Homework 15. Due November 8

Problem 1. 30 points. Statistical definition of beam emittance.

We consider a statistical distribution of non-interacting particles in phase space (x, x'). Let $\rho(x, x')$ be the distribution function with

$$\int \rho(x, x')dxdx' = 1$$

the first and second moments of beam distribution are

$$\langle x \rangle = \int x\rho(x, x')dxdx' \qquad \langle x' \rangle = \int x'\rho(x, x')dxdx'$$

$$\langle x^2 \rangle = \int x^2\rho(x, x')dxdx' \qquad \langle x'^2 \rangle = \int x'^2\rho(x, x')dxdx'$$

$$\sigma_x{}^2 = \langle x^2 \rangle - \langle x \rangle^2 \qquad \sigma_{x'}{}^2 = \langle x'^2 \rangle - \langle x' \rangle^2$$

$$\sigma_{xx'}{}^2 = \langle xx' \rangle - \langle x \rangle\langle x' \rangle \stackrel{\text{def}}{=} r\sigma_x\sigma_{x'}$$

here σ_x and $\sigma_{x'}$ are rms beam widths, and r is the correlation coefficient. The rms emittance is therefore defined as

$$\varepsilon_{rms}{}^2 = \sigma_x{}^2\sigma_{x'}{}^2 - \sigma_{xx'}{}^2 = \sigma_x{}^2\sigma_{x'}{}^2(1 - r^2)$$

a) Assuming that particles are uniformly distributed in an ellipse

$$\frac{x^2}{a^2} + \frac{x'^2}{b^2} = 1$$

show that the total phase space area is $A = \pi ab = 4\pi\varepsilon_{rms}$

b) Show that the rms emittance defined above is invariant under a coordinate rotation

$$X = xcos\theta + x'sin\theta \qquad X' = -xsin\theta + x'cos\theta$$

and show that the correlation coefficient r is 0 if we choose the rotation angle to be

$$tan2\theta = \frac{2\sigma_x\sigma_{x'}r}{\sigma_x{}^2 - \sigma_{x'}{}^2}$$

show that σ_X and $\sigma_{X'}$ reach extrema at this rotation angle.

c) In accelerators, particles are distributed in Courant-Snyder ellipse

$$I(x, x') = \gamma x'^2 + 2\alpha xx' + \beta x^2$$

where $1 + \alpha^2 = \beta\gamma$. Apply the coordinate rotation you did above to this invariant to show that

$$\varepsilon_{rms} = \frac{\sigma_x{}^2}{\beta} = \frac{\sigma_{x'}{}^2}{\gamma}$$

and $$r = -\frac{\alpha}{\sqrt{\beta\gamma}}$$

or $$\begin{pmatrix} \sigma_x{}^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'}{}^2 \end{pmatrix} = \varepsilon_{rms}\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

show that $$\boldsymbol{x^T\sigma x} = \frac{\mathbf{1}}{\varepsilon_{rms}}(\gamma x'^2 + 2\alpha xx' + \beta x^2)$$

where $\boldsymbol{x} = \begin{pmatrix} x \\ x' \end{pmatrix}$ thus $\boldsymbol{x^T\sigma x}$ is invariant.

d) For a linear Hamiltonian, particle motion in accelerator obeys Hamiltonian dynamics

$$\frac{dx'}{ds} = -\frac{\partial H}{\partial x} = -Kx$$

where K(s) is focusing function. Show that the rms emittance is conserved (hint: write what is $\frac{d\varepsilon_{rms}{}^2}{ds}$ in terms of $\frac{dx'}{ds}$ and $\frac{\partial H}{\partial x}$)

solution:

a)

For a uniform distribution in an ellipse $x^2/a^2 + x'^2/b^2 = 1$, we have $\rho = 1/\pi ab$, $\langle x \rangle = 0$, $\langle x' \rangle = 0$, $\sigma_{xx'} = 0$, and

$$\sigma_x^2 = \int x^2 \rho dx' dx = \frac{4}{\pi ab} \int_0^a x^2 \cdot b \left(1 - \frac{x^2}{a^2}\right)^{1/2} dx = \frac{a^2}{4}.$$

Similarly, we find $\sigma_{x'}^2 = b^2/4$ and $\sigma_{xx'} = 0$. Thus $r = 0$ and $\epsilon_{\rm rms} = \pi \sigma_x \sigma_{x'} = \pi ab/4 = \mathcal{A}/4 = \epsilon_{\max}/4$, where $\mathcal{A} = \pi ab$ is the phase space area of the beam.

b)

Assuming $\langle x \rangle = \langle x' \rangle = 0$, we find $\langle X \rangle = \langle X' \rangle = 0$ after the coordinate rotation; and

$$\begin{aligned}
\sigma_X^2 &= \frac{1}{N}\sum X_i^2 = \langle (x\cos\theta + x'\sin\theta)^2 \rangle = \sigma_x^2 \cos^2\theta + \sigma_{x'}^2 \sin^2\theta + \sin 2\theta \sigma_{xx'} \\
\sigma_{X'}^2 &= \sigma_x^2 \sin^2\theta + \sigma_{x'}^2 \cos^2\theta - \sin 2\theta \sigma_{xx'} \\
\sigma_{XX'} &= -\sigma_x^2 \sin\theta\cos\theta + \sigma_{x'}^2 \sin\theta\cos\theta + \cos 2\theta \sigma_{xx'}
\end{aligned}$$

$$\epsilon_{\rm rms}(XX') = \sigma_X\sigma_{X'}\sqrt{1-R^2} = \sqrt{\sigma_X^2\sigma_{X'}^2 - \sigma_{XX'}^2} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2} = \epsilon_{\rm rms}(xx').$$

The statistical definition of the emittance is invariant under coordinate rotation.

When

$$\tan 2\theta = \frac{2\sigma_{xx'}}{\sigma_x^2-\sigma_{x'}^2} = \frac{2r\sigma_x\sigma_{x'}}{\sigma_x^2-\sigma_{x'}^2},$$

we have

$$\begin{aligned}
\sigma_{XX'} &= -\frac{1}{2}(\sigma_x^2-\sigma_{x'}^2)\sin 2\theta + \sigma_{xx'}\cos 2\theta = 0,\\
\sigma_X\frac{d\sigma_X}{d\theta} &= -\frac{1}{2}(\sigma_x^2-\sigma_{x'}^2)\sin 2\theta + \sigma_{xx'}\cos 2\theta = 0,\\
\sigma_{X'}\frac{d\sigma_{X'}}{d\theta} &= \frac{1}{2}(\sigma_x^2-\sigma_{x'}^2)\sin 2\theta - \sigma_{xx'}\cos 2\theta = 0,
\end{aligned}$$

i.e. σ_X and $\sigma_{X'}$ reach their extrema.

Under the rotation,

$$\begin{aligned}
I &= \gamma x^2 + 2\alpha x x' + \beta x'^2\\
&= (\gamma\cos^2\theta + \beta\sin^2\theta + \alpha\sin 2\theta)X^2 + (\gamma\sin^2\theta + \beta\cos^2\theta - \alpha\sin 2\theta)X'^2\\
&\quad +(1/2)(-\gamma\sin 2\theta + \beta\sin 2\theta + 2\alpha\cos 2\theta)XX'
\end{aligned}$$

To reach an upright position, we set $(\beta-\gamma)\sin 2\theta + 2\alpha\cos 2\theta = 0$. Thus we obtain

$$\tan 2\theta = -\frac{2\alpha}{\beta-\gamma} = \frac{2r\sigma_x\sigma_{x'}}{\sigma_x^2-\sigma_{x'}^2} \qquad \text{or} \qquad \frac{2r(\sigma_x/\sigma_{x'})}{(\sigma_x/\sigma_{x'})^2-1} = -\frac{2\alpha}{\beta-\gamma},$$

i.e.

$$\frac{\sigma_x^2}{\beta} = \frac{\sigma_{x'}^2}{\gamma} = \epsilon_{\rm rms}, \qquad r = -\frac{\alpha}{\sqrt{\beta\gamma}}.$$

Here we have used the definition $\epsilon_{\rm rms} = \sigma_x^2/\beta = \sigma_x\sigma_{x'}\sqrt{1-r^2}$. Thus we obtain

$$\sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'}^2 \end{pmatrix} = \epsilon_{\rm rms}\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix},$$

and

$$\mathbf{x}^\dagger\sigma^{-1}\mathbf{x} = \frac{1}{\epsilon_{\rm rms}}(\gamma x^2 + 2\alpha x x' + \beta x'^2).$$

d)

Using the definitions:

$$\begin{aligned}
\epsilon^2 &= \sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2\\
\sigma_x^2 &= \left\langle x^2\right\rangle - \langle x\rangle^2, \quad \sigma_{x'}^2 = \left\langle x'^2\right\rangle - \langle x'\rangle^2 \quad \sigma_{xx'} = \langle xx'\rangle - \langle x\rangle\langle x'\rangle,
\end{aligned}$$

we find

$$
\begin{aligned}
\frac{d\sigma_x^2}{ds} &= 2\langle xx'\rangle - 2\langle x\rangle\langle x'\rangle \\
\frac{d\sigma_{x'}^2}{ds} &= 2\langle x'x''\rangle - 2\langle x'\rangle\langle x''\rangle \\
\frac{d\sigma_{xx'}}{ds} &= \left\langle x'^2\right\rangle - \langle x'\rangle^2 - \langle x\rangle\langle x''\rangle + \langle xx''\rangle
\end{aligned}
$$

Using Hamilton's equation $x'' = -\frac{\partial H}{\partial x}$, we find

$$
\begin{aligned}
\frac{d\epsilon^2}{ds} &= \sigma_x^2\frac{d\sigma_{x'}^2}{ds} + \sigma_{x'}^2\frac{d\sigma_x^2}{ds} - 2\sigma_{xx'}\frac{d\sigma_{xx'}}{ds} \\
&= -2\sigma_x^2\left(\left\langle x'\frac{\partial H}{\partial x}\right\rangle - \langle x'\rangle\left\langle\frac{\partial H}{\partial x}\right\rangle\right) + 2\sigma_{xx'}\left(\left\langle x\frac{\partial H}{\partial x}\right\rangle - \langle x\rangle\left\langle\frac{\partial H}{\partial x}\right\rangle\right)
\end{aligned}
$$

If $\partial H/\partial x = Kx$, then $\frac{d\epsilon^2}{ds} = -2\sigma_x^2\sigma_{xx'} + 2\sigma_{xx'}\sigma_x^2 = 0$, i.e. ϵ is conserved. If the Hamiltonian is nonlinear, ϵ is not invariant.