Center for Accelerator Science and Education

PHY 564
Advanced Accelerator Physics
Lecture 10
Periodic systems and parameterization of linearized particle's motion

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Recap:

As we discussed in previous lectures, motion in a linear Hamiltonian system is fully described by transport matrix:

$$H = \frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n} h_{ij}(s)x_i x_j \equiv \frac{1}{2}X^T \cdot \mathbf{H}(s)\cdot X,\; X^T = \{q_1, P_1 ..., q_n, P_n\} \equiv \{x_1, x_2 ..., x_{2n-1}, x_{2n}\};$$
$$\frac{d}{ds}X = \mathbf{D}(s)\cdot X;\ \mathbf{D}(s) \equiv \mathbf{S}\cdot\mathbf{H}(s) \;\rightarrow X(s) = \mathbf{M}(s_o|s)X(s_o);\; \frac{d}{ds}\mathbf{M}(s_o|s) = \mathbf{D}(s)\cdot\mathbf{M}(s_o|s);\; \mathbf{M}(s_o|s_o) = \mathbf{I}. ; \quad \text{(r1)}$$

We proved that the matrix is symplectic

$$\mathbf{M}^T\cdot\mathbf{S}\cdot\mathbf{M} = \mathbf{M}\cdot\mathbf{S}\cdot\mathbf{M}^T = \mathbf{S} \quad \rightleftarrows \quad \mathbf{M}^{-1} = -\mathbf{S}\cdot\mathbf{M}^T\cdot\mathbf{S}; \quad \text{(r2)}$$

We also found the order of matrices multiplication: the first transport matrix is on the right and the last one is on the left:

$$X(s_1) = \mathbf{M}(s_o|s_1)X(s_o);\; X(s_2) = \mathbf{M}(s_1|s_2)X(s_1) = \mathbf{M}(s_1|s_2)\mathbf{M}(s_o|s_1)X(s_o);$$
$$\mathbf{M}(s_0|s_2) \equiv \mathbf{M}(s_1|s_2)\mathbf{M}(s_o|s_1) \rightarrow X(s_2) = \mathbf{M}(s_0|s_2)X(s_o). ; \quad \text{(r3)}$$

It the transport line consist of N element, the matrix of the line will be just an ordered product of its matrices:

$$\mathbf{M}_{tl} = \prod_{ordered\ n=1}^{N}\mathbf{M}_n \equiv \mathbf{M}_N\mathbf{M}_{N-1}....\mathbf{M}_2\mathbf{M}_1;; \quad \text{(r4)}$$

We found analytical expression for the transport matrix of an arbitrary *s*-dependent linear motion:

$$\frac{d}{ds}\mathbf{M}(s_o|s) = \mathbf{D}(s)\cdot\mathbf{M}(s_o|s) \rightarrow \mathbf{M}(s_o|s) = \lim_{\max|s_n - s_{n-1}| -> 0}\left(\prod_{n=1,\ ordered}^{N} e^{\mathbf{D}(s_n^*)(s_n - s_{n-1})}\right); \quad ; \qquad \text{(r5)}$$

$$\{s_o, s\} \equiv \{s_o, s_1, s_2 ....., s_{N-1}, s_N\};\ s_N = s;\ s_n^* \in \{s_{n-1}, s_n\}; \quad e^{\mathbf{A}} = \exp(\mathbf{A}) = \sum_{n=0}^{\infty}\frac{\mathbf{A}^n}{n!};$$

using matrix exponent. Finally, we found a way of expressing exponent of square $m \times m$ matrix as a polynomial matrix with power *m-1* containing *m* terms:

$$\exp(\mathbf{A}) = \sum_{n=0}^{m-1} c_n \mathbf{A}^n;\ \ \mathbf{A} = \mathbf{A}_{m\times m}; \qquad \text{(r6)}$$

and found the way of determining exact way of evaluating this expression using Sylester formulae and eigen values of the matrix. Naturally, for Hamiltonian system the matrix size is always even and equal twice the number of dimensions: m=2 for 1D and m=6 for 3D.

$$\exp[\mathbf{D}s] = \sum_{k=1}^{m}\left[e^{\lambda_k s}\prod_{i\neq k}\left\{\frac{\mathbf{D}-\lambda_i\mathbf{I}}{\lambda_k-\lambda_i}\sum_{j=0}^{n_k-1}\left(\frac{\mathbf{D}-\lambda_k\mathbf{I}}{\lambda_i-\lambda_k}\right)^j\right\}^{n_i}\sum_{p=0}^{n_k-1}\frac{s^p}{p!}(\mathbf{D}-\lambda_k\mathbf{I})^p\right] \qquad \text{(r6)}$$

Thus, we are fully equipped with the tools to calculate transport matrix of any accelerator structure. In practice, for any sizable accelerator this calculation is done by computer, which can evaluated (r5) to any given precision.

Here is worth noting that in a general case using truncated Taylor series for exponent expansion

$$e^{\mathbf{D}\Delta s} \approx \sum_{n=0}^{N} \frac{\mathbf{D}^n \Delta s^n}{n!};$$

or high order Runge-Kutta solution of differential equation (r1) will likely result in matrix which is not symplectic $\mathbf{M}\cdot\mathbf{S}\cdot\mathbf{M}^T \neq \mathbf{S}$ and result in non-conservation of the phase volume and other highly undesirable artificial effects. As we discussed before, the exception for use of truncated series is nilpotent matrices $\mathbf{D}^k = 0$ with all eigen values being zero, which is violated for a general case. Hence, from linear Hamiltonian system use symplectic matrices at each step.

One more note: We can write analytical expression for most of hard-edge element in accelerator. We can multiply matrices of various elements (either in analytical or digital form). Computers are multiplying digital matrices much faster and – unless the program is wrong – without errors and typos. Thus competing with computes in this dull process is indeed unnecessary.

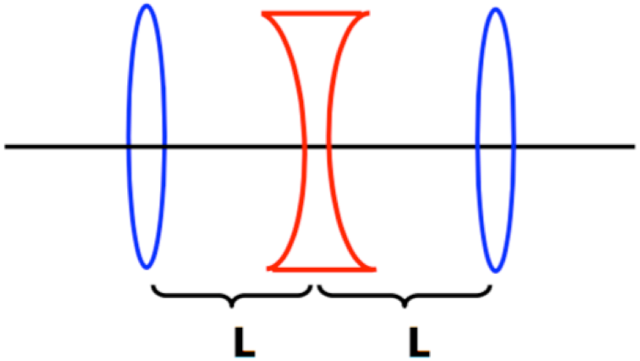
Analytical form of matrix is a different case – here we can gain some important insights. But try it – even 2x2 matrix of four-five-six-seven… elements becoming so convoluted and analytical extensions growing so fast into an unmanageable size.

$$\mathbf{M}_{tl} = \prod_{ordered\ n=1}^{N} \mathbf{M}_n \equiv \mathbf{M}_N \mathbf{M}_{N-1} \ldots . \mathbf{M}_2 \mathbf{M}_1;$$

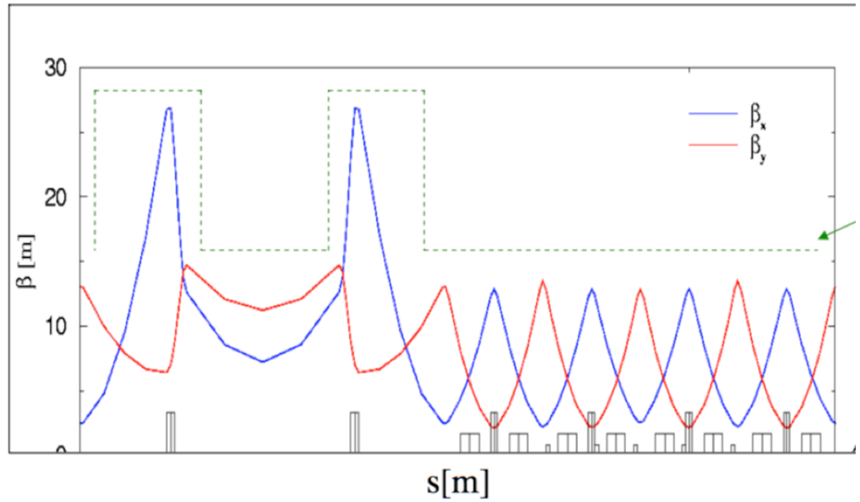
This is the reason that only relatively simple accelerator structures – frequency called cells – are fully evaluated analytically. Hence, with few notable exceptions, many of accelerator structures are built from repeatable cells. FODO, triplet, double-bend achromat and triple-bend achromat are among the most popular cells. A FODO cell comprised of two quadrupoles F and D, separated by drift spaces O. It is customary to call F quadrupole focusing in horizontal (x, radial) direction and, naturally, defocusing in vertical (y) direction. Vice versa, D is a defocusing quadrupole focuses in y direction and defocuses in x. FODO is a simple and still very popular cell. For example, eRHIC energy recovery linac (ERL) arcs will be comprised of many hundreds of FODO cells.

Periodic linear Hamiltonian systems are of special interest for accelerators. As we discussed before, it is a natural way of making big systems, such as an arc of accelerator or a transport line, from periodic cells with well-defined properties. Some accelerator beam-lines (e.g. a part of an accelerator) frequently comprising hundreds (or even thousands) of magnets. Physicist and engineers like using a relatively simple cell and repeat it multiple times. This allows one to study this cell in detail and then "match" the beam into the entire beamline. But since cells are repeated many times, stability of the particle's motion in a cell is important for particles staying confined around the beam axis.

Further more, one of most popular accelerator designs is a circular accelerator (called synchrotrons and storage rings) where particles going around for millions and billions of turns. At each path they go through the same sequence of the elements, e.g. they see periodic structure with period equal to accelerator circumference. Stability of the particle's motion is of a paramount importance for their proper operation.



(a)



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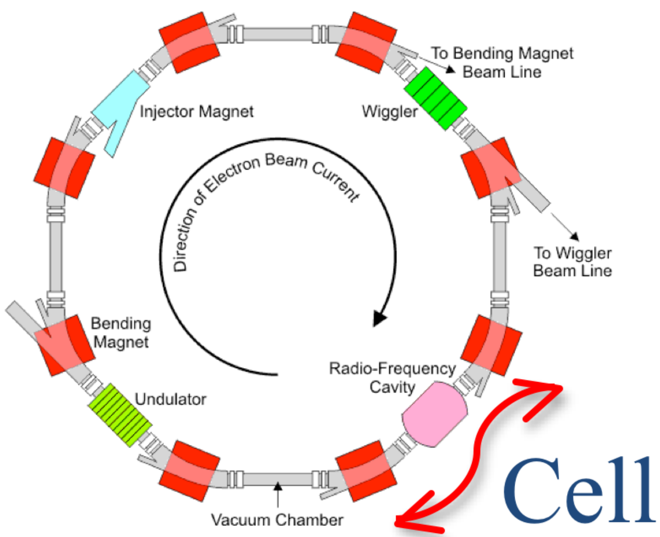

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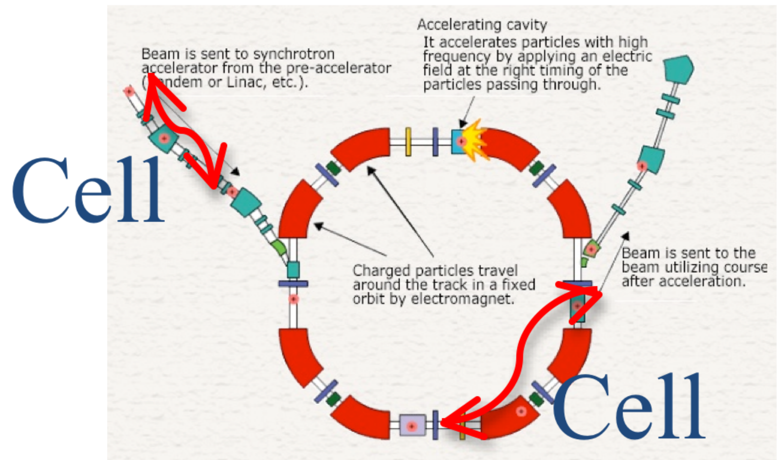
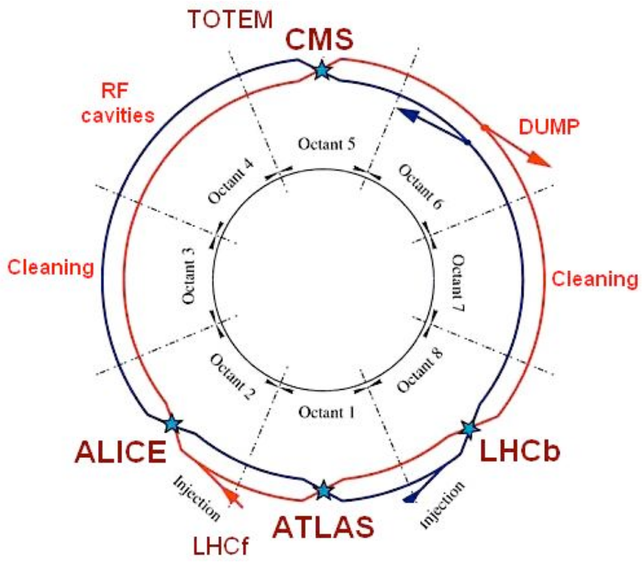
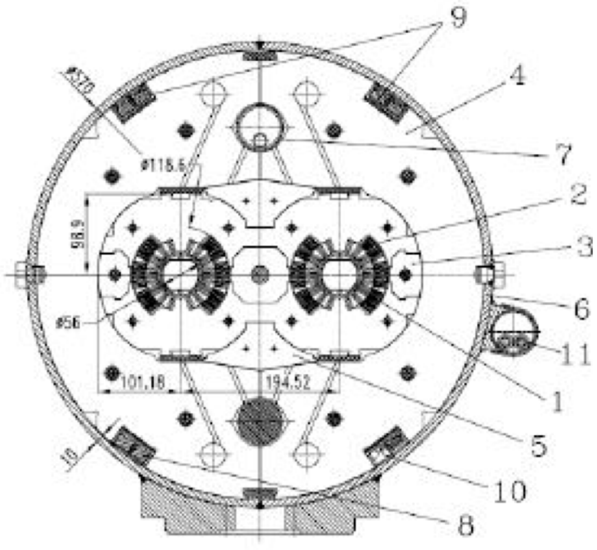
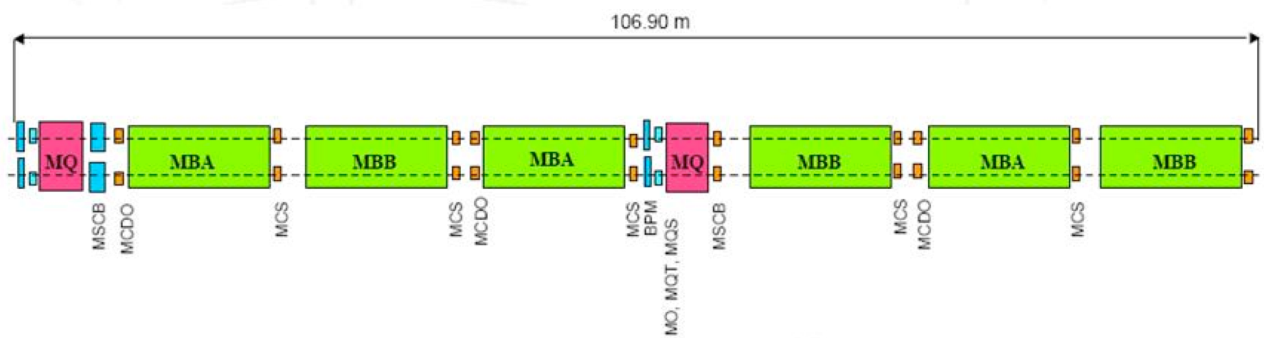


(d)

(a) FODO cell; (b) Periodic structure matched to an interaction region; (c) 8-cell storage ring; d(d) 4-cell double-bend achromat storage ring with an injection beamline using a two cell structure with opposite sign of the bending field.







The LHC is not a perfect circle. It is made of eight arcs and eight ‘insertions’. LHC consists of eight 2.45-km-long arcs, and eight 545-m-long straight sections. The arcs contain the dipole ‘bending’ magnets, with 154 in each arc. An insertion consists of a long straight section plus two (one at each end) transition regions — the so-called ‘dispersion suppressors’. The exact layout of the straight section depends on the specific use of the insertion: physics (beam collisions within an experiment), injection, beam dumping or beam cleaning. Each arc, with a regular lattice structure, contains 23 arc cells, and each arc cell has a FODO structure (main dipole magnets + quadrupole magnets + other multipoles magnets), 106.9 m long

https://www.lhc-closer.es/taking_a_closer_look_at_lhc/0.lhc_layout

Stability and Parameterization of motion in periodic systems

A Hamiltonian periodic system with period C, is described by periodic Hamiltonian: $H(X,s+C)=H(X,s)$. For linear Hamiltonian system is means that (elements of) matrix of the Hamiltonian is (are) a periodic function of s.:

$$H=\frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n}h_{ij}(s)x_i x_j \equiv \frac{1}{2}X^T\cdot\mathbf{H}(s)\cdot X,\ \mathbf{H}(s+C)=\mathbf{H}(s); \quad (1)$$

In this case, a one-turn (or one period) transport matrix

$$\mathbf{T}(s)=\mathbf{M}(s|s+C) \quad (2)$$

plays a very important role. Its eigen values, λ_i,

$$\det[\mathbf{T}-\lambda_i\cdot\mathbf{I}]=0 \quad (3)$$

determine if the motion is stable, e.g. that all $|\lambda_i|\le 1$ or is unstable, e.g. some $|\lambda_i|>1$. Before making specific statements about the stability, we look at the properties of the eigen vectors.

First, eigen values are a function of periodic system and do not depend on the azimuth, s. It is easy to show that a one-turn matrix is transformed by the transport matrix as

$$\mathbf{T}(s_1)=\mathbf{M}(s|s_1)\mathbf{T}(s)\mathbf{M}^{-1}(s|s_1) \quad (4)$$

$$\mathbf{T}(s_1)=\mathbf{M}(s_1|s_1+C)=\mathbf{M}(s+C|s_1+C)\mathbf{M}(s_1|s+C)=\mathbf{M}(s+C|s_1+C)\mathbf{M}(s|s+C)\mathbf{M}(s_1|s)$$

$$\mathbf{M}(s+C|s_1+C)\equiv\mathbf{M}(s|s_1);\ \mathbf{M}(s|s+C),\ \mathbf{M}(s_1|s)\equiv\mathbf{M}^{-1}(s|s_1)\Rightarrow\mathbf{T}(s_1)=\mathbf{M}(s|s_1)\mathbf{T}(s)\mathbf{M}^{-1}(s|s_1)\#$$

It means that $\mathbf{T}(s_1)$ has the same eigen values (3); thus, the eigen values of $\mathbf{T}(s)$ do not depend upon s because

$$\det\left[\mathbf{M}\mathbf{T}\mathbf{M}^{-1}-\lambda_i\cdot\mathbf{I}\right]=\det\left[\mathbf{M}(\mathbf{T}-\lambda_i\cdot\mathbf{I})\mathbf{M}^{-1}\right]=\det[\mathbf{T}-\lambda_i\cdot\mathbf{I}]$$
$$\Rightarrow\left[\mathbf{T}(s_1)-\lambda_i\cdot\mathbf{I}\right]=\left[\mathbf{T}(s)-\lambda_i\cdot\mathbf{I}\right]=0 \quad (5)$$

The matrix $\mathbf{T}$ is a real, complex conjugate of eigen value λ_i^* which is also eigen value of $\mathbf{T}$

$$[\mathbf{T}-\lambda_i \cdot \mathbf{I}]^* = [\mathbf{T}-\lambda^*_i \cdot \mathbf{I}] = 0$$

Furthermore, the symplecticity of $\mathbf{T}$ requires that λ_i^{-1} also is eigen value of $\mathbf{T}$. Proving that the inverse matrix $\mathbf{T}^{-1}$ has λ_i^{-1} as a eigen value is easy.

$$\mathbf{T}Y_i = \lambda_i Y_i; \;\; \mathbf{T}^{-1}\mathbf{T} = \mathbf{I} \rightarrow \left(\mathbf{T}^{-1}\mathbf{T}\right)Y_i = \mathbf{I}Y_i = Y_i$$

$$\mathbf{T}^{-1}\mathbf{T}Y_i = \lambda_i \mathbf{T}^{-1}Y_i = Y_i \rightarrow \mathbf{T}^{-1}Y_i = \lambda_i^{-1}Y_i$$

At the same time

$$0 = \det\left[\mathbf{T}^{-1} - \lambda_i^{-1}\mathbf{I}\right] = \det\left(\mathbf{S}\left[\mathbf{T}^T - \lambda_i^{-1}\mathbf{I}\right]\mathbf{S}^{-1}\right) = \det\left[\mathbf{T}^T - \lambda_i^{-1}\mathbf{I}\right] = \det\left[\mathbf{T} - \lambda_i^{-1}\mathbf{I}\right] \tag{5'}$$

and here, symplectic conditions help us again. Thus, the real symplectic matrix has n pairs of eigen values as follows: a) inverse $\{\lambda_i, \lambda_i^{-1}\}$, and b) complex conjugate $\{\lambda_i, \lambda_i^*\}$.

Let's assume that matrix T can be diagonalized. Therefore, repeating the matrix T again and again undoubtedly will cause an exponentially growing solution if $|\lambda_i| > 1$. The set of eigen vectors Y_i of matrix $\mathbf{T}$

$$\mathbf{T} \cdot Y_i = \lambda_i \cdot Y_i; \qquad i = 1,2....2n \tag{6}$$

is complete and an arbitrary vector X can be expanded about this basis:

$$X = \sum_{i=1}^{2n} a_i Y_i \equiv \mathbf{U} \cdot A, \quad \mathbf{U} = [Y_1 ......Y_{2n}], \qquad A^T = [a_1 ......a_{2n}]. \tag{7}$$

where we introduces matrix $\mathbf{U}$ built from eigen vector of the matrix $\mathbf{T}$:

$$\mathbf{T} \cdot \mathbf{U} = \mathbf{U} \cdot \Lambda, \;\; \Lambda = \begin{bmatrix} \lambda_1 & ... & 0 \\ ... & ... & .... \\ 0 & .... & \lambda_{2n} \end{bmatrix} \tag{8}$$

Diagonalization of the matrix **T** gives:

$$\mathbf{U}^{-1}\cdot\mathbf{T}\cdot\mathbf{U}=\Lambda,\ or\ \ \mathbf{T}=\mathbf{U}\cdot\Lambda\cdot\mathbf{U}^{-1} \tag{9}$$

Multiple application of matrix **T** (i.e., passes around the ring)

$$\mathbf{T}^n\cdot X=\sum_{i=1}^{2n}\lambda_i^{\ n}a_iY_i \tag{10}$$

exhibit exponentially growing terms if the module of even one eigen value is larger than 1, $\lambda_k=|\lambda|e^{i\mu},\ |\lambda|>1$; we easily observe that a solution with the initial condition $X_o=\operatorname{Re}a_kY_k$ grows exponentially:

$$\mathbf{T}^nX_o=|\lambda|^n\operatorname{Re}a_kY_ke^{in\mu}.$$

Immediately this suggests that the only possible stable system is when all eigen values are uni-modular

$$|\lambda_i|=1. \tag{11}$$

otherwise assuming $|\lambda_i|<1$ means that there is eigen value $\lambda_k=\lambda_i^{-1};\ |\lambda_k|=1/|\lambda_i|>1$.

In general case of multiplicity of eigen vectors, the matrix cannot be diagonalized but can be brought to Jordan normal form $\{Y_{k,1},\ldots,Y_{k,h}\}$ that belong to a eigen value λ_k with multiplicity h:

$$\mathbf{T}\cdot Y_{k,h}=\lambda_kY_{k,h};\ \mathbf{T}\cdot Y_{k,m}=\lambda_kY_{k,m}+Y_{k,m+1};\ m=1...h-1.$$

The result is even stronger than in the diagonal case: motion is unstable even when $|\lambda_k|=1$:

$$\mathbf{T}\cdot Y_{k,h-1}=\lambda_kY_{k,h-1}+Y_{k,h}\Rightarrow\mathbf{T}^n\cdot Y_{k,h-1}=Y_{k,h-1}+n\cdot Y_{k,h}$$

There is no good reason to study exotic case of unstable periodic system, unless you are interested in blowing up the beam size and loose particles. Hence. Let's focus on case of stable motion with 2N linearly independent eigen vectors. In other words, there are *n* pairs of eigen vectors, which determine modes of oscillations:

$$\lambda_k \equiv 1/\lambda_{k+n} \equiv \lambda^*{}_{k+n} \equiv e^{i\mu_k}; \;\; \mu_k \equiv 2\pi\nu_k, \;\; \{k=1,...n\}. \tag{12}$$

where the complex conjugate pairs are identical to the inverse pairs.

Eq. (9) can be rewritten as

$$\mathbf{T}(s) = \mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s); \; \Lambda = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_1^* & & 0 \\ & & ... & 0 \\ 0 & 0 & 0 & \lambda_n^* \end{pmatrix}; \;\; \mathbf{T}(s)\cdot\mathbf{U}(s) = \mathbf{U}(s)\cdot\Lambda \tag{13}$$

and matrix **U** built from complex conjugate eigen vectors of **T**:

$$\mathbf{U}(s) = \left[Y_1, Y_1^* ... Y_n, Y_n^*\right]; \qquad \mathbf{T}(s)Y_k(s) = \lambda_k Y_k(s) \quad \Leftrightarrow \quad \mathbf{T}(s)Y_k^*(s) = \lambda_k^* Y_k^*(s) \tag{14}$$

Thus, eigen vectors can be transported from one azimuth to another by the transport matrix:

$$\tilde{Y}_k(s_1) = \mathbf{M}(s|s_1)\tilde{Y}_k(s) \;\Leftrightarrow\; \frac{d}{ds}\tilde{Y}_k = \mathbf{D}(s)\cdot\tilde{Y}_k \tag{15}$$

It is eigen vector of $\mathbf{T}(s_1)$. - just add (4) to (14):

$$\mathbf{T}(s_1)\tilde{Y}_k(s_1) = \mathbf{M}(s|s_1)\mathbf{T}(s)\mathbf{M}^{-1}(s|s_1)\mathbf{M}(s|s_1)\tilde{Y}_k(s) = \mathbf{M}(s|s_1)\mathbf{T}(s)\tilde{Y}(s) = \lambda_k\mathbf{M}(s|s_1)\tilde{Y}(s) = \lambda_k\tilde{Y}_k(s_1)\#$$

Similarly,

$$\tilde{\mathbf{U}}(s_1) = \mathbf{M}(s|s_1)\tilde{\mathbf{U}}(s) \Leftrightarrow \frac{d}{ds}\tilde{\mathbf{U}} = \mathbf{D}(s)\cdot\tilde{\mathbf{U}} \tag{16}$$

with the obvious follow-up by

$$\tilde{\mathbf{U}}(s+C) = \tilde{\mathbf{U}}(s)\cdot\Lambda,\ \tilde{Y}_k(s+C) = \lambda_k\tilde{Y}_k(s) = e^{i\mu_k}\tilde{Y}_k(s) \tag{17}$$

The k^{th} eigen vectors are multiplied by $e^{i\mu_k}$ after each pass through the period. Hence, we can write

$$\tilde{Y}_k(s) = Y_k(s)e^{\psi_k(s)};\quad Y_k(s+C) = Y_k(s);\qquad \psi_k(s+C) = \psi_k(s) + \mu_k \tag{18}$$

$$\tilde{\mathbf{U}}(s) = \mathbf{U}(s)\cdot\Psi(s),\ \Psi(s) = \begin{pmatrix} e^{i\psi_1(s)} & 0 & & 0 \\ 0 & e^{-i\psi_1(s)} & & 0 \\ & & \dots & 0 \\ 0 & 0 & 0 & e^{-i\psi_n(s)} \end{pmatrix} \tag{19}$$

It is remarkable that the symplectic products (12) of the eigen vectors are non-zero only for complex conjugate pairs: in other words, the structure of the Hamiltonian metrics is preserved here. $Y_k^T\cdot\mathbf{S}\cdot Y_k^T \equiv 0$ is obvious. Using only the symplecticity of **T** gives us desirable yields

$$Y_k^T\cdot\mathbf{S}\cdot Y_j^T = Y_k^T\cdot\mathbf{T}^T\mathbf{S}\mathbf{T}\cdot Y_j^T = \lambda_k\lambda_j\left(Y_k^T\cdot\mathbf{S}\cdot Y_j^T\right) \Rightarrow (1-\lambda_k\lambda_j)\left(Y_k^T\cdot\mathbf{S}\cdot Y_j^T\right) = 0$$

for $\lambda_k\lambda_j \neq 1$

$$Y_k^{T*}\cdot\mathbf{S}\cdot Y_{j\neq k} = 0;\quad Y_k^T\cdot\mathbf{S}\cdot Y_j = 0;\ . \tag{20}$$

and only the nonzero products for $\lambda_k = 1/\lambda_j = \lambda^*{}_j$ are clearly pure imaginary*:

$$Y_k^{T*}\cdot\mathbf{S}\cdot Y_k = 2i, \tag{21}$$

where we chose the calibration of purely imaginary values as *2i* for the following expansion to be symplectic.

$$*\left(A^{*T}\cdot\mathbf{S}\cdot A\right)^* = \left(A^T\cdot\mathbf{S}\cdot A^*\right) = -\left(A^{*T}\cdot\mathbf{S}\cdot A\right)^T = -\left(A^{*T}\cdot\mathbf{S}\cdot A\right)$$

Eqs. (20-21) in compact matrix form is

$$\mathbf{U}^T \cdot \mathbf{S} \cdot \mathbf{U} \equiv \tilde{\mathbf{U}}^T \cdot \mathbf{S} \cdot \tilde{\mathbf{U}} = -2i\mathbf{S}, \;\; \mathbf{U}^{-1} = \frac{1}{2i}\mathbf{S} \cdot \mathbf{U}^T \cdot \mathbf{S}. \tag{22}$$

The expressions for the transport matrices through β, α-functions, and phase advances often derived as a "miraculous" result, and hence called matrix gymnastics, is just a trivial consequence of equations (16), (19), and (22):

$$\mathbf{M}(s|s_1) = \tilde{\mathbf{U}}(s_1)\tilde{\mathbf{U}}^{-1}(s) = \frac{1}{2i}\tilde{\mathbf{U}}(s_1)\cdot\mathbf{S}\cdot\tilde{\mathbf{U}}^T(s)\cdot\mathbf{S} = \frac{1}{2i}\mathbf{U}(s_1)\cdot\Psi(s_1)\cdot\mathbf{S}\cdot\Psi^{-1}(s)\cdot\mathbf{U}^T(s_1) \tag{16'}$$

with a specific case of a one-turn matrix:

$$\mathbf{T} = \mathbf{U}\Lambda\mathbf{U}^{-1} = \frac{1}{2i}\mathbf{U}\Lambda\mathbf{S}\mathbf{U}^T\mathbf{S} \tag{13'}$$

S-orthogonality (20) provides an excellent tool of finding complex coefficients in the expansion eq. (7) of an arbitrary solution X(s)

$$X_o = \sum_{i=1}^{2n} a_i Y_i \Rightarrow X(s) = \frac{1}{2}\sum_{k=1}^{n}\left(a_k\tilde{Y}_k + a_k^*\tilde{Y}_k^*\right) \equiv \mathrm{Re}\sum_{k=1}^{n} a_k Y_k e^{i\psi_k} \equiv \frac{1}{2}\tilde{\mathbf{U}}\cdot A = \frac{1}{2}\mathbf{U}\cdot\Psi\cdot A = \frac{1}{2}\mathbf{U}\cdot\tilde{A} \tag{23}$$

where *2n* complex coefficients, which are constants of motion*! for linear Hamiltonian system, can be found by a simple multiplications (instead of solving a system of 2n linear equations (7))

$$\begin{gathered} a_i = \frac{1}{2i}Y_i^{*T}SX; \; \tilde{a}_i \equiv a_i e^{i\psi_i} = \frac{1}{2i}Y_i^{*T}SX; \\ A = 2\tilde{\mathbf{U}}^{-1}\cdot X = -i\Psi^{-1}\cdot\mathbf{S}\cdot\mathbf{U}^{T*}\cdot\mathbf{S}\cdot X; \; \tilde{A} = \Psi A = -i\cdot\mathbf{S}\cdot\mathbf{U}^{T*}\cdot\mathbf{S}\cdot X. \end{gathered} \tag{24}$$

Equation (23) is nothing else but a general parameterization of motion in the linear Hamiltonian system. It is very powerful tool and we will use this many times in this course.

* in matrix form using (16) we have $X = \frac{1}{2}\tilde{\mathbf{U}}A, X' = \frac{1}{2}\left(\tilde{\mathbf{U}}'A + \tilde{\mathbf{U}}A'\right), = \mathbf{D}X = \frac{1}{2}\mathbf{D}\tilde{\mathbf{U}}\cdot A = \frac{1}{2}\tilde{\mathbf{U}}'\cdot A \Rightarrow A' = 0$

We consider next a specific case of a 1D system with a linear periodical Hamiltonian:

$$\tilde{h} = \frac{p^2}{2} + K_1(s)\frac{y^2}{2};\ \mathbf{H} = \begin{bmatrix} K_1 & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{D} = \mathbf{SH} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix}. \qquad (25)$$

The equations of motion are simple

$$\frac{d}{ds}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} p \\ -K_1 x \end{bmatrix} \ (i.e.\ x' \equiv p). \qquad (26)$$

A one-turn matrix within its determinant (*ad-bc=1*)

$$\mathbf{T}(s) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s);\ \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} = \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{pmatrix}; \qquad (27)$$

$$Y = \begin{bmatrix} \mathrm{w} \\ u + i/\mathrm{w} \end{bmatrix}; \tilde{Y} = \begin{bmatrix} \mathrm{w} \\ u + i/\mathrm{w} \end{bmatrix} e^{i\psi}; \mathbf{U} = \begin{bmatrix} \mathrm{w} & \mathrm{w} \\ u + i/\mathrm{w} & u - i/\mathrm{w} \end{bmatrix}; \tilde{\mathbf{U}} = \mathbf{U} \cdot \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \qquad (28)$$

where w(s)* and u(s) are real functions and calibration was used for (21). **T** has a trace

$$Trace(\mathbf{T}) = Trace(\Lambda) = 2\cos\mu \qquad (29)$$

(because $Trace(ABA^{-1}) = Trace(B)$). Thus, the stability of motion (when μ is real!) is easy to check:

$$-2 < Trace(\mathbf{T}) < 2 \qquad (30)$$

where some well-know resonances are excluded: The integer $\mu = 2\pi m$, and the half-integer $\mu = 2(m+1)\pi$ as being unstable (troublesome!).

*We are free to multiply the eigen vector **Y** by $e^{i\phi}$ to make **a** real number. In other words we define the choice of our phase as $\tilde{Y}(s) = \begin{pmatrix} \tilde{y}_1(s) \\ \tilde{y}_2(s) \end{pmatrix}; \mathrm{w}(s) = |\tilde{y}_1(s)|; \psi(s) = \arg(\tilde{y}(s))$.

Combining (28) into the equations of motion (25)

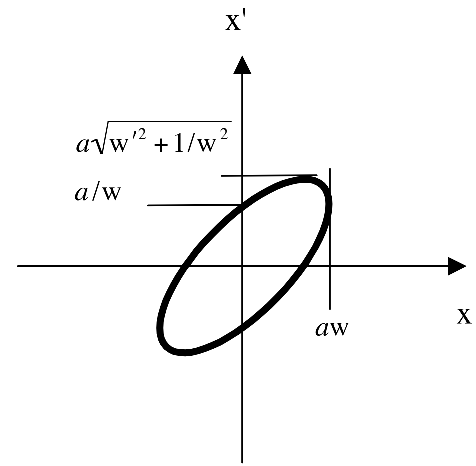
$$\frac{d}{ds}\begin{bmatrix} \mathrm{w} \\ u+i/\mathrm{w} \end{bmatrix} e^{i\psi} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix} \cdot \tilde{Y} = \begin{bmatrix} \mathrm{w} \\ u+i/\mathrm{w} \end{bmatrix} e^{i\psi} \Rightarrow \begin{matrix} \mathrm{w}' + i\mathrm{w}\psi' = u + i/\mathrm{w} \\ u' - i\mathrm{w}'/\mathrm{w}^2 + i\psi'(u+i/\mathrm{w}) = -K_1 \mathrm{w} \end{matrix}. \quad (31)$$

Then, separating the real and imaginary parts, we have from the first equation:

$$u = \mathrm{w}'; \quad \psi' = 1/\mathrm{w}^2. \quad (32)$$

Plugging these into the second equation yields one nontrivial equation on the envelope function, w(s):

$$\mathrm{w}'' + K_1(\mathrm{s})\mathrm{w} = \frac{1}{\mathrm{w}^3}. \quad (33)$$



Thus, the final form of the eigen vector can be rewritten as

$$Y = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix}; \psi' = \frac{1}{\mathrm{w}^2}; \tilde{Y} = Ye^{i\psi} \quad (34)$$

The parameterization of the linear 1D motion is

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \mathrm{Re}\left(ae^{i\varphi} \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix} e^{i\psi} \right);$$

$$\begin{matrix} x = a \cdot \mathrm{w}(\mathrm{s}) \cdot \cos(\psi(s) + \varphi) \\ x' = a \cdot (\mathrm{w}'(\mathrm{s}) \cdot \cos(\psi(s)+\varphi) - sin(\psi(s)+\varphi)/\mathrm{w}(\mathrm{s})) \end{matrix} \quad (35)$$

where ***a*** and ***φ*** are the constants of motion.

Tradition in accelerator physics calls for using the so-called β-function, which simply a square of the envelope function:

$$\beta \equiv \mathrm{w}^2 \;\Rightarrow\; \psi' = 1/\beta. \tag{36}$$

and a wavelength of oscillations divided by 2π. Subservient functions are defined as

$$\alpha \equiv -\beta'/2 \equiv -\mathrm{w}\,\mathrm{w}',\ \gamma \equiv \frac{1+\alpha^2}{\beta}. \tag{37}$$

While α,β,γ are frequently used in accelerator physics, unless they are equiped with indicies $\alpha_{x,y},\beta_{x,y},\gamma_{x,y}$, they can be easily mistaken with relativistic factors β and γ. Beware of this possibility and see in what contest α,β,γ are used.

Manipulations with them is much less transparent, and oscillation (35) looks like

$$\begin{aligned} x &= a\cdot\sqrt{\beta(\mathrm{s})}\cdot\cos\left(\psi(s)+\varphi\right) \\ x' &= -\frac{a}{\sqrt{\beta(\mathrm{s})}}\cdot\left(\alpha(\mathrm{s})\cdot\cos\left(\psi(s)+\varphi\right)+\sin\left(\psi(s)+\varphi\right)\right) \end{aligned} \tag{38}$$

Finally, (13') gives us a well-known feature in AP parameterization of a one-turn matrix:

$$\mathbf{T} = \mathbf{U}\Lambda\mathbf{U}^{-1} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;\quad \mathbf{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}; \mathbf{J}^2 = -\mathbf{I} \tag{39}$$

Summary of 1D treatment

1D - ACCELERATOR

$$\tilde{h}=\frac{p^2}{2}+K_1(s)\frac{y^2}{2};\ \mathbf{H}=\begin{vmatrix}K_1 & 0\\ 0 & 1\end{vmatrix};\mathbf{D}=\mathbf{SH}=\begin{bmatrix}0 & 1\\ -K_1 & 0\end{bmatrix};\ \frac{d}{ds}\begin{bmatrix}x\\ p\end{bmatrix}=\begin{bmatrix}0 & 1\\ -K_1 & 0\end{bmatrix}\cdot\begin{bmatrix}x\\ p\end{bmatrix}=\begin{bmatrix}p\\ -K_1 x\end{bmatrix}\ (i.e.\ x'\equiv p).$$

$$\mathbf{T}(s)=\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s);\ \Lambda=\begin{pmatrix}\lambda & 0\\ 0 & 1/\lambda\end{pmatrix}=\begin{pmatrix}e^{i\mu} & 0\\ 0 & e^{-i\mu}\end{pmatrix};$$

$$\mathbf{T}=\mathbf{I}\cos\mu+\mathbf{J}\sin\mu;$$

$$\mathbf{J}=\begin{bmatrix}-ww' & w^2\\ -w'^2-\frac{1}{w^2} & ww'\end{bmatrix}=\begin{bmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{bmatrix};\quad \mathbf{J}^2=-\mathbf{I}$$

HENCE:

$$\mathbf{T}=\mathbf{I}\cos\mu+\mathbf{J}\sin\mu;$$

$$\mathbf{J}=\begin{bmatrix}-ww' & w^2\\ -w'^2-\frac{1}{w^2} & ww'\end{bmatrix}=\begin{bmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{bmatrix};\quad \mathbf{J}^2=-\mathbf{I};\ \gamma=\left(1+\alpha^2\right)/\beta$$

$$\cos\mu=Trace(\mathbf{T})/2=\frac{T_{11}+T_{22}}{2}$$

$$Stability\ if:-1<Trace(\mathbf{T})/2<1$$

$$w^2\equiv\beta=\frac{T_{12}}{\sin\mu}=\frac{|T_{12}|}{\sqrt{1-(Trace(\mathbf{T})/2)^2}};w=\sqrt{\frac{|T_{12}|}{\sqrt{1-(Trace(\mathbf{T})/2)^2}}};$$

$$ww'\equiv-\alpha=\frac{T_{22}-T_{11}}{2\cos\mu}=-\frac{T_{11}-T_{22}}{T_{11}+T_{22}}$$

1D - ACCELERATOR

$$Y=\begin{bmatrix}\mathrm{w}\\ \mathrm{w}'+i/\mathrm{w}\end{bmatrix};\tilde{Y}=\begin{bmatrix}\mathrm{w}\\ \mathrm{w}'+i/\mathrm{w}\end{bmatrix}e^{i\psi};\mathbf{U}=\begin{bmatrix}\mathrm{w} & \mathrm{w}\\ \mathrm{w}'+i/\mathrm{w} & \mathrm{w}'-i/\mathrm{w}\end{bmatrix};\tilde{\mathbf{U}}=\mathbf{U}\cdot\begin{pmatrix}e^{i\psi} & 0\\ 0 & e^{-i\psi}\end{pmatrix}$$

$$\mathrm{w}''+K_1(s)\mathrm{w}=\frac{1}{\mathrm{w}^3},\quad \psi'=1/\mathrm{w}^2;\begin{bmatrix}x\\ x'\end{bmatrix}=\mathrm{Re}\left(ae^{i\varphi}\begin{bmatrix}\mathrm{w}\\ \mathrm{w}'+i/\mathrm{w}\end{bmatrix}e^{i\psi}\right);$$

$$x=a\cdot\mathrm{w}(s)\cdot\cos(\psi(s)+\varphi)$$

$$x'=a\cdot\left(\mathrm{w}'(s)\cdot\cos(\psi(s)+\varphi)-\sin(\psi(s)+\varphi)/\mathrm{w}(s)\right)$$

$$\beta\equiv\mathrm{w}^2\ \Rightarrow\ \psi'=1/\beta;\alpha\equiv-\beta'\equiv-\mathrm{w}\,\mathrm{w}',\ \gamma\equiv\frac{1+\alpha^2}{\beta}\ \text{- definitions}$$

$$x=a\cdot\sqrt{\beta(s)}\cdot\cos(\psi(s)+\varphi)$$

$$x'=-\frac{a}{\sqrt{\beta(s)}}\cdot\left(\alpha(s)\cdot\cos(\psi(s)+\varphi)+\sin(\psi(s)+\varphi)\right)$$

Complex amplitude and real amplitude and phase are easy to calculate. Expression for a^2 is called Currant-Snyder invariant.

$$X=\mathrm{Re}\,\tilde{a}Y;$$

$$ae^{i\varphi}=-iY^{*T}SX=\begin{bmatrix}\mathrm{w}\\ \mathrm{w}'-i/\mathrm{w}\end{bmatrix}^T\cdot\begin{bmatrix}\mathrm{x}'\\ \mathrm{-x}\end{bmatrix}=\mathrm{x}/\mathrm{w}+i(\mathrm{w}'\mathrm{x}-\mathrm{wx}')$$

$$a^2=\frac{\mathrm{x}^2}{\mathrm{w}^2}+(\mathrm{w}'\mathrm{x}-\mathrm{wx}')^2\equiv\frac{\mathrm{x}^2+(\alpha\mathrm{x}+\beta\mathrm{x}')^2}{\beta}\equiv\gamma\mathrm{x}^2+2\alpha\mathrm{xx}'+\beta\mathrm{x}'^2$$

$$\varphi=\arg(\mathrm{x}/\mathrm{w}+i(\mathrm{w}'\mathrm{x}-\mathrm{wx}'))=\tan^{-1}\frac{\mathrm{ww}'\mathrm{x}-\mathrm{w}^2\mathrm{x}'}{\mathrm{x}}=-\tan^{-1}\frac{\alpha\mathrm{x}+\beta\mathrm{x}'}{\mathrm{x}};$$

$$\varphi=\sin^{-1}\frac{\mathrm{w}'\mathrm{x}-\mathrm{wx}'}{\sqrt{\mathrm{x}^2+\mathrm{w}^2(\mathrm{w}'\mathrm{x}-\mathrm{wx}')^2}}=-\sin^{-1}\frac{\alpha\mathrm{x}+\beta\mathrm{x}'}{\sqrt{\gamma\mathrm{x}^2+2\alpha\mathrm{xx}'+\beta\mathrm{x}'^2}}$$

$$M(s_1|s_2)=\frac{1}{2i}\begin{bmatrix} \mathrm{w}_2 & \mathrm{w}_2 \\ \mathrm{w}_2'+i/\mathrm{w}_2 & \mathrm{w}_2'-i/\mathrm{w}_2 \end{bmatrix}\cdot\begin{pmatrix} e^{i\Delta\psi} & 0 \\ 0 & e^{-i\Delta\psi} \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\cdot\begin{bmatrix} \mathrm{w}_1 & \mathrm{w}_1'+i/\mathrm{w}_1 \\ \mathrm{w}_1 & \mathrm{w}_1'-i/\mathrm{w}_1 \end{bmatrix}\cdot\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Delta\psi=\psi(s_2)-\psi(s_1);$$

$$M(s_1|s_2)=\frac{1}{2i}\begin{bmatrix} \mathrm{w}_2 & \mathrm{w}_2 \\ \mathrm{w}_2'+i/\mathrm{w}_2 & \mathrm{w}_2'-i/\mathrm{w}_2 \end{bmatrix}\cdot\begin{bmatrix} -(\mathrm{w}_1'-i/\mathrm{w}_1)e^{i\Delta\psi} & \mathrm{w}_1 e^{i\Delta\psi} \\ (\mathrm{w}_1'+i/\mathrm{w}_1)e^{-i\Delta\psi} & \text{-}\mathrm{w}_1 e^{-i\Delta\psi} \end{bmatrix}=$$

$$\begin{bmatrix} \mathrm{w}_2/\mathrm{w}_1\cos\Delta\psi-\mathrm{w}_2\mathrm{w}_1'\sin\Delta\psi & \mathrm{w}_1\mathrm{w}_2\sin\Delta\psi \\ (\mathrm{w}_2'/\mathrm{w}_1-\mathrm{w}_1'/\mathrm{w}_2)\cos\Delta\psi & \mathrm{w}_1/\mathrm{w}_2\cos\Delta\psi+\mathrm{w}_1\mathrm{w}_2'\sin\Delta\psi \\ -\left(\mathrm{w}_1'\mathrm{w}_2'+1/(\mathrm{w}_2\mathrm{w}_1)\right)\sin\Delta\psi & \end{bmatrix}$$

use $\mathrm{w}=\sqrt{\beta}$; $\mathrm{w}'=\text{-}\alpha/\sqrt{\beta}$ to get a standard

$$M(s_1|s_2)=\begin{bmatrix} \dfrac{\cos\Delta\psi+\alpha_1\sin\Delta\psi}{\sqrt{\beta_1/\beta_2}} & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\dfrac{(\alpha_2-\alpha_1)\cos\Delta\psi+(1+\alpha_1\alpha_2)\sin\Delta\psi}{\sqrt{\beta_1\beta_2}} & \dfrac{\cos\Delta\psi-\alpha_2\sin\Delta\psi}{\sqrt{\beta_2/\beta_1}} \end{bmatrix}$$

A little bit more complex is fully coupled 2D case. First, the equation for eigen values of symplectic matrix has symmetric coefficients:

$$\det[\mathbf{T}-\lambda\mathbf{I}] = c_4\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0 \Rightarrow$$
$$\lambda^4 \det[\mathbf{T}-\lambda^{-1}\mathbf{I}] = \lambda^4\left(c_4\lambda^{-4} + c_3\lambda^{-3} + c_2\lambda^{-2} + c_1\lambda^{-1} + c_0\right) = 0$$
$$\Rightarrow c_4 + c_3\lambda + c_2\lambda^2 + c_1\lambda^3 + c_0\lambda^4 = 0$$
$$\Rightarrow c_0 = c_4; c_3 = c_1; c_0 = 1; c_3 = -Trace[\mathbf{T}]; \tag{40}$$

with only one coefficient c_2 remaining unknown.

$$T(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \ \det[T-\lambda I] = \lambda^4 - \lambda^3 Tr[T] + \lambda^2 c_2 - \lambda Tr[T] + 1 = 0;$$
$$c_2 = \underbrace{\mathrm{a}_{11}\ \mathrm{a}_{22}\ - \mathrm{a}_{12}\ \mathrm{a}_{21}}_{\det A} + \underbrace{\mathrm{d}_{11}\ \mathrm{d}_{22}\ - \mathrm{d}_{12}\ \mathrm{d}_{21}}_{\det D} - (\underbrace{\mathrm{b}_{11}\ \mathrm{c}_{11}\ \mathrm{b}_{12}\ \mathrm{c}_{21} + \mathrm{b}_{21}\ \mathrm{c}_{12} + \mathrm{b}_{22}\ \mathrm{c}_{22}}_{Tr[B\cdot C]})$$
$$+ \underbrace{\mathrm{a}_{11}\ \mathrm{d}_{11} + \mathrm{a}_{11}\ \mathrm{d}_{22} + \mathrm{a}_{22}\ \mathrm{d}_{11} + \mathrm{a}_{22}\ \mathrm{d}_{22}}_{Tr[A]\cdot Tr[B]} \tag{41}$$

$$\mathbf{T}(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \ \det[\mathbf{T}-\lambda\mathbf{I}] = \lambda^4 - \lambda^3 Tr[\mathbf{T}] + a\lambda^2 - \lambda Tr[\mathbf{T}] + 1 = 0$$
$$a = 2\det A + Tr[A]\cdot Tr[D] - Tr[BC]$$
$$(note \det A = \det D; \det C = \det B = 1 - \det A) \tag{42}$$

Finding roots:

$$d(\lambda)=\lambda^4-\lambda^3 Tr[\mathbf{T}]+a\lambda^2-\lambda Tr[\mathbf{T}]+1=(\lambda-\lambda_1)(\lambda-\lambda_1^{-1})(\lambda-\lambda_2)(\lambda-\lambda_2^{-1});z_k=\lambda_k+\lambda_k^{-1}$$

$$d(\lambda)=(\lambda^2-\lambda z_1+1)(\lambda^2-\lambda z_2+1)=\lambda^4-\lambda^3(z_1+z_2)+\lambda^2(z_1z_2+2)-\lambda(z_1+z_2)+1$$

$$z_1+z_2=Tr[\mathbf{T}]=Tr[A+B];z_1z_2=a-2\rightarrow(z-z_1)(z-z_2)=z^2-z(z_1+z_2)+z_1z_2=0$$

$$z^2-z\cdot Tr[A+B]+a-2=0\Rightarrow z_k=\frac{Tr[A+B]}{2}\pm\left\{\frac{Tr[A+B]^2}{4}+2-a\right\}^{1/2};$$

$$\frac{Tr[A+B]^2}{4}-Tr[A]\cdot Tr[D]+2(1-\det A)+Tr[BC]=\frac{Tr[A-B]^2}{4}+2\det C+Tr[BC]; \tag{43}$$

$$z_k=2\cos\mu_k=\frac{Tr[A+B]}{2}\pm\sqrt{\frac{Tr[A-B]^2}{4}+2\det C+Tr[BC]}.$$

Stability conditions for coupled 2D motions is:

$$|\cos\mu_k|\le 1;k=1,2 \tag{44}$$

And finally, the parameterization

$$X=\begin{bmatrix}x\\P_x\\y\\P_y\end{bmatrix}=\operatorname{Re}\tilde{a}_1Y_1+\operatorname{Re}\tilde{a}_1Y_2=\operatorname{Re}a_1\tilde{Y}_1+\operatorname{Re}\tilde{a}_1\tilde{Y}_2$$

$$Y_k=R_k+iQ_k;\quad \tilde{Y}_k=\begin{bmatrix}w_{kx}e^{i\psi_{kx}}\\(u_{kx}+iv_{kx})e^{i\psi_{kx}}\\w_{ky}e^{i\psi_{ky}}\\(u_{ky}+iv_{ky})e^{i\psi_{ky}}\end{bmatrix};\quad \psi_{kx}(s+C)=\psi_{kx}(s)+\mu_k;\psi_{ky}(s+C)=\psi_{ky}(s)+\mu_k;$$

$$w_{kx}v_{kx}+w_{ky}v_{ky}=1; \tag{45}$$

Conditions: there are

$$Y_k^{*T} S Y_k = 2i;\ \ Y_1^{*T} S Y_2 = 0;\ Y_1^T S Y_2 = 0;\quad \theta_k = \psi_{kx} - \psi_{ky}$$

$$a)\ \ w_{1x} v_{1x} = w_{2y} v_{2y} = 1 - q \quad \Rightarrow v_{1x} = \frac{1-q}{w_{1x}};\ \ v_{2y} = \frac{1-q}{w_{2y}}$$

$$b)\ \ w_{1y} v_{1y} = w_{2x} v_{2x} = q \quad \Rightarrow v_{2x} = \frac{q}{w_{2x}};\ \ w_{1y} = \frac{q}{w_{1y}}$$

$$c)\ \ c = w_{1x} w_{1y} \sin\theta_1 = -w_{2x} w_{2y} \sin\theta_2$$

$$d)\ \ d = w_{1x}\left(u_{1y}\sin\theta_1 - v_{1y}\cos\theta_1\right) = -w_{2x}\left(u_{2y}\sin\theta_2 - v_{2y}\cos\theta_2\right)$$

$$e)\ \ e = w_{1y}\left(u_{1x}\sin\theta_1 + v_{1x}\cos\theta_1\right) = -w_{2y}\left(u_{2x}\sin\theta_2 + v_{2x}\cos\theta_2\right) \qquad (46)$$

Conditions are result of symplecticity. Conditions a) and b) are equivalent to Poincaré's invariants conserving sum of projections on (x-px) and (y-py) planes.

$$Y_1 = \begin{bmatrix} w_{1x} e^{i\varphi_{1x}} \\ \left(u_{1x} + i\dfrac{q}{w_{1x}}\right) e^{i\varphi_{1x}} \\ w_{1y} e^{i\varphi_{1y}} \\ \left(u_{1y} + i\dfrac{1-q}{w_{1y}}\right) e^{i\varphi_{1y}} \end{bmatrix};\ \ Y_2 = \begin{bmatrix} w_{2x} e^{i\varphi_{2x}} \\ \left(u_{2x} + i\dfrac{1-q}{w_{2x}}\right) e^{i\varphi_{2x}} \\ w_{2y} e^{i\varphi_{2y}} \\ \left(u_{2y} + i\dfrac{q}{w_{2y}}\right) e^{i\varphi_{2y}} \end{bmatrix} \qquad (47)$$

One should note for completeness that there is another way of parameterization of coupled motion proposed by Edward and Teng (*D.A.Edwards, L.C.Teng, IEEE Trans. Nucl. Sci.* ***NS-20*** *(1973) 885*), which differs from what we are discussing here.

What we learned today

- We fond stability criteria for periodic linear Hamiltonian system via its single period transport matrix
- We found that eigen vectors of this matrix is natural parameterization of the particle's motion
- It reduces it to something similar of harmonic oscillator with variable frequency of oscillations