pi}\oint_C \beta_y(s)\left\{D(s)K_2(s) + K_y(s)\right\}ds.$$

Summary

- We calculated (using perturbation approach) tune and β-function variation caused by errors (variation) of the focusing strength of quadrupoles – to be exact by variation of K(s) in Hill's equations
- Using this equations we found additional parametric resonances, where particles motion would be unstable
- We used the method to describe tunes variation of off-momentum particles and introduced chromaticity
- Finally, we discussed the way to compensate chromaticity using nonlinear elements called sextupoles
- We will return to discussing both chromatic effects as part of collective effect studies and sextupoles, as drivers of non-linear resonances