Coherent electron Cooling

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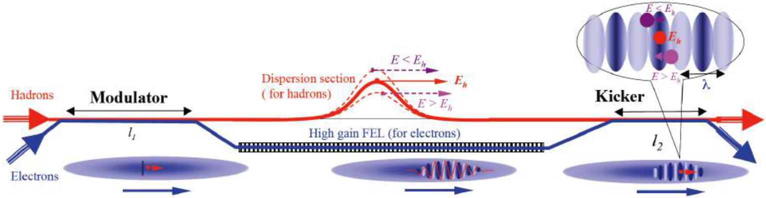
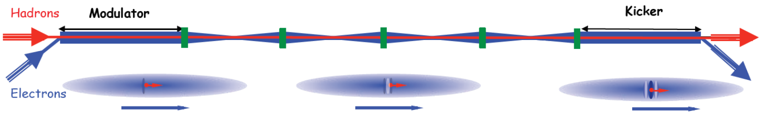
- In the Electron-Ion Collider (EIC), Strong Hadron Cooling (SHC) is needed to reach high luminosity. Present baseline approach for SHC is based on Coherent electron Cooling (CeC).
- A general CeC scheme consists of three main sections: Modulator, Amplifier, Kicker

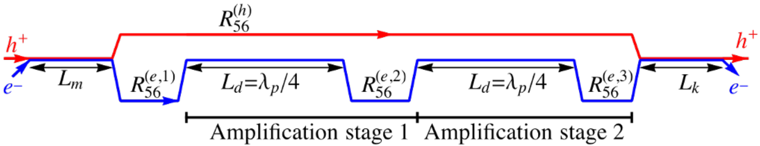


(a) CeC with free electron laser (FEL) amplifier

Other implementations of amplifier



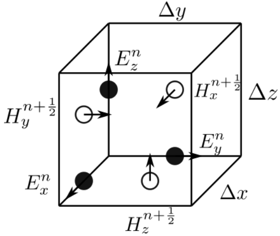
(a) Plasma cascade amplifier (PCA)



(b) Microbunched coherent electron cooling (MBEC)

Simulation tool

- The SPACE code is a parallel, relativistic, three-dimensional (3D) electromagnetic (EM) Particle-in-Cell (PIC) code.
  Finite-difference time-domain (FDTD) or Yee's method



  Uniform mesh, adaptive mesh, adaptive Particle-in-Cloud

- The GENESIS code is a three-dimensional, time-dependent code developed for high-gain FEL simulations.

Analytical tools for modulation process

Cold uniform electron beam ©V. N. Litvinenko

$$q = -Ze \cdot (1 - \cos\varphi_1) \qquad \varphi_1 = \omega_p l_1 / c\gamma_0$$

(a) Density modulation

$$\left\langle \frac{\delta E}{E} \right\rangle \cong -2Z \frac{r_e}{a^2} \cdot \frac{L_{pol}}{\gamma} \cdot \left(\frac{z}{|z|} - \frac{z}{\sqrt{a^2/\gamma^2 + z^2}} \right)$$

(b) Energy modulation

G. Wang, and M. Blaskiewicz. Physical Review E 78.2 (2008): 026413.
Linearized Vlasov Equation

$$\frac{\partial}{\partial t} f_1(\vec{x},\vec{v},t) + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} f_1(\vec{x},\vec{v},t) - \frac{e\vec{E}}{m_e} \cdot \frac{\partial}{\partial \vec{v}} f_0(\vec{v}) = 0$$

$$\vec{\nabla} \cdot \vec{E}(\vec{x},t) = \frac{\rho(\vec{x},t)}{\epsilon_0}$$

$$\rho(\vec{x},t) = Z_i e\,\delta(\vec{x}) - e\tilde{n}_1(\vec{x},t)$$

$$\tilde{n}_1(\vec{x},t) = \int f_1(\vec{x},\vec{v},t) d^3 v$$

Fourier transform

$$\frac{\partial}{\partial t} f_1(\vec{k},\vec{v},t) + i\vec{k}\cdot\vec{v} f_1(\vec{k},\vec{v},t) + i\frac{e\Phi(\vec{k},t)}{m_e}\vec{k}\cdot\frac{\partial}{\partial \vec{v}} f_0(\vec{v}) = 0$$

$$\Phi(\vec{k},t) = \frac{e}{\epsilon_0 k^2}[Z_i - \tilde{n}_1(\vec{k},t)]$$

Multiply both sides by $e^{i\vec{k}\cdot\vec{v}t}$

$$\frac{\partial}{\partial t}[e^{i\vec{k}\cdot\vec{v}t} f_1(\vec{k},\vec{v},t)] = -i\frac{e}{m_e}\Phi(\vec{k},t)e^{i\vec{k}\cdot\vec{v}t}\left(\vec{k}\cdot\frac{\partial}{\partial \vec{v}} f_0(\vec{v})\right)$$

Analytical tools for modulation process

Initial condition $f_1(\vec{k},0) = 0$

$$f_1(\vec{k},\vec{v},t) = -i\frac{e}{m_e}\int_0^t \Phi(\vec{k},t_1)e^{i\vec{k}\cdot\vec{v}(t_1-t)}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0(\vec{v})dt_1$$

Note relation

$$i\int\frac{\vec{k}}{k^2}\cdot\frac{\partial}{\partial\vec{v}}f_0(\vec{v})e^{i\vec{k}\cdot\vec{v}\tau}d^3v = \int f_0(\vec{v})e^{i\vec{k}\cdot\vec{v}\tau}\tau\, d^3v$$

We have

$$\tilde{n}_1(\vec{k},t) = \omega_p^2 \int_0^t [\tilde{n}_1(\vec{k},t_1) - Z_i](t_1 - t) g(\vec{k}(t - t_1)) dt_1$$

$$g(\vec{u}) \equiv \frac{1}{n_0} \int f_0(\vec{v}) e^{-i\vec{u}\cdot\vec{v}} d^3 v$$

$$\omega_p = \sqrt{n_0 e^2 / m_e \epsilon_0}$$

Analytical tools for modulation process

For cold electrons, the velocity distribution in the rest frame of the ion reads $f_0(\vec{v}) = n_0\delta^3(\vec{v})$, which gives $g(\vec{u}) = 1$
The integral equation reduces to 2nd order ODE

$$\frac{d^2}{dt^2}\tilde{n}_1(\vec{k}, t) = -\omega_p^2\tilde{n}_1(\vec{k}, t) + Z_i\omega_p^2$$

Analytical tools for modulation process

Without ion, with initial perturbation

$$\frac{d^2}{dt^2}\tilde{n}_1\left(\vec{k},t\right) = -\omega_p^2\tilde{n}_1\left(\vec{k},t\right)$$

$$\Rightarrow \tilde{n}_1\left(\vec{k},t\right) = \tilde{n}_1\left(\vec{k},0\right)\cos\left(\omega_p t\right) + \frac{\dot{\tilde{n}}_1\left(\vec{k},0\right)}{\omega_p}\sin\left(\omega_p t\right)$$

With ion, without initial perturbation

$$\tilde{n}_1\left(\vec{k},t\right) = Z_i\left[1-\cos\left(\omega_p t\right)\right]$$

Analytical tools for modulation process

Warm uniform electron beam with $\kappa - 2$ velocity distribution:

$$f_0(\vec{v}) = \frac{1}{\pi^2 \beta_x \beta_y \beta_z} \left(1 + \frac{v_x^2}{\beta_x^2} + \frac{v_y^2}{\beta_y^2} + \frac{v_z^2}{\beta_z^2} \right)^{-2}$$

(a) $\kappa - 2$

G. Wang, and M. Blaskiewicz. Physical Review E 78.2 (2008): 026413.

$$\tilde{n}_1(\vec{x},t) = \frac{Z_i}{\pi^2 a_x a_y a_z} \int_0^{\omega_p t} \frac{\tau \sin \tau \cdot d\tau}{\left[\tau^2 + \left(\frac{x}{a_x} + \frac{v_{0,x}}{\beta_x} \tau \right)^2 + \left(\frac{y}{a_y} + \frac{v_{0,y}}{\beta_y} \tau \right)^2 + \left(\frac{z}{a_z} + \frac{v_{0,z}}{\beta_z} \tau \right)^2 \right]^2}$$

(a) Density modulation

Analytical tools for modulation process

Warm uniform electron beam with $\kappa - 2$ velocity distribution. G. Wang, V. N. Litvinenko, and M. Blaskiewicz. "Energy Modulation in Coherent Electron Cooling." Proceedings of IPAC (2013).

$$\left\langle \frac{\delta E}{E_0} \right\rangle = \frac{\langle v_z \rangle}{c} = -\frac{1}{e n_0 \pi a^2 c} I_d\left(\gamma_0 z_l, \frac{L_{\mathrm{mod}}}{\beta_0 \gamma_0 c} \right)$$

(a) Energy modulation

$$I_d(z,t) = -\frac{Z_i e \omega_p^2}{\pi} \int_0^t d\tau \left(z + v_{0,z}\tau\right) \left\{ \frac{a_z \sin(\omega_p \tau)}{\left[\bar{\beta}^2\tau^2 + \left(z+v_{0,z}\tau\right)^2\right]\left[1 + \bar{\beta}^2\tau^2 + \left(z+v_{0,z}\tau\right)^2 / a^2\right]} - \cos(\omega_p \tau)\left[\frac{\arctan\left(\left|z+v_{0,z}\tau\right| / \left(\bar{\beta}\tau\right)\right)}{\left|z+v_{0,z}\tau\right|} - \frac{\arctan\left(\sqrt{\left(z+v_{0,z}\tau\right)^2 + a^2} / \left(\bar{\beta}\tau\right)\right)}{\sqrt{\left(z+v_{0,z}\tau\right)^2 + a^2}} \right] \right\}$$

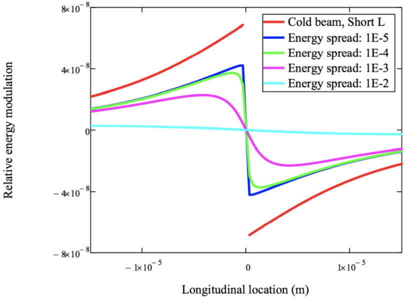
(b) Energy modulation

Analytical tools for modulation process

The warm beam result reduces to the previously derived cold beam result at the corresponding limits

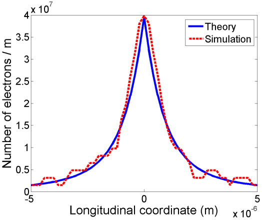
$$\bar{\beta} = 0 \quad v_{0,z} = 0 \quad L_{\text{mod}} << \beta_0 \gamma_0 c / \omega_p$$

(a)

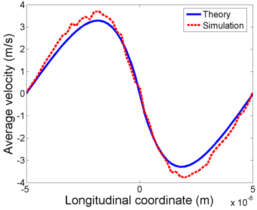


(b) Energy modulation

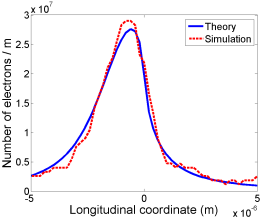
Simulation using uniform beam



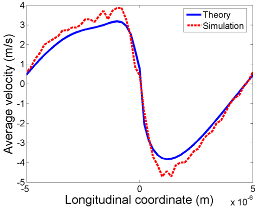
(a) Density, stationary ion



(b) Velocity, stationary ion



(c) Density, moving ion



(d) Velocity, moving ion

Simulation using Gaussian beam

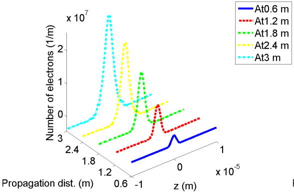
Continuous focusing field

$$\vec{E}_1(\vec{r}) = \frac{m_e}{e}\frac{\sigma_v^2}{\sigma_r^2}(\vec{r} - \vec{r}_0)$$

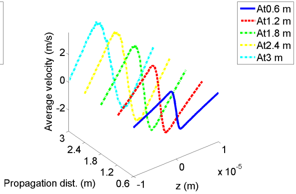
$$\vec{E}_2(\vec{r}) = \frac{q}{2\pi\varepsilon_0|\vec{r} - \vec{r}_0|}\left(1 - e^{-|\vec{r}-\vec{r}_0|^2/2\sigma_r^2}\right)$$

where $\vec{r} = (x, y)$ is the radial coordinate in transverse plane, $\vec{r}_0 = (x_0, y_0)$ is the center of the Gaussian distribution, σ_r is the RMS of the Gaussian distribution in both horizontal and vertical directions and σ_v is the RMS velocity of the electron distribution.
Transverse beam size is constant in the modulator.

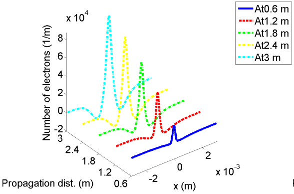
Simulation using Gaussian beam, continuous focusing



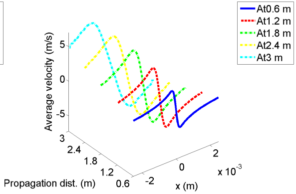
(a) Longitudinal density



(b) Longitudinal velocity

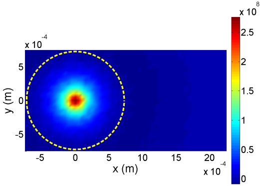


(c) Transverse density

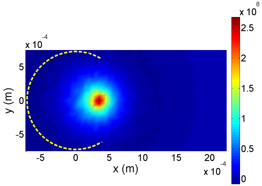


(d) Transverse velocity

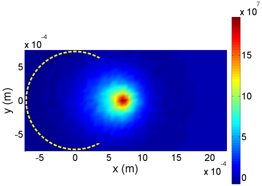
Simulation using Gaussian beam, continuous focusing



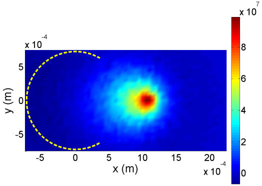
(a) Ion at center



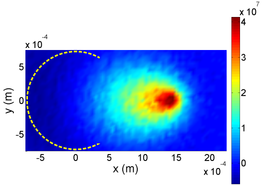
(b) Ion 0.5σ off center



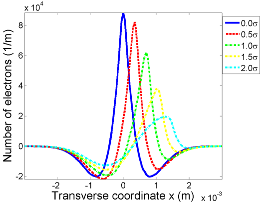
(c) Ion 1.0σ off center



(d) Ion 1.5σ off center

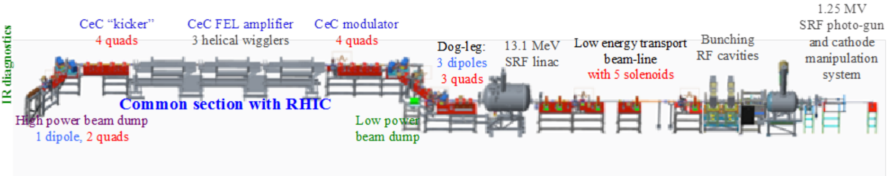


(e) Ion 2.0σ off center

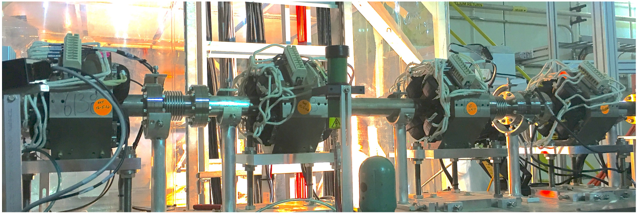


(f) Transverse density

FEL-based CeC experiment

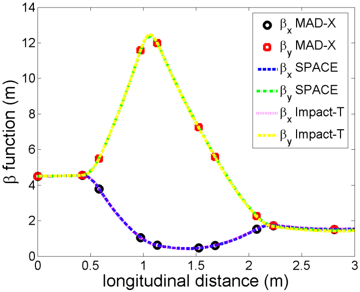


Modulator of FEL-based CeC experiment

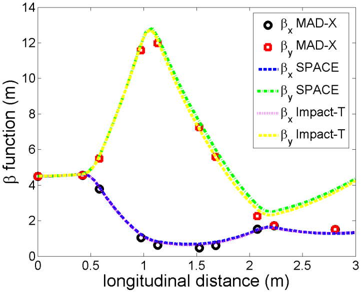


Q4 Q3 Q2 Q1

$$
\begin{aligned}
B_x &= G \cdot y \\
B_y &= G \cdot x \\
\kappa &= \frac{G}{B\rho} \\
B\rho(T \cdot m) &= 3.3356pc(GeV)
\end{aligned}
$$

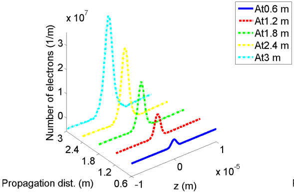


(a) No space charge

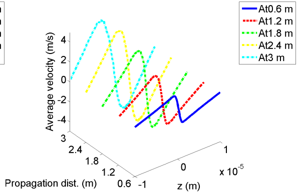


(b) With space charge

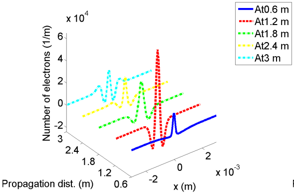
Modulation, quadrupole beam line



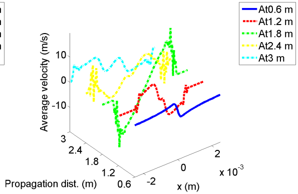
(a) Longitudinal density



(b) Longitudinal velocity



(c) Transverse density



(d) Transverse velocity

Transport in quadrupole channel

$$\langle x_o \delta x_o' \rangle = -\varepsilon, \varepsilon > 0.$$

(a) Initial correlation

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \begin{pmatrix} x_o \\ x_o' \end{pmatrix}, \quad ad - bc = 1$$

(b) Transport

$$\begin{pmatrix} \delta x(s) \\ \delta x'(s) \end{pmatrix} = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix} \begin{pmatrix} 0 \\ \delta x_o' \end{pmatrix}$$

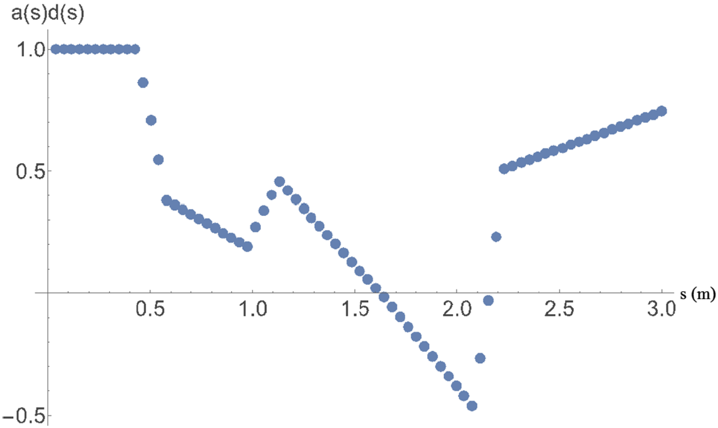
(c) Transport

$$\begin{aligned} x &= a x_o + b x_o' \\ \delta x' &= d \delta x_o' \\ \langle x \delta x' \rangle &= ad \cdot \langle x_o \delta x_o' \rangle \\ &= -ad \cdot \varepsilon \end{aligned}$$

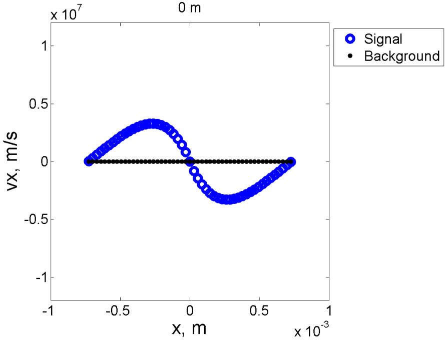
(d) Final correlation

Transport in quadrupole channel

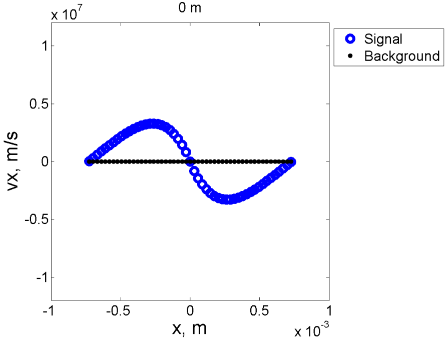
J. Ma, et al. Physical Review Accelerators and Beams 21.11 (2018): 111001.



Transverse phase advance in quadrupole beam line



(a) No space charge



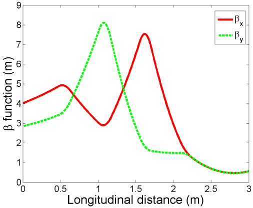
(b) With space charge

Bunching factor

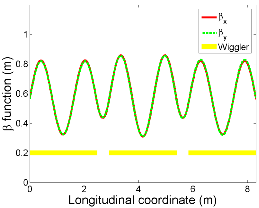
$$b \equiv \frac{1}{N_\lambda} \sum_{k=1}^{N_\lambda} e^{i \frac{2\pi}{\lambda_{opt}} z_k}, -\frac{\lambda_{opt}}{2} \leq z_k \leq \frac{\lambda_{opt}}{2},$$

where λ_{opt} is the optical wavelength, the sum is taken over a slice of λ_{opt} width, centered at the location of the ion, and N_λ is the total number of electrons within that slice.

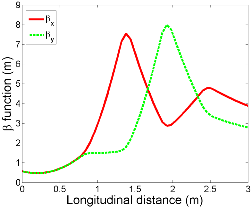
Beam envelope in FEL-based CeC



(a) Modulator

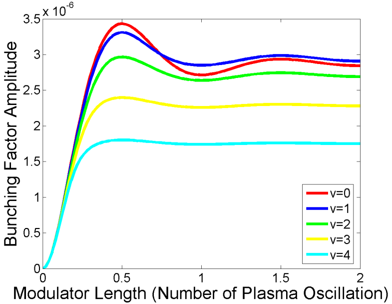


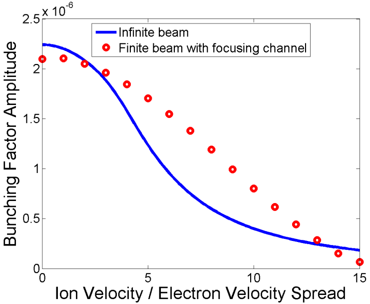
(b) FEL amplifier



(c) Kicker

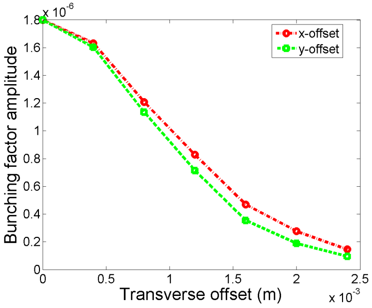
Dependence on ion velocity and modulator length

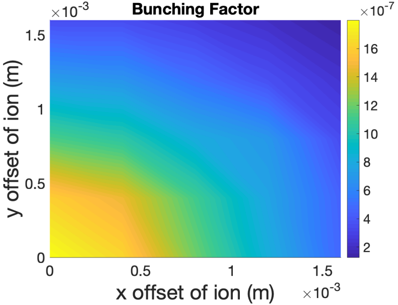




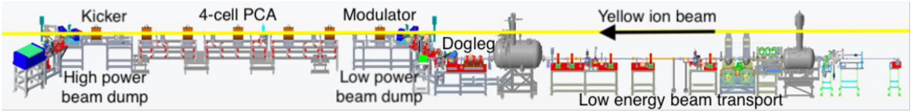
The ion velocity is in unit of electron longitudinal velocity spread.

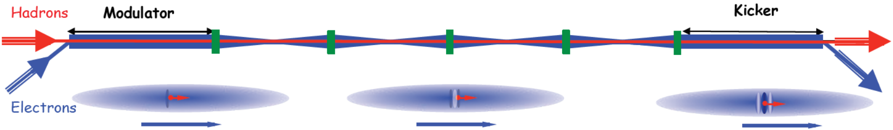
Dependence on ion transverse offset





PCA-based CeC





Solenoid field

An example of on-axis magnetic field:

$$B_{z,0} = \frac{B_0}{2}\left(\frac{L/2 - z}{\sqrt{(z - L/2)^2 + R^2}} + \frac{L/2 + z}{\sqrt{(z + L/2)^2 + R^2}}\right)$$

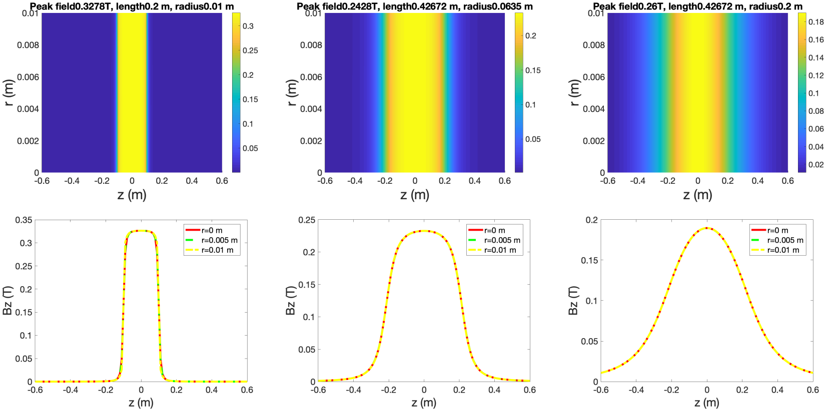
Off-axis magnetic field:

$$\begin{aligned} B_z(r) &= B_{z,0} - \frac{r^2}{4} B''_{z,0} + \frac{r^4}{64} B''''_{z,0} - \frac{r^6}{2304} B''''''_{z,0} \cdots \\ B_r(r) &= -\frac{r}{2} B'_{z,0} + \frac{r3}{16} B'''_{z,0} - \frac{r^5}{384} B'''''_{z,0} \cdots \end{aligned}$$

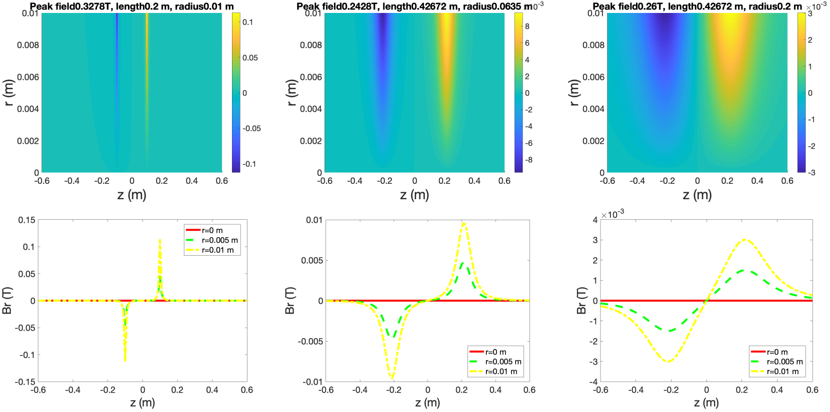
Lorentz transformation of the fields

$$
\begin{aligned}
E_x^* &= \gamma E_x - \gamma\beta c B_y \\
E_y^* &= \gamma E_y + \gamma\beta c B_x \\
E_z^* &= E_z \\
B_x^* &= \gamma B_x + \frac{\gamma\beta}{c} E_y \\
B_y^* &= \gamma B_y - \frac{\gamma\beta}{c} E_x \\
B_z^* &= B_z
\end{aligned}
$$

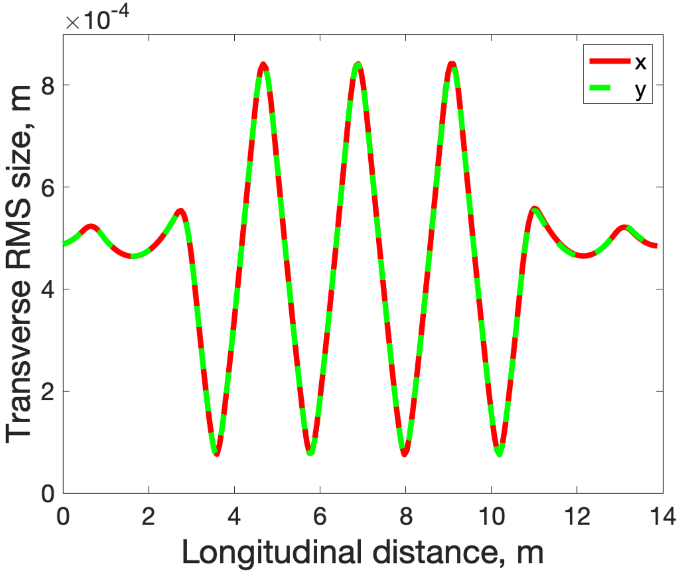
Solenoid field B_z



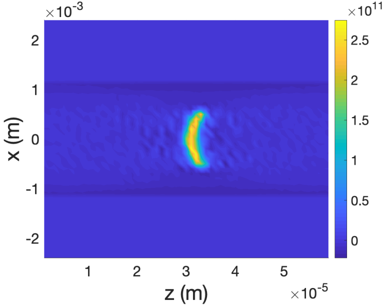
Solenoid field B_r



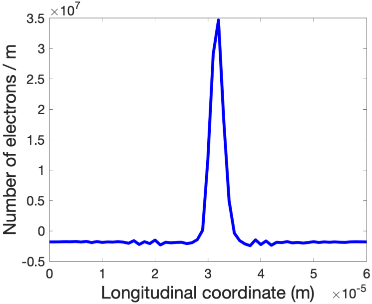
Beam envelope in PCA-based CeC



Density modulation in PCA-based CeC

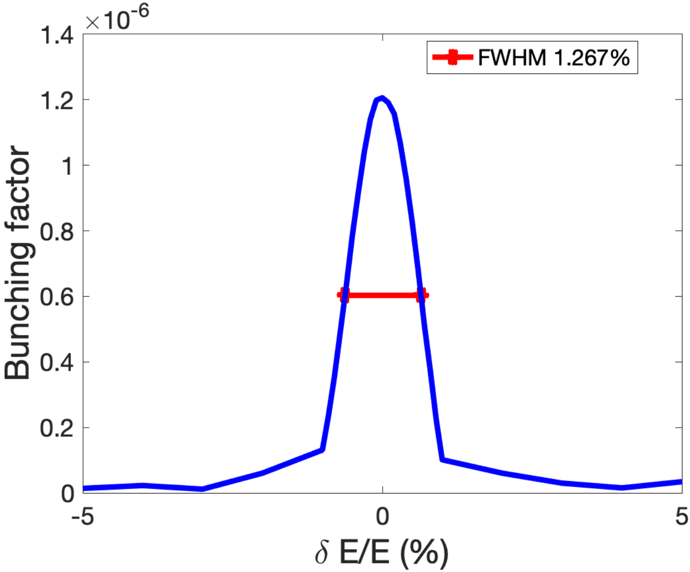


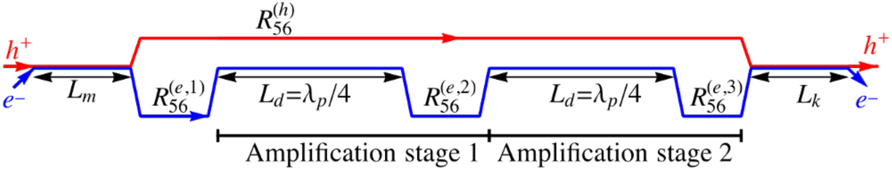
(a) 2D plot



(b) 1D plot

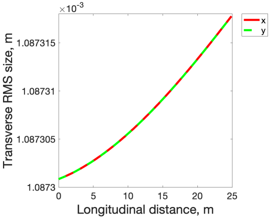
Dependence on energy difference



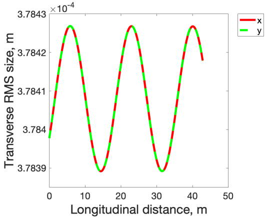


$R_{56}^{(h)}$
h^+
h^+
e^-
e^-
L_m
$R_{56}^{(e,1)}$
$L_d = \lambda_p/4$
$R_{56}^{(e,2)}$
$L_d = \lambda_p/4$
$R_{56}^{(e,3)}$
L_k
Amplification stage 1
Amplification stage 2

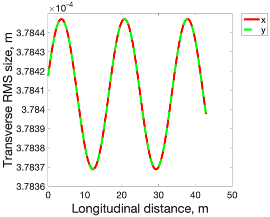
Beam envelope in MBEC



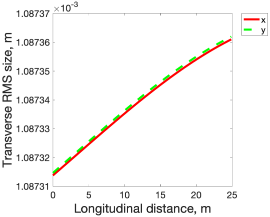
(a) Modulator



(b) First stage

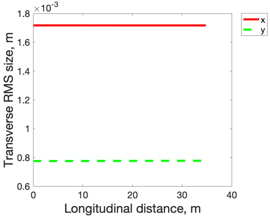


(c) Second stage

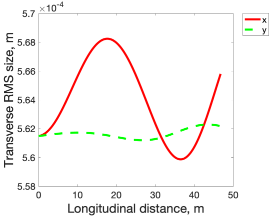


(d) Kicker

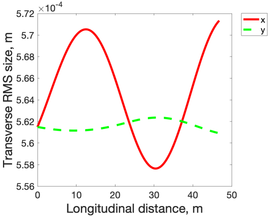
Beam envelope in MBEC



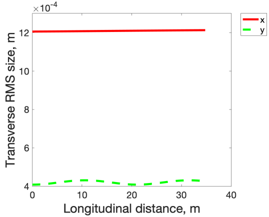
(a) Modulator



(b) First stage

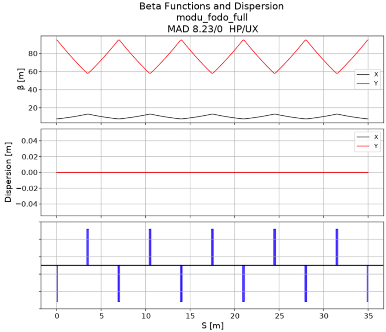


(c) Second stage

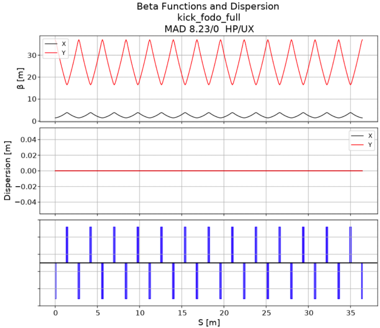


(d) Kicker

Beam envelope in MBEC

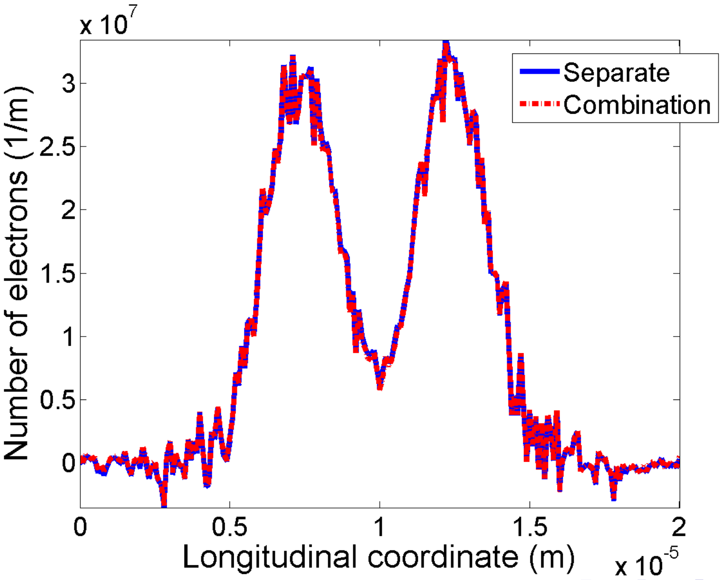


(a) Modulator

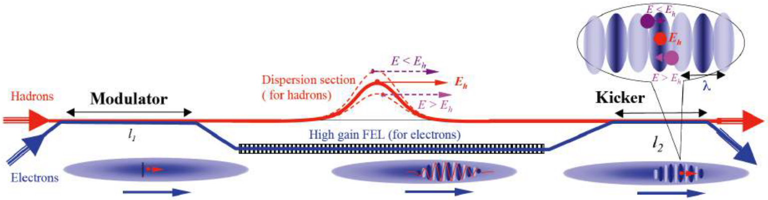


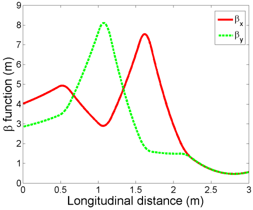
(b) Kicker

Superposition principle in density modulation

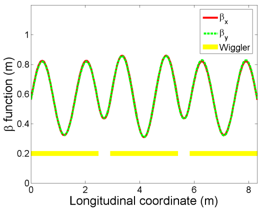


FEL-based CeC

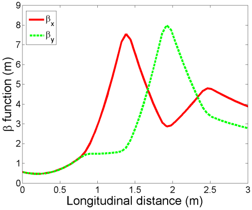




(a) Modulator

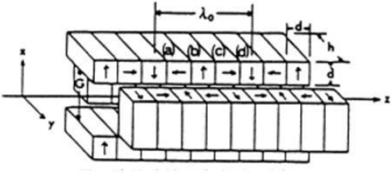


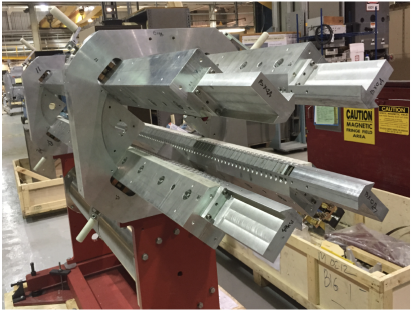
(b) FEL amplifier



(c) Kicker

Helical undulator





$$
\begin{aligned}
B_x(x,y,z) &= B_0\cos(k_u z)\\
B_y(x,y,z) &= B_0\sin(k_u z)
\end{aligned}
$$

Electron motion in helical wiggler without radiation field

$$\vec{B}_w(x,y,z) = B_w\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right]$$

$$\vec{F}(x,y,z) = -e\vec{v}\times\vec{B} = -ev_z\hat{z}\times\vec{B} = -ev_z B_w\left[\cos(k_u z)\hat{y} + \sin(k_u z)\hat{x}\right]$$

$$\frac{d(m\gamma v_x)}{dt} = m\gamma\frac{dv_x}{dt} = -ev_z B_w\sin(k_u z) \qquad \frac{d(m\gamma v_y)}{dt} = m\gamma\frac{dv_y}{dt} = -ev_z B_w\cos(k_u z)$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \qquad v = \sqrt{v_x^2+v_y^2+v_z^2} \qquad \tilde{v} \equiv v_x + iv_y$$

$$m\gamma\frac{d\tilde{v}}{dt} = -iev_z B_w\left(\cos(k_u z) - i\sin(k_u z)\right) = -iev_z B_w e^{-ik_u z}$$

$$m\gamma\frac{d\tilde{v}}{dt} = m\gamma\frac{dz}{dt}\frac{d\tilde{v}}{dz} = -iev_z B_w e^{-ik_u z} \Rightarrow m\gamma\frac{d\tilde{v}}{dz} = -ieB_w e^{-ik_u z}$$

Electron motion in helical wiggler without radiation field

$$\frac{\tilde{v}(z)}{c} = \frac{-ieB_w}{mc\gamma}\int e^{-ik_u z_1}\,dz_1 = \frac{eB_w}{mc\gamma k_u}e^{-ik_u z} = \frac{K}{\gamma}e^{-ik_u z}$$

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right] \qquad v_z = const.$$

$$K \equiv \frac{eB_w\lambda_w}{2\pi mc} \qquad \theta_s = K/\gamma$$

Energy change of electrons due to radiation field

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right]$$

$$\vec{E}_\perp(z,t) = E\left[\cos(kz - \omega t)\hat{x} + \sin(kz - \omega t)\hat{y}\right] \qquad E_z = 0$$

$$= E\left[\cos(k(z - ct))\hat{x} + \sin(k(z - ct))\hat{y}\right] \qquad \omega = kc$$

$$\frac{d\mathcal{E}}{dt} = \vec{F}\cdot\vec{v} = -e\vec{v}_\perp \cdot \vec{E}_\perp$$

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \cos\left[\left(k_w + k - k\frac{c}{v_z}\right)z + \psi_0\right]$$

Resonant radiation wavelength

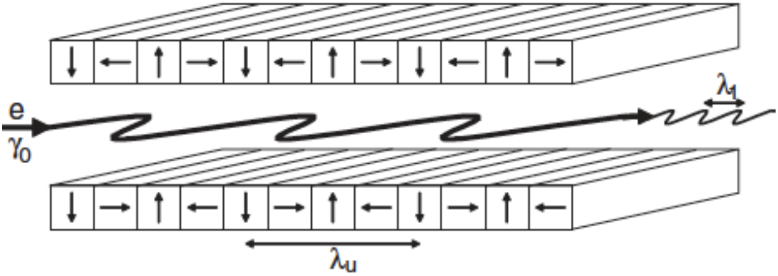
$$k_w + k_0 - k_0 \frac{c}{v_z} = 0 \Rightarrow \lambda_0 = \lambda_w \left(\frac{c}{v_z} - 1 \right) \approx \frac{\lambda_w}{2\gamma_z^2}$$

$$\gamma_z^{-2} \equiv 1 - v_z^2 / c^2 = 1 - \left(v_z^2 + v_\perp^2 \right) / c^2 + v_\perp^2 / c^2 = \gamma^{-2} + \theta_s^2 = \gamma^{-2} \left(1 + K^2 \right)$$

$$\lambda_0 \approx \frac{\lambda_w \left(1 + K^2 \right)}{2\gamma^2}$$

$$K \equiv \frac{eB_w \lambda_w}{2\pi mc}$$

Planar undulator



$$B_y(x,y,z) = B_0 \sin(k_u z)$$
$$\lambda_0 = \frac{\lambda_w}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$

FEL gain

Low gain

$$g_l = \frac{(E_{ext} + \Delta E)^2 - E_{ext}^2}{E_{ext}^2} \approx \frac{2\Delta E}{E_{ext}} = \tau \cdot f(\hat{C}_l) = 2\Gamma^3 l_w^3 f_l(\hat{C}_l)$$

$$f_l(\hat{C}_l) = \frac{2}{\hat{C}_l^3}\left(1 - \cos\hat{C}_l - \frac{\hat{C}_l}{2}\sin\hat{C}_l\right)$$

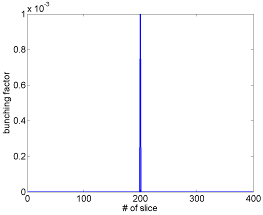
High gain

$$g_h(\hat{C}_l) = \frac{\widetilde{E}^2 - E_{ext}^2}{E_{ext}^2} = \left|\frac{\lambda_2\lambda_3 e^{\lambda_1 \hat{l}_w}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2 \hat{l}_w}}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3 \hat{l}_w}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\right|^2 - 1$$

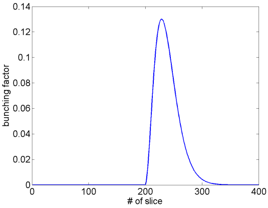
$$= 2\Gamma^3 l_w^3 f_h(\hat{C}_l) \qquad \hat{l}_w = l_w \Gamma$$

$$f_h(\hat{C}_l) = \frac{1}{2\hat{l}_w^3}\left\{\left|\frac{\lambda_2\lambda_3 e^{\lambda_1 \hat{l}_w}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2 \hat{l}_w}}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3 \hat{l}_w}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\right|^2 - 1\right\}$$

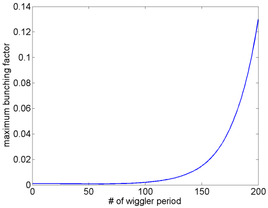
FEL gain, no saturation



(a) Initial

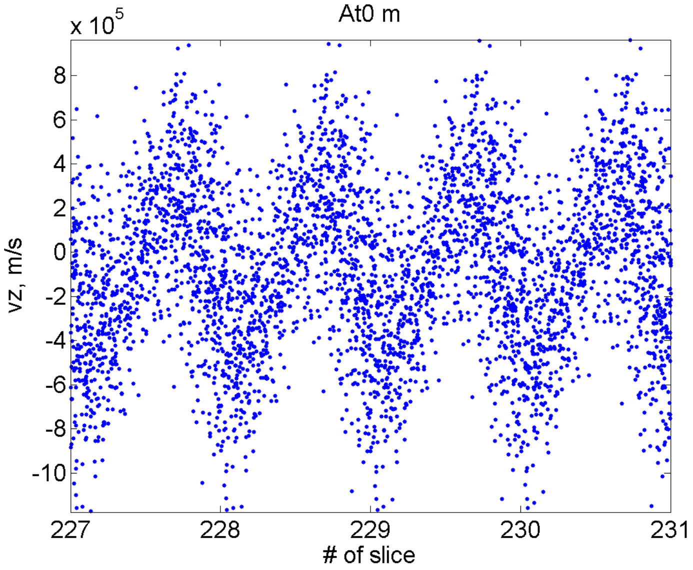


(b) Final

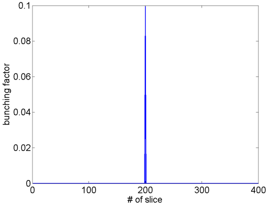


(c) Growth

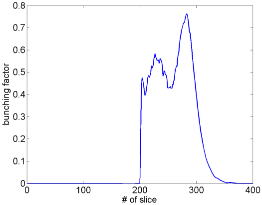
FEL gain, no saturation



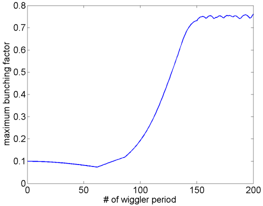
FEL gain, saturation



(a) Initial

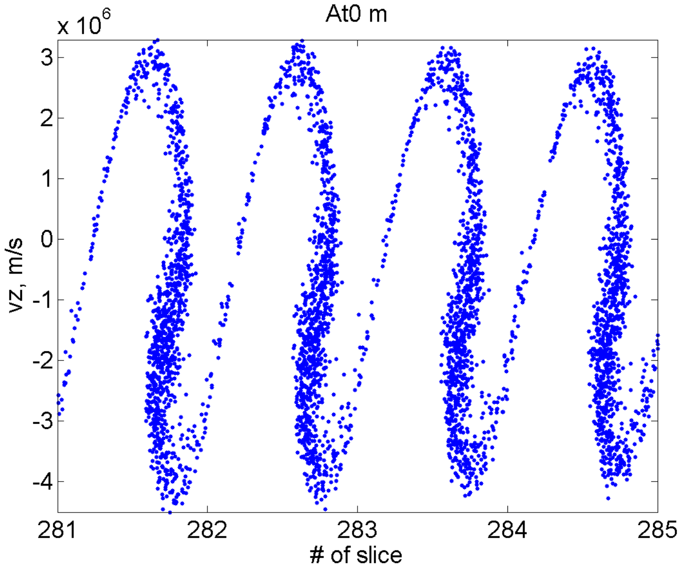


(b) Final

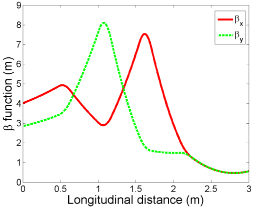


(c) Growth

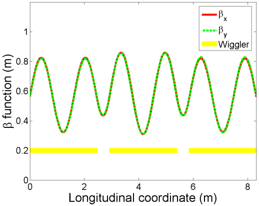
FEL gain, saturation



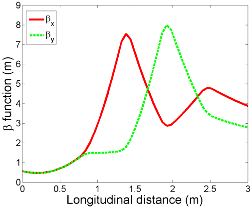
Beam envelope in FEL-based CeC



(a) Modulator

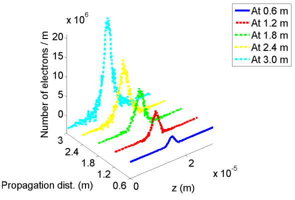


(b) FEL amplifier

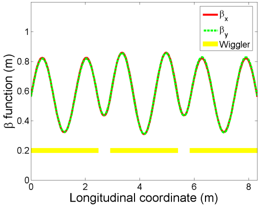


(c) Kicker

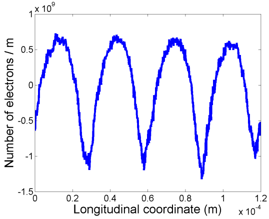
Density modulation in FEL-based CeC



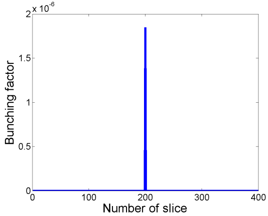
(a) Exit of modulator



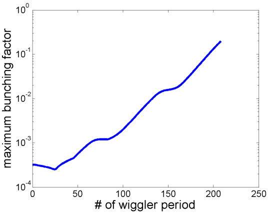
(b) FEL amplifier



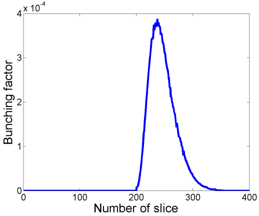
(c) Entrance of kicker



(d) Entrance of FEL

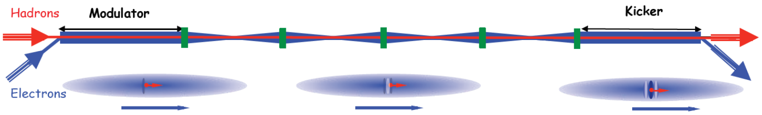


(e) FEL amplifier



(f) Exit of FEL

PCA-based CeC



Working principle of PCA is the plasma cascade instability (PCI).
Litvinenko, Vladimir N., et al. Physical Review Accelerators and Beams 24.1 (2021): 014402.

Plasma oscillation

$$\frac{\partial n}{\partial t} + \mathrm{div}(n\vec{v}) = 0.$$

$$\frac{\partial \vec{v}}{\partial t} = \frac{e\vec{E}}{m}; \qquad \mathrm{div}\vec{E} = 4\pi e\tilde{n} \rightarrow \mathrm{div}\frac{\partial \vec{v}}{\partial t} = \frac{4\pi e^2}{m}\tilde{n},$$

$$\frac{\partial^2 \tilde{n}}{\partial t^2} + 4\pi\frac{e^2 n_o}{m}\tilde{n} = \left(\frac{\partial \vec{v}}{\partial t}\cdot\vec{\nabla}\tilde{n} + \frac{\partial \tilde{n}}{\partial t}\cdot\mathrm{div}\vec{v}\right) = O\left(\left|\frac{\tilde{n}}{n_o}\right|^2\right)$$

$$\ddot{\tilde{n}} + \omega_p^2\tilde{n} = 0,$$

$$\tilde{n} = \delta n(\vec{r})\cdot\cos[\omega_p t + \varphi(\vec{r})];$$
$$\dot{\tilde{n}} = -\omega_p \delta n(\vec{r})\cdot\sin[\omega_p t + \varphi(\vec{r})],$$

Varying plasma frequency

$$\ddot{\tilde{n}} + \omega_p^2(t)\tilde{n} = 0$$

$$\begin{bmatrix} \tilde{n}(t_2) \\ \dot{\tilde{n}}(t_2) \end{bmatrix} = \mathbf{M}(t_1|t_2) \begin{bmatrix} \tilde{n}(t_1) \\ \dot{\tilde{n}}(t_1) \end{bmatrix},$$

$$\mathbf{M}(t_1|t_2) = \underset{\text{ordered}}{\exp}\left[\int_{t_1}^{t_2} \mathbf{D}(t)dt\right]; \quad \mathbf{D}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_p^2(t) & 0 \end{bmatrix}$$

$$\underset{\text{ordered}}{\exp}\left[\int_{t_1}^{t_2} \mathbf{D}(t)\right]dt = \lim_{N\to\infty} \prod_{\substack{n=1 \\ \text{ordered}}}^{N} M_n \equiv M_N \ldots M_2 M_1;$$

$$M_n = \exp[\mathbf{D}(t_n)\Delta t]$$

$$= \begin{bmatrix} \cos\omega_p(t_n)\Delta t & \frac{\sin\omega_p(t_n)\Delta t}{\omega_p(t_n)} \\ -\omega_p(t_n)\sin\omega_p(t_n)\Delta t & \cos\omega_p(t_n)\Delta t \end{bmatrix};$$

$$\Delta t = \frac{t_2 - t_1}{N}; \qquad t_n \in \{t_1 + (n-1)\Delta t, t_1 + n\Delta t\}.$$

Stability

$$\det[\mathbf{M}-\lambda\mathbf{I}]=0; \quad \mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}; \quad \lambda_1\lambda_2=\det\mathbf{M}=1$$

Periodic system $\omega_p(t+T)=\omega_p(T)$

$$\mathbf{M}_c=\mathbf{M}(0|T)=\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}; \quad \mathbf{M}(0|nT)=\mathbf{M}_c{}^n;$$

Stable

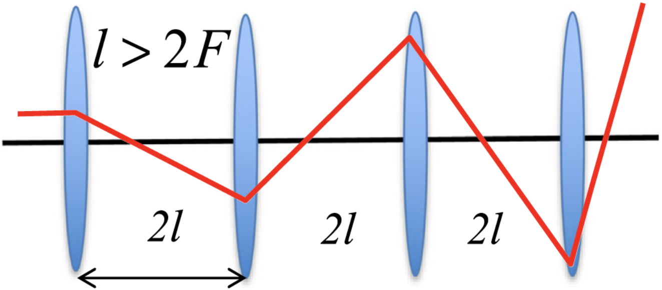
$$-2 < TraceM_c < 2$$

Stability

Unstable

$$|TraceM_c| > 2$$

$$\lambda_1 = \lambda_2^{-1} = \frac{\mathrm{Trace}\mathbf{M}_c}{2}\left(1 + \sqrt{1 - \frac{4}{(\mathrm{Trace}\mathbf{M}_c)^2}}\right); \quad |\lambda_1| > 1;$$



$$\tilde{n}(nT) = \tilde{n}(0)\frac{{\lambda_1}^n + {\lambda_1}^{-n}}{2} + \left(m_{12}\dot{\tilde{n}}(0) + \frac{m_{11} - m_{22}}{2}\tilde{n}(0)\right)\frac{{\lambda_1}^n - {\lambda_1}^{-n}}{\lambda_1 - {\lambda_1}^{-1}};$$

$$\dot{\tilde{n}}(nT) = \dot{\tilde{n}}(0)\frac{{\lambda_1}^n + {\lambda_1}^{-n}}{2} + \left(m_{21}\tilde{n}(0) + \frac{m_{22} - m_{11}}{2}\dot{\tilde{n}}(0)\right)\frac{{\lambda_1}^n - {\lambda_1}^{-n}}{\lambda_1 - {\lambda_1}^{-1}};$$

Instability

$$\tilde{n}(nT) \to \tilde{n}(0)\frac{{\lambda_1}^n}{2} + \left(m_{12}\dot{\tilde{n}}(0) + \frac{m_{11}-m_{22}}{2}\tilde{n}(0)\right) \times \frac{{\lambda_1}^n}{\lambda_1 - {\lambda_1}^{-1}};$$

$$\dot{\tilde{n}}(nT) \to \dot{\tilde{n}}(0)\frac{{\lambda_1}^n}{2} + \left(m_{21}\tilde{n}(0) + \frac{m_{22}-m_{11}}{2}\dot{\tilde{n}}(0)\right) \times \frac{{\lambda_1}^n}{\lambda_1 - {\lambda_1}^{-1}};$$

Note the alternating sign when $\lambda_1 < -1$

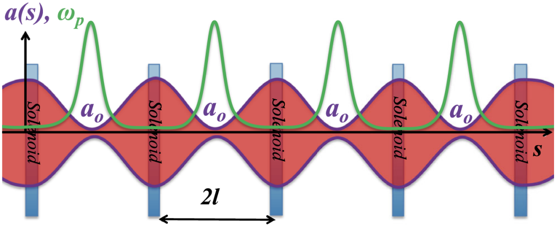
Transverse Kapchinsky-Vladimirsky (KV) distribution

$$f_\perp(x, x', y, y') = N \cdot \delta\left(\frac{x^2 + y^2}{a(s)^2} + \frac{[a(s)x' - a'(s)x]^2 + [a(s)y' - a'(s)y]^2}{{\varepsilon_{KV}}^2} - 1\right);$$

$$a'(s) = \frac{da(s)}{ds},$$

$$\varepsilon_{\mathrm{KV}} = 4\varepsilon_{\mathrm{rms}}$$

Beam envelope



$$a'' + K(s)a - \frac{2}{\beta_o{}^3\gamma_o{}^3}\frac{I_o}{I_A}\frac{1}{a} - \frac{\varepsilon_{KV}{}^2}{a^3} = 0;$$

$$K(s) = \left(\frac{eB_{\rm sol}(s)}{2p_o c}\right)^2 \equiv \left(\frac{eB_{\rm sol}(s)}{2\beta_o\gamma_o mc^2}\right)^2,$$

Beam density

$$\rho(z) = \frac{I_o}{e_o c} \frac{1}{\pi a^2(s)},$$

(a) Lab frame

$$n_o(t) = \frac{I_o}{e\beta_o \gamma_o c} \frac{1}{\pi a^2(\gamma_o \beta_o c t)},$$

(b) Beam frame

Beam density

$$\frac{\partial}{\partial t}\tilde{f}_k + ikv\tilde{f}_k + \frac{eE_s}{m}\cdot\frac{\partial f_o}{\partial v} = 0;$$

$$ikE_s = 4\pi e n_o(t)\cdot\int_{-\infty}^{\infty}\tilde{f}_k(v,t)dv$$

$$f_o(v) = \sigma_v/\pi({\sigma_v}^2 + v^2)$$

$$\frac{d^2\tilde{g}_k}{dt^2} + \omega_p^2(t)\tilde{g}_k = 0; \qquad \tilde{g}_k = \tilde{f}_k e^{|k\sigma_v t|};$$

$$\rho_k(t) = \int_{-\infty}^{\infty}\tilde{f}_k(v,t)dv = \tilde{g}_k(t)\cdot\exp(-|k\sigma_v t|).$$

Periodic lattice

$$\frac{d^2\hat{a}}{d\hat{s}^2} - k_{sc}{}^2\hat{a}^{-1} - k_\beta{}^2\hat{a}^{-3} = 0;$$

$$\frac{d^2}{d\hat{s}^2}\tilde{g}_k + 2\frac{k_{sc}^2}{\hat{a}(\hat{s})^2}\cdot\tilde{g}_k = 0; \qquad \hat{a} = \frac{a}{a_o} \geq 1;$$

$$\hat{s} = \frac{s}{l}; \qquad \hat{s} \in \{-1, 1\},$$

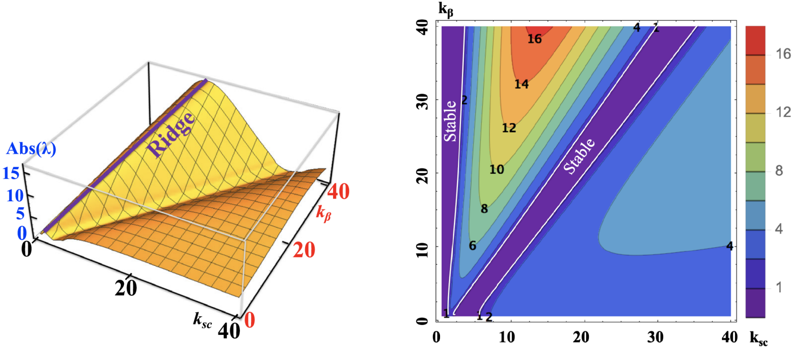
$$k_{sc} = \sqrt{\frac{2}{\beta_o{}^3\gamma_o{}^3}\frac{I_o}{I_A}\frac{l^2}{a_o{}^2}}; \qquad k_\beta = \frac{\varepsilon l}{a_o{}^2}.$$

$$\ddot{\tilde{n}} + \omega_p^2(t)\tilde{n} = 0$$

Cell

$$\mathbf{M}_c = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{11} \end{bmatrix}; \quad \lambda_1 = \lambda_2{}^{-1} = m_{11} - \sqrt{m_{11}^2 - 1};$$

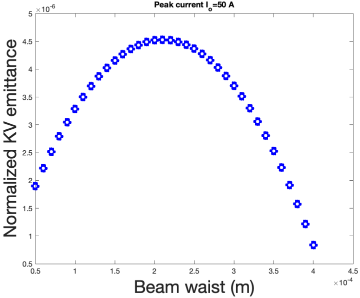
$$\tilde{g}_k(mT) = \frac{\lambda_1^m + \lambda_1^{-m}}{2} \tilde{g}_k(0) - \frac{m_{12}}{\sqrt{m_{11}^2 - 1}} \frac{\lambda_1^m - \lambda_1^{-m}}{2} \dot{\tilde{g}}_k(0),$$

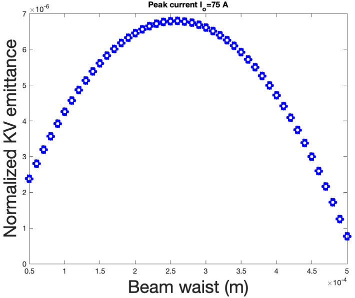


$$k_\beta \approx 3 \cdot (k_{sc} - 1.2) \quad \lambda \propto 1.25 k_{sc} \approx 1.5 + 0.413 k_\beta$$

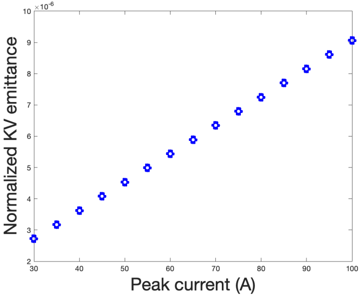
Cell

$$k_{sc} = \sqrt{\frac{2}{\beta_o{}^3 \gamma_o{}^3} \frac{I_o}{I_A} \frac{l^2}{a_o{}^2}}; \qquad k_\beta = \frac{\varepsilon l}{a_o{}^2}.$$

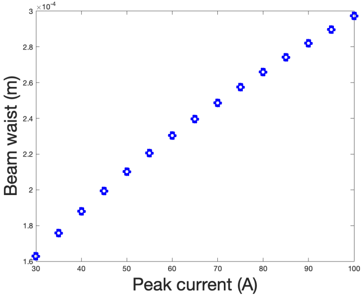




Cell



×10^-6
10
9
8
7
6
5
4
3
2
Normalized KV emittance
30
40
50
60
70
80
90
100
Peak current (A)



×10^-4
3
2.8
2.6
2.4
2.2
2
1.8
1.6
Beam waist (m)
30
40
50
60
70
80
90
100
Peak current (A)

Landau damping

$$k_{\max} = \frac{\ln \lambda}{T\sigma_v}; \qquad \omega_{\max} = \frac{v_o}{2l} \cdot \frac{{\gamma_o}^3}{\sigma_\gamma} \ln \lambda,$$

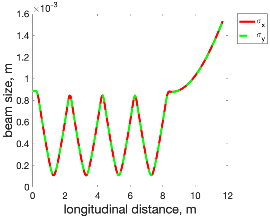
PCI, general 3D case

$$\tilde{\rho}_{\vec{k}}(t) = e\int e^{-i\vec{k}\cdot\overleftrightarrow{\mathbf{B}}(t)\cdot\vec{P}}\breve{f}_{\vec{k}(0)}(P,0)dP^3 - 4\pi e^2 n_o \beta_o(t)\int_o^t \frac{\tilde{\rho}_{\vec{k}(\tau)}(\tau)}{\det \mathbf{A}(\tau)}\frac{d\tau}{\beta_o{}^2(\tau)}\frac{\vec{k}\cdot\overleftrightarrow{\mathbf{U}}(t)\cdot\vec{k} - \vec{k}(\tau)\cdot\overleftrightarrow{\mathbf{U}}(\tau)\cdot\vec{k}(\tau)}{\gamma_o(\tau)^2[\vec{k}^2(\tau) - k_z{}^2(\tau)\beta_o{}^2(\tau)]}$$
$$\times\int e^{i(\vec{k}(\tau)\cdot\overleftrightarrow{\mathbf{B}}(\tau) - \vec{k}\cdot\overleftrightarrow{\mathbf{B}}(t))\cdot\vec{P}}F_o(P)dP^3,$$

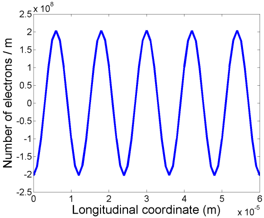
where $\mathbf{U} = \mathbf{U}^T = \mathbf{A}^{-1}\mathbf{B}^T$ and

$$k \equiv \{k_x, k_y, k_z\} \equiv \{k_1, k_2, k_3\}; \quad k(t) = k(0)\mathbf{A}^{-1}(t)$$

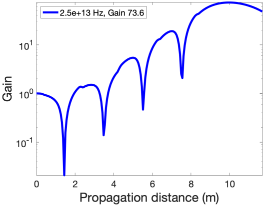
Periodic PCA



(a) Beam size

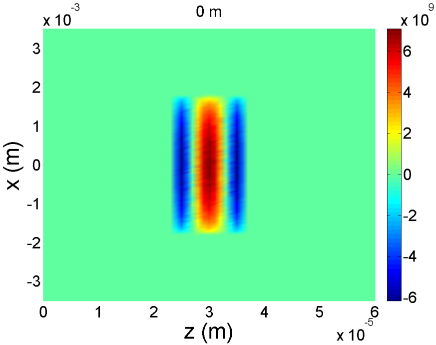


(b) Initial signal at 2.5e+13 Hz

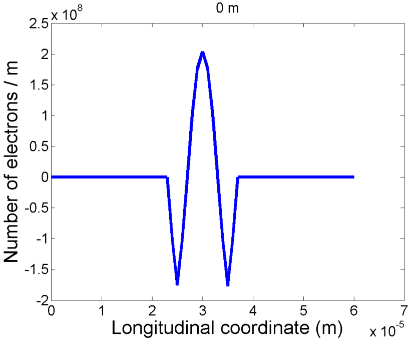


(c) Gain of 2.5e+13 Hz signal

Evolution of density modulation

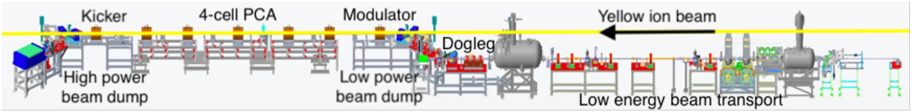


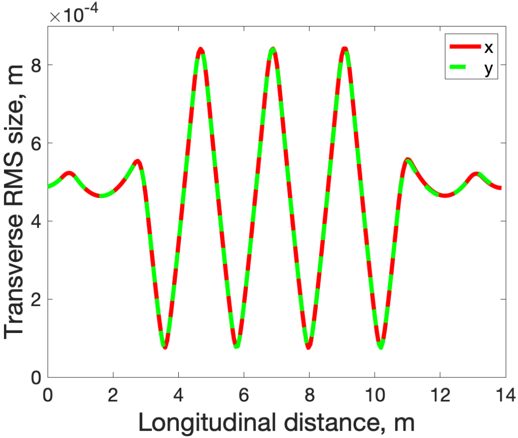
(a) z-x plot



(b) z plot

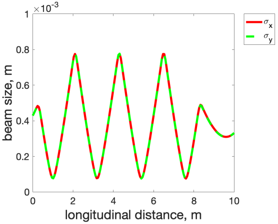
PCA-based CeC experiment



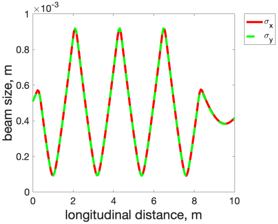


CeC PCA lattice design

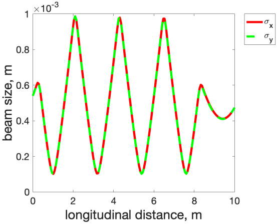
- I=50 A, a_{min}=1.5e-4 m, $\varepsilon_{n,KV}$=5 μm
- I=75 A, a_{min}=1.8e-4 m, $\varepsilon_{n,KV}$=7 μm
- I=100 A, a_{min}=2e-4 m, $\varepsilon_{n,KV}$=8 μm



(a) 50 A

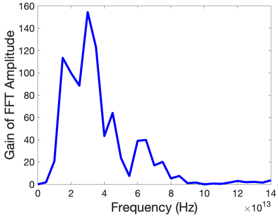


(b) 75 A

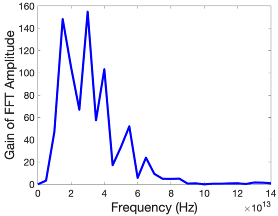


(c) 100 A

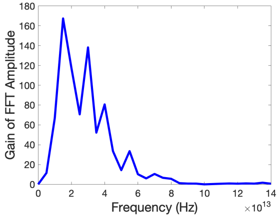
CeC PCA gain



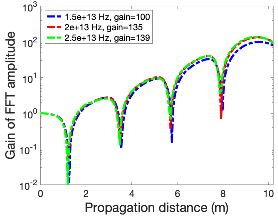
(a) 50 A



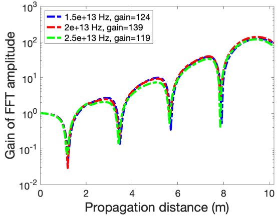
(b) 75 A



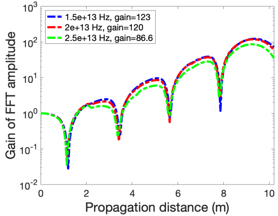
(c) 100 A



(d) 50 A

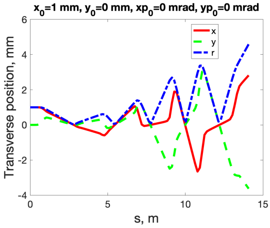


(e) 75 A

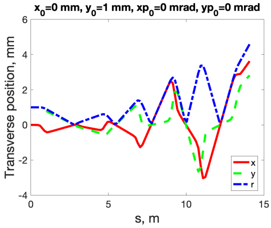


(f) 100 A

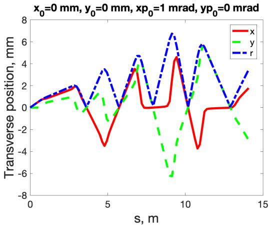
Single particle orbit with initial offsets



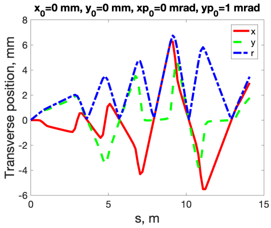
(a) x_0=1 mm



(b) y_0=1 mm

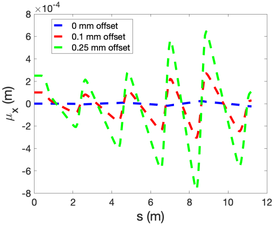


(c) x'_0=1 mrad

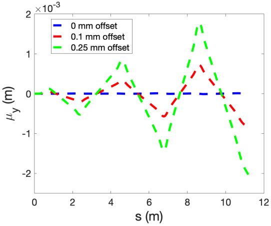


(d) y'_0=1 mrad

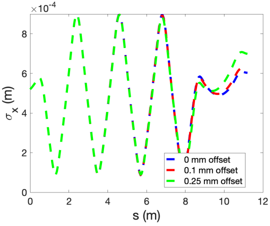
Transverse beam position and size with initial offsets



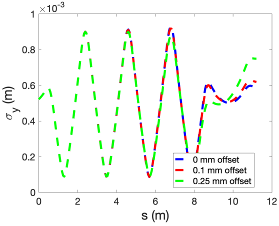
(a) x position



(b) y position

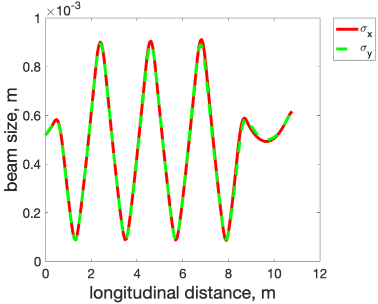


(c) x size

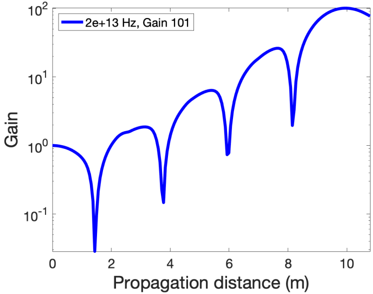


(d) y size

Sensitivity study of orbit, no offset

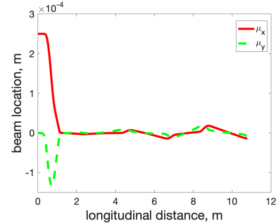


(a) Beam size

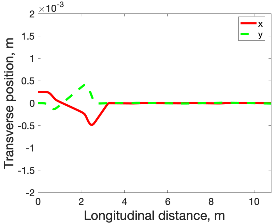


(b) PCA gain

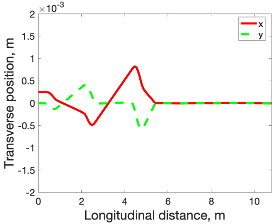
Sensitivity study of orbit, with initial offset



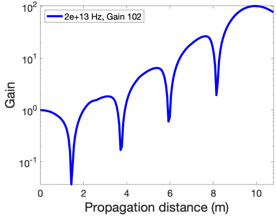
(a) Position



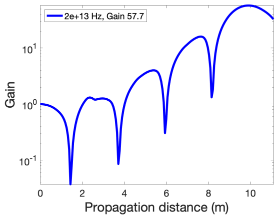
(b) Position



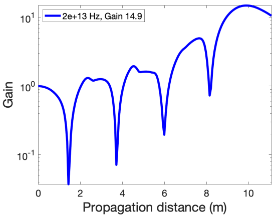
(c) Position



(d) PCA gain

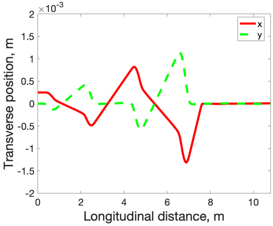


(e) PCA gain

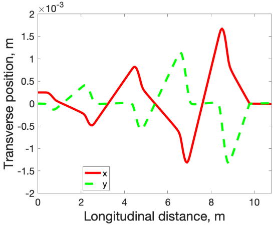


(f) PCA gain

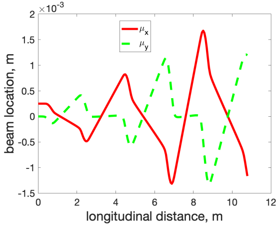
Sensitivity study of orbit, with initial offset



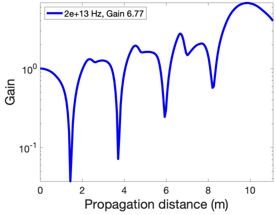
(a) Position



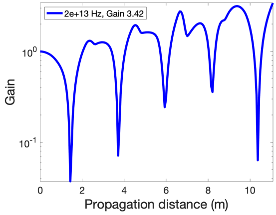
(b) Position



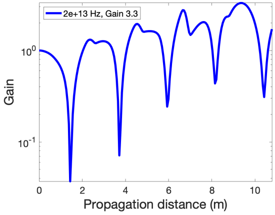
(c) Position



(d) PCA gain

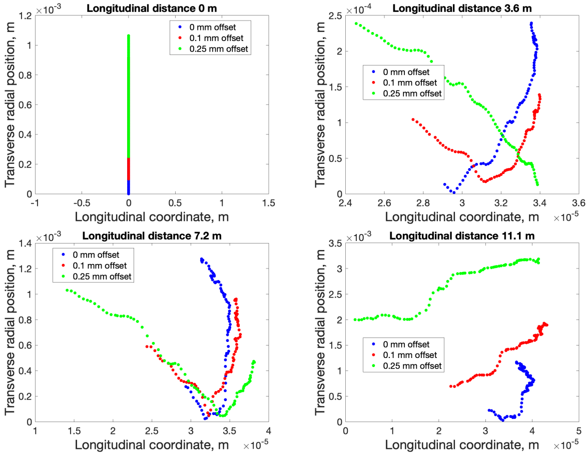


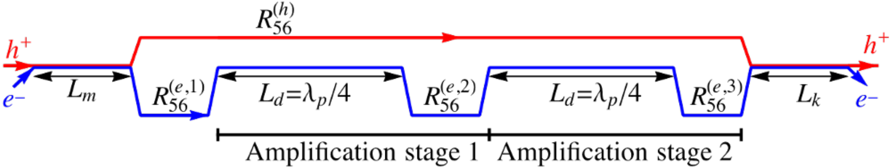
(e) PCA gain

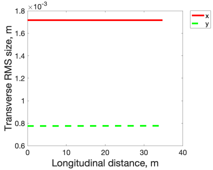


(f) PCA gain

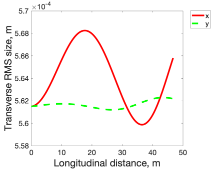
Track a line of particles



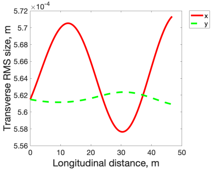




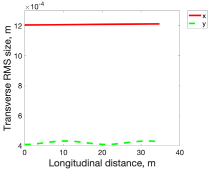
(a) Modulator



(b) First stage



(c) Second stage



(d) Kicker

G. Stupakov, and P. Baxevanis. Physical Review Accelerators and Beams 22.3 (2019): 034401.

Interaction of charged Gaussian disk

$$f_z(z) = -\frac{Ze^2}{\Sigma^2}\Phi\left(\frac{z\gamma}{\Sigma}\right),$$

$$F_z(z) = \frac{e^2}{\Sigma_p^2}\Phi\left(\frac{\gamma z}{\Sigma_p}\right).$$

$$\Phi(x) = \frac{1}{2}\left[\frac{x}{|x|} - \frac{x\sqrt{\pi}}{2}\exp\left(\frac{1}{4}x^2\right)\mathrm{erfc}\left(\frac{1}{2}|x|\right)\right],$$

Interaction of charged Gaussian disk

$$f_z(z) = -\frac{Ze^2}{\Sigma^2}\Phi\left(\frac{z\gamma}{\Sigma}\right),$$

$$F_z(z) = \frac{e^2}{\Sigma_p^2}\Phi\left(\frac{\gamma z}{\Sigma_p}\right).$$

$$\Delta\eta(z) = -\frac{Zr_e L_m}{\gamma\Sigma^2}\Phi\left(\frac{z\gamma}{\Sigma}\right),$$

Interaction of charged Gaussian disk

$$\Phi(x) = \frac{1}{2}\left[\frac{x}{|x|} - \frac{x\sqrt{\pi}}{2}\exp\left(\frac{1}{4}x^2\right)\operatorname{erfc}\left(\frac{1}{2}|x|\right)\right]$$

$$H(\varkappa) = \frac{i}{2}\int_{-\infty}^{\infty} dx\Phi(x)e^{-i\varkappa x} = \int_0^{\infty} dx\Phi(x)\sin(\varkappa x).$$

Plasma oscillation

$$\frac{\partial \delta f}{\partial t} + \frac{c\eta}{\gamma^2}\frac{\partial \delta f}{\partial z} + \dot{\eta} n_0 F_0'(\eta) = 0,$$

$$\dot{\eta} = \frac{1}{\gamma m_e c}\int_{-\infty}^{\infty} dz' \delta n(z', t) F_z(z - z'),$$

$$\delta n(z, t) = \int_{-\infty}^{\infty} d\eta\, \delta f(z, \eta, t)$$

Plasma oscillation

$$\delta\hat{f}_k(\eta,t) = \int_{-\infty}^{\infty} dz\, e^{-ikz}\delta f(z,\eta,t),$$

$$\delta\hat{n}_k(t) = \int_{-\infty}^{\infty} dz\, e^{-ikz}\delta n(z,t),$$

$$\frac{\partial\delta\hat{f}_k}{\partial t} + \frac{ikc\eta}{\gamma^2}\delta\hat{f}_k + \zeta(k)\delta\hat{n}_k n_0 F_0'(\eta) = 0,$$

$$\zeta(k) = \frac{1}{\gamma mc}\int_{-\infty}^{\infty} d\xi e^{-ik\xi}F_z(\xi) = -\frac{2ie^2}{\Sigma_p\gamma^2 mc}H\left(\frac{k\Sigma_p}{\gamma}\right)$$

Plasma oscillation

$$\frac{\partial \delta \hat{f}_k}{\partial t} + \frac{ikc\eta}{\gamma^2}\delta \hat{f}_k + \zeta(k)\delta \hat{n}_k n_0 F_0'(\eta) = 0,$$

Neglect second term

$$\delta \hat{f}_k(\eta, t) = \delta \hat{f}_k(\eta, 0) - \zeta(k) n_0 F_0'(\eta) \int_0^t dt' \delta \hat{n}_k(t').$$

$$\frac{\partial \delta \hat{f}_k}{\partial t} + \frac{ikc\eta}{\gamma^2}\delta \hat{f}_k + \zeta(k)\delta \hat{n}_k n_0 F_0'(\eta) = 0,$$

Integrate over η

$$\frac{d\delta \hat{n}_k}{dt} + \frac{ikc}{\gamma^2}\delta \hat{q}_k = 0,$$

$$\delta \hat{q}_k = \int_{-\infty}^{\infty} d\eta\, \eta \delta \hat{f}_k$$

$$\frac{d\delta \hat{q}_k}{dt} - \zeta(k)\delta \hat{n}_k n_0 = 0,$$

Plasma oscillation

$$\frac{d^2\delta\hat{n}_k}{dt^2} + \frac{ikc}{\gamma^2}\zeta(k)n_0\delta\hat{n}_k = 0,$$

$$\omega_p^2 = \frac{ikcn_0}{\gamma^2}\zeta(k) = \frac{2kn_0e^2}{\Sigma_p\gamma^4 m}H\left(\frac{k\Sigma_p}{\gamma}\right) = 2\Omega^2\varkappa_p H(\varkappa_p),$$

$$\Omega^2 = \frac{n_0e^2}{m\Sigma_p^2\gamma^3}$$

MBEC gain

$$\delta\hat{n}_k = \delta\hat{n}_k(0)\cos(\omega_p t) - \frac{ikc}{\gamma^2\omega_p}\delta\hat{q}_k(0)\sin(\omega_p t),$$

$$\frac{c\sigma_e}{\gamma\Omega\Sigma_p} \sim \sigma_e\sqrt{\frac{\gamma I_A}{I_e}} \ll 1,$$

$$\delta\hat{n}_k \approx \delta\hat{n}_k(0)\cos(\omega_p t).$$

MBEC gain

$$\delta\hat{f}_k(\eta,t) = \delta\hat{f}_k(\eta,0) - \frac{1}{\omega_p}\zeta(k)n_0F_0'(\eta)\delta\hat{n}_k(0)\sin(\omega_p t).$$

$$\delta\hat{n}_k^{(2)} = \int_{-\infty}^{\infty} d\eta\delta\hat{f}_k e^{-ikR_{56}^{(e,2)}\eta}$$

$$= -\frac{1}{\omega_p}\zeta(k)n_0\delta\hat{n}_k(0)g(k)\sin\left(\frac{\omega_p L_d}{c}\right),$$

$$g(k) = \int_{-\infty}^{\infty} d\eta F_0'(\eta)e^{-ikR_{56}^{(e,2)}\eta} = ikR_{56}^{(e,2)}e^{-k^2(R_{56}^{(e,2)})^2\sigma_e^2/2}.$$

MBEC gain

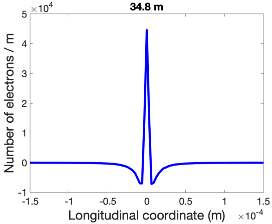
$$G = \delta \hat{n}_k^{(2)} / \delta \hat{n}_k(0)$$

$$G = -\frac{1}{\sigma_e} \sqrt{\frac{2I_e}{\gamma I_A}} \sqrt{\varkappa_p H(\varkappa_p)} q_p e^{-\varkappa_p^2 q_p^2 / 2} \sin\left(\frac{\omega_p L_d}{c}\right),$$

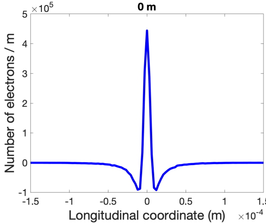
$$G_{\max} = -\frac{1}{\sigma_e} \sqrt{\frac{2I_e}{\gamma I_A}} \sqrt{\frac{2H(\varkappa_p)}{e\varkappa_p}} \sin\left(\frac{\omega_p L_d}{c}\right),$$

$$q_p = 1/\varkappa_p$$

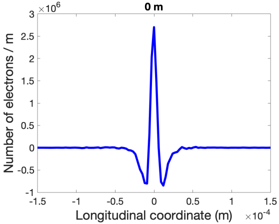
MBEC gain



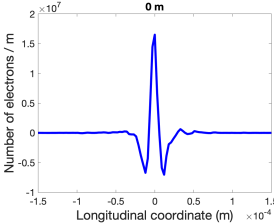
(a) Exit of modulator



(b) After 1st chicane

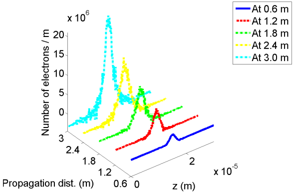


(c) After 2nd chicane

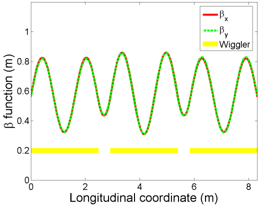


(d) After 3rd chicane

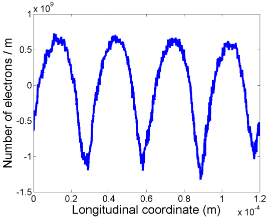
Density modulation in FEL-based CeC



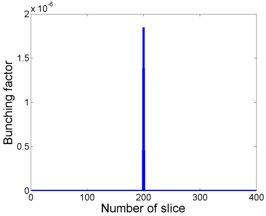
(a) Exit of modulator



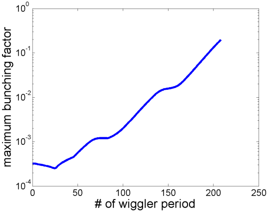
(b) FEL amplifier



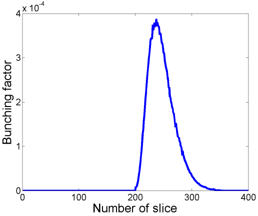
(c) Entrance of kicker



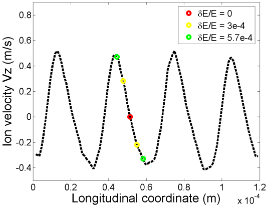
(d) Entrance of FEL



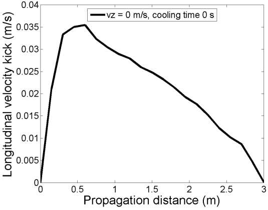
(e) FEL amplifier



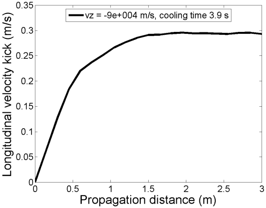
(f) Exit of FEL



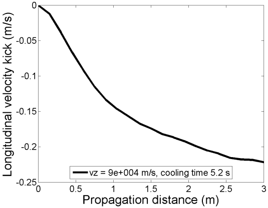
(a) Ions at different locations



(b) Ion with reference energy

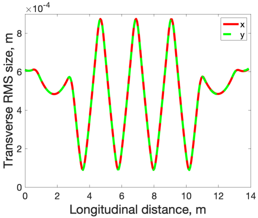


(c) Ion with lower energy

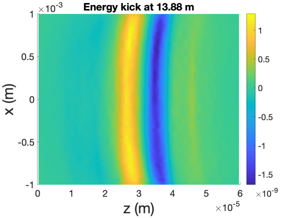


(d) Ion with higher energy

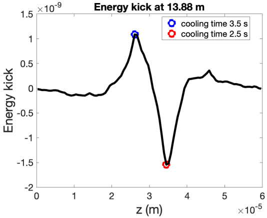
PCA-based CeC, beam current 75 A



(a) Beam size

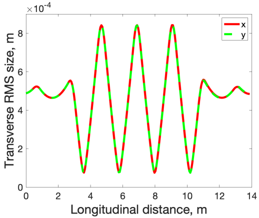


(b) Cooing force

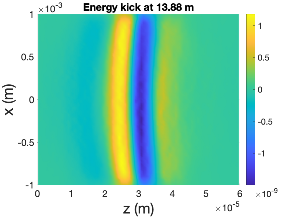


(c) Cooing force

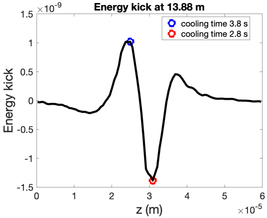
PCA-based CeC, beam current 50 A



(a) Beam size

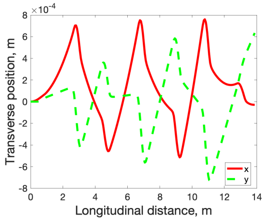


(b) Cooing force

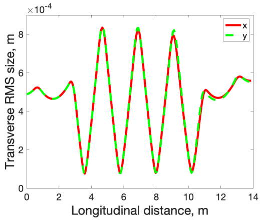


(c) Cooing force

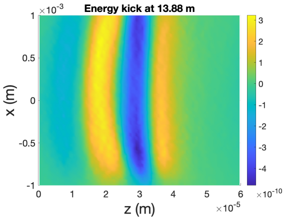
PCA-based CeC, earth field



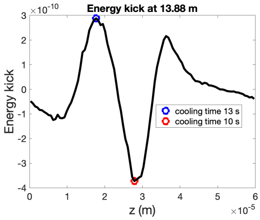
(a) Beam position



(b) Beam size

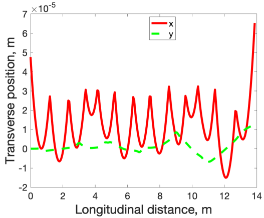


(c) Cooing force

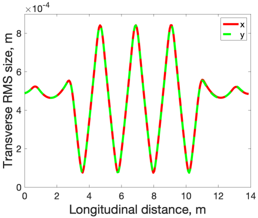


(d) Cooing force

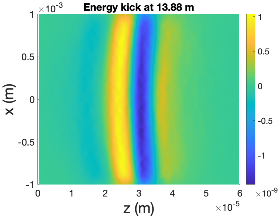
PCA-based CeC, earth field, orbit correction



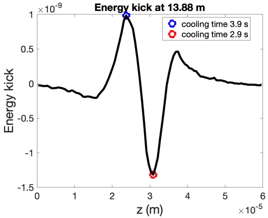
(a) Beam position



(b) Beam size



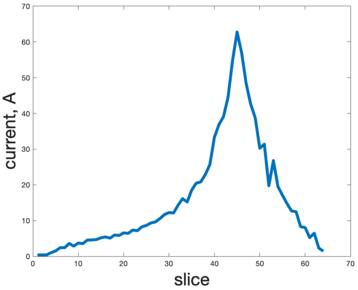
(c) Cooing force



(d) Cooing force

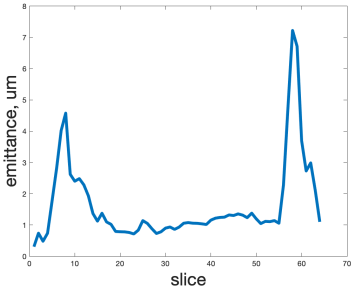
PCA-based CeC

- Beam dynamics simulations propagate beam starting from the gun.
- Take slice parameters from beam dynamics simulations at the entrance of modulator.



current, A
slice
0
10
20
30
40
50
60
70

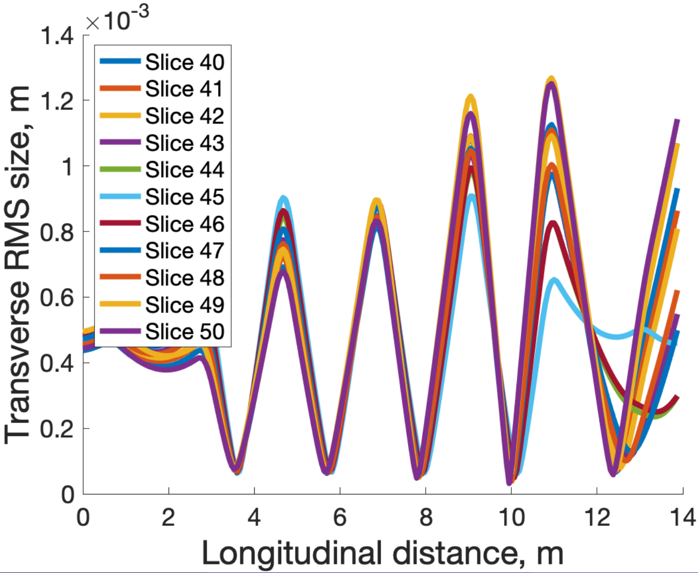
(a)



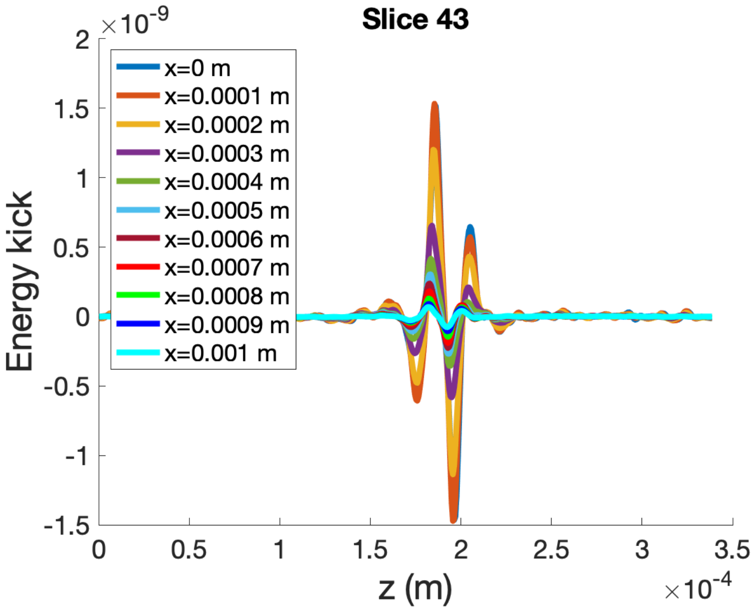
emittance, um
slice
0
1
2
3
4
5
6
7
8
10
20
30
40
50
60
70

(b)

Beam size

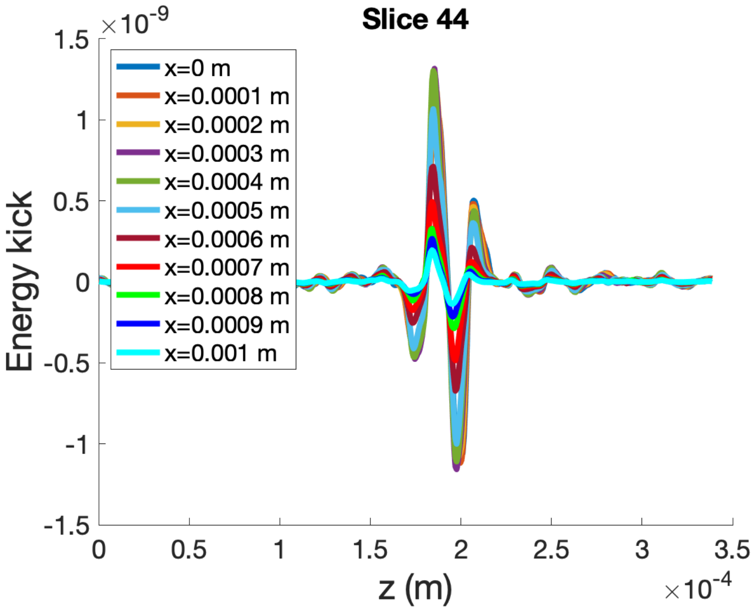


×10^-3
Transverse RMS size, m
Longitudinal distance, m
Slice 40
Slice 41
Slice 42
Slice 43
Slice 44
Slice 45
Slice 46
Slice 47
Slice 48
Slice 49
Slice 50
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
2
4
6
8
10
12
14

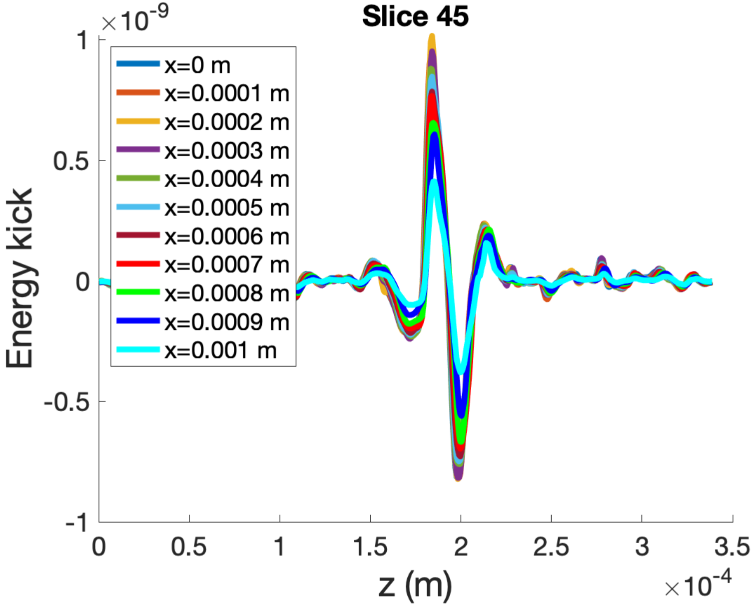


Slice 43
×10^-9
2
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

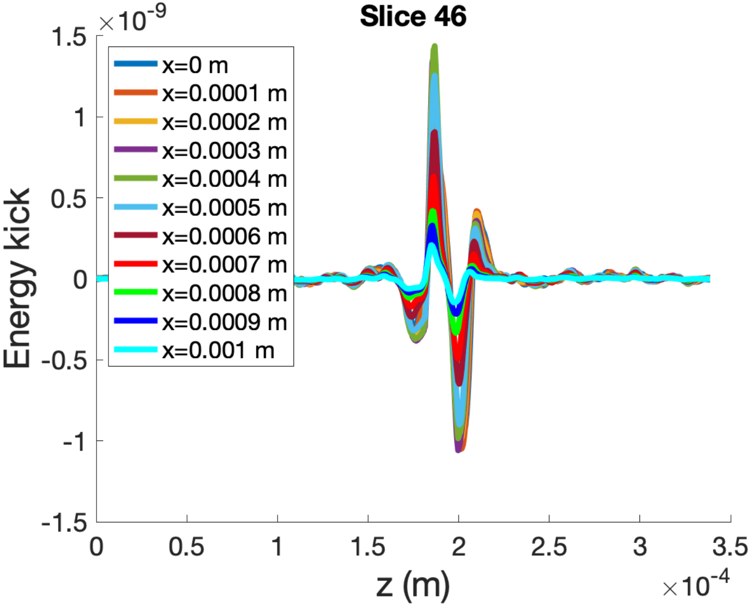
(a)


Slice 44
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

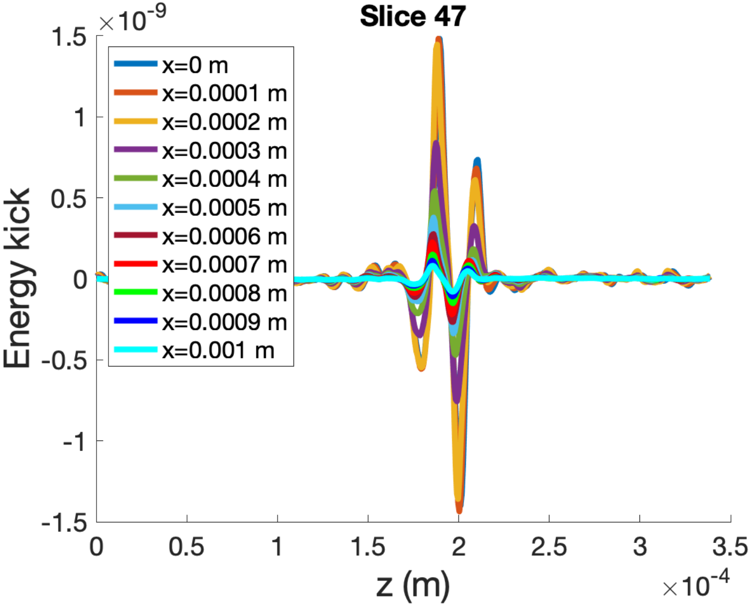
(a)


Slice 45
$\times 10^{-9}$
1
0.5
0
-0.5
-1
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
$\times 10^{-4}$

(a)


Slice 46
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

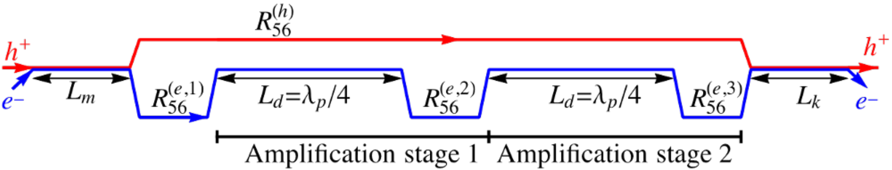
(a)

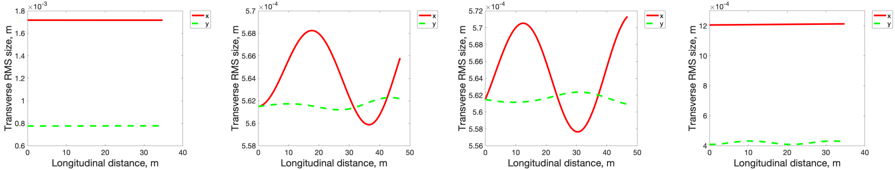


Slice 47
×10^-9
1.5
1
0.5
0
-0.5
-1
-1.5
Energy kick
x=0 m
x=0.0001 m
x=0.0002 m
x=0.0003 m
x=0.0004 m
x=0.0005 m
x=0.0006 m
x=0.0007 m
x=0.0008 m
x=0.0009 m
x=0.001 m
0
0.5
1
1.5
2
2.5
3
3.5
z (m)
×10^-4

(a)

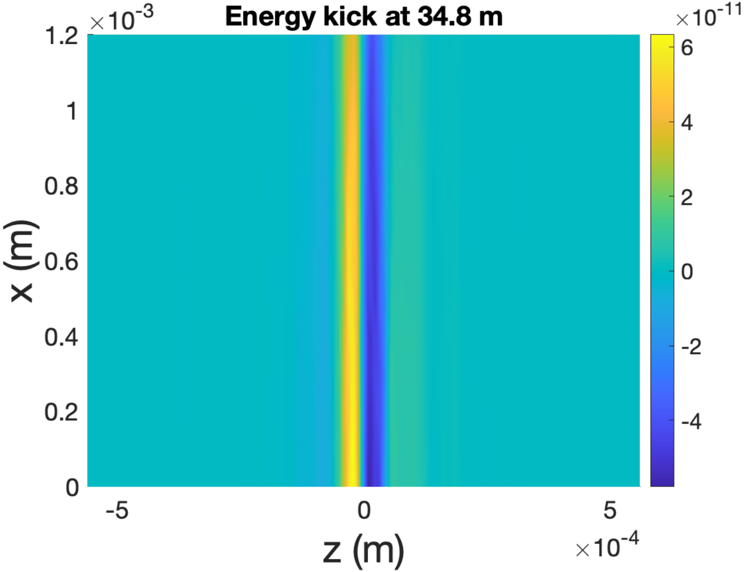
MBEC



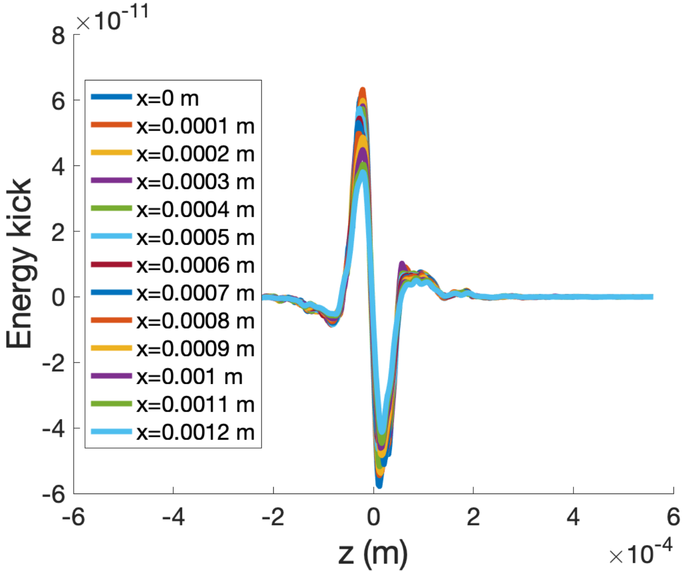


(a) Modulator (b) First stage (c) Second stage (d) Kicker

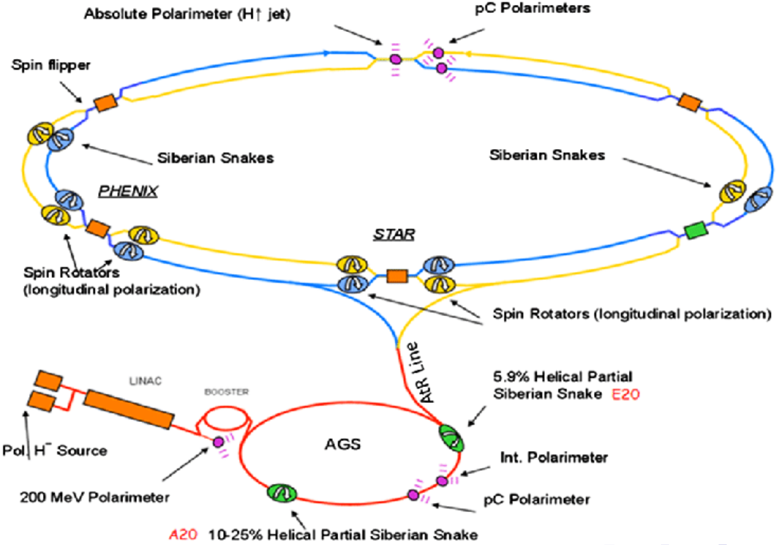
MBEC cooling force



MBEC cooling force



The Relativistic Heavy Ion Collider at the Brookhaven National Laboratory



Cooling time

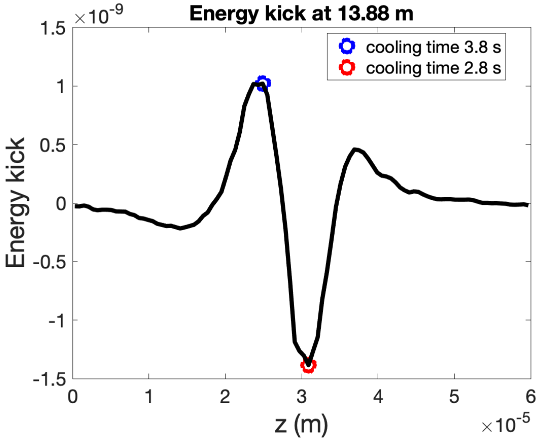
$$\frac{1}{\tau_{v_i}} \equiv -\frac{1}{v_i}\frac{dv_i}{dt}$$

If τ is independent of t

$$v_i(t) = v_i(0)\exp\left(-\frac{t}{\tau_{v_i}}\right)$$

τ is cooling time, $1/\tau$ is cooling rate.

PCA-based CeC, beam current 50 A



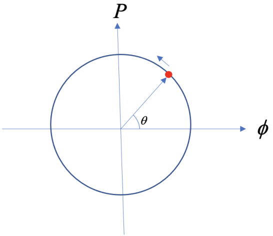
(a) Cooing force

x-axis $z = R_{56} * \delta\gamma$

y-axis $\Delta\delta\gamma = f(\delta\gamma) \approx k_0\delta\gamma$

$$
\begin{aligned}
\frac{1}{\tau} &= -\frac{1}{\delta\gamma}\frac{d\delta\gamma}{dt} \\
&= -\frac{1}{\delta\gamma}\frac{\Delta\delta\gamma}{T_{rev}} \\
&\approx -\frac{1}{\delta\gamma}\frac{k_0\delta\gamma}{T_{rev}} \\
&= -\frac{k_0}{T_{rev}}
\end{aligned}
$$

Averaging longitudinal cooling over synchrotron oscillation (linear cooling force)



Action-angle variable: $P \equiv -h\frac{|\eta|}{\nu_s}\frac{\Delta p}{p} = \sqrt{2I}\sin\theta \qquad \phi \equiv \omega_{rf}\tau = \sqrt{2I}\cos\theta$

Reduction of action due to cooling: $\Delta I_c = \frac{1}{2}\Delta\left(P^2+\phi^2\right) = P\Delta P_c \qquad \Delta P_c = \frac{h|\eta|}{\nu_s\gamma}\Delta\delta\gamma_c$

Assuming cooling force is linear, $\Delta\delta\gamma_c = -\zeta_0 T_{rev}\delta\gamma \qquad \Delta P_c = -\zeta_0 T_{rev} P$

the action reduction becomes $\Delta I_c = -\zeta_0 T_{rev} P^2 = -2I\zeta_0 T_{rev}\sin^2\theta$

The average cooling rate is given by

$$\zeta(I) = -\frac{1}{I}\left\langle\frac{\Delta I_c}{T_{rev}}\right\rangle_{T_s} = \zeta_0\bar{\zeta}(I)$$

$$\bar{\zeta}(I) = \frac{1}{2\pi}\int_0^{2\pi}\sin^2\theta d\theta = \frac{1}{2}$$

For the linear cooling force, synchrotron motion reduces the averaged cooling rate by a factor of 2.

G. Wang. Physical Review Accelerators and Beams 22.11 (2019): 111002
1D Fokker-Planck equation

$$\frac{\partial}{\partial t}F(I,t)-\frac{\partial}{\partial I}[\zeta(I)\cdot I\cdot F(I,t)]-\frac{\partial}{\partial I}\left(I\cdot D(I)\cdot\frac{\partial F(I,t)}{\partial I}\right)=0,$$

$$\zeta(I)=-\frac{1}{I}\left\langle\frac{\Delta I_c}{T_{\mathrm{rev}}}\right\rangle_{T_s}\quad D(I)=\frac{1}{I}\frac{\langle\Delta I_d^2\rangle_{T_s}}{2T_{\mathrm{rev}}}$$

$F(I,t)$ is longitudinal phase space density, $\zeta(I)$ is cooling rate, $D(I)$ is diffusion coefficient.

Fokker-Planck equation

$$\phi = \sqrt{2I}\sin w,$$

$$P \equiv -h\frac{|\eta|}{\nu_s}\frac{\Delta p}{p}$$

$$H_0 = \frac{1}{2}\omega_0\nu_s P^2 + \omega_0\nu_s\frac{1}{2}\phi^2 = \omega_0\nu_s I,$$

Fokker-Planck equation

$$\int_0^\infty F(I,t)dI = \frac{N}{2\pi},$$

$$\begin{aligned}\int_{-\infty}^{\infty} F\left(\frac{P^2+\phi^2}{2},t\right)dPd\phi &= \int_{-\infty}^{\infty} F\left(\frac{P^2+\phi^2}{2},t\right)\frac{\partial(P,\phi)}{\partial(I,w)}dIdw \\ &= \int_0^{2\pi}\int_0^{\infty} F(I,t)dIdw \\ &= N,\end{aligned}$$

$$K(P,\phi,t) = F\left(\frac{1}{2}P^2+\frac{1}{2}\phi^2,t\right).$$

$$F_{\text{eq}}(I) = A\exp\left\{-\int\frac{\zeta(I)}{D(I)}dI\right\}.$$

Zero diffusion

$$\frac{\partial}{\partial t}\tilde{F}(I,t) - \alpha(I)\frac{\partial}{\partial I}\tilde{F}(I,t) = 0,$$

$$\tilde{F}(I,t) \equiv \zeta(I) \cdot I \cdot F(I,t),$$

$$\alpha(I) \equiv I \cdot \zeta(I).$$

$$F(I,t) = \frac{h^{-1}(C)\zeta[h^{-1}(C)]F_0[h^{-1}(C)]}{\zeta(I) \cdot I},$$

$$C = t - t_0 + \int \frac{dI}{\alpha(I)},$$

$$h(I) \equiv \int \frac{dI}{\alpha(I)}.$$

Zero diffusion

$$\zeta(I) = \zeta_0 \frac{I_e}{I + I_e},$$

$$h(I) = \frac{1}{\zeta_0} \int \frac{1 + \frac{I}{I_e}}{I} dI = \frac{1}{\zeta_0} \ln \left[\frac{I}{I_e} \exp\left(\frac{I}{I_e} \right) \right],$$

$$h(I) = \frac{1}{\zeta_0} \ln \left[\frac{I}{I_e} \exp\left(\frac{I}{I_e} \right) \right] = C.$$

$$I = h^{-1}(C) = I_e P_{\log}[\exp(\zeta_0 C)],$$

$P_{log}(x) = w^{-1}(x)$ is the inverse function of $w(x) = xe^x$

Zero diffusion

$$F(I,t) = \left(1+\frac{I_e}{I}\right)\cdot\frac{P_{\log}[\exp(\zeta_0 C)]F_0\{I_e P_{\log}[\exp(\zeta_0 C)]\}}{1+P_{\log}[\exp(\zeta_0 C)]}.$$

$$F_0(I) = \frac{N}{2\pi I_{\text{ion}}}\exp\left(-\frac{I}{I_{\text{ion}}}\right),$$

$$C = t + \frac{1}{\zeta_0}\ln\left[\frac{I}{I_e}\exp\left(\frac{I}{I_e}\right)\right].$$

$$F(I,t) = \frac{N}{2\pi I_{\text{ion}}} g\left(\frac{I}{I_e}\right),$$

$$g(\eta,t) = \left(1+\frac{1}{\eta}\right)\frac{P_{\log}[\eta\exp(\zeta_0 t+\eta)]}{1+P_{\log}[\eta\exp(\zeta_0 t+\eta)]} \times\exp\left(-\frac{I_e}{I_{\text{ion}}}P_{\log}[\eta\exp(\zeta_0 t+\eta)]\right).$$

Zero diffusion

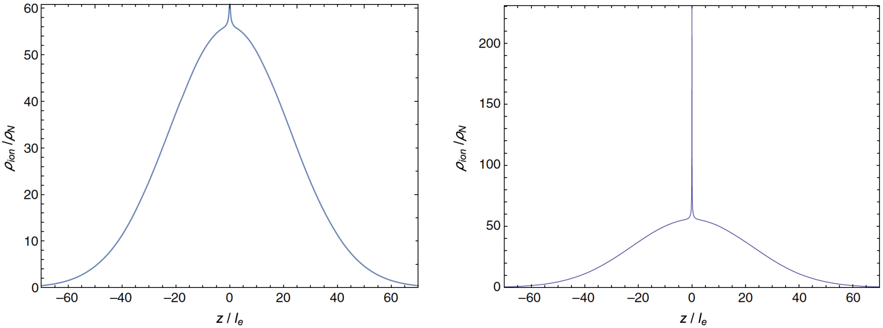
$$\rho_{\text{ion}}(t,z) = k_{rf} \int_{-\infty}^{\infty} F\left(\frac{1}{2}k_{rf}^2 z^2 + \frac{1}{2}P^2, t\right) dP$$

$$= \rho_N \int_{-\infty}^{\infty} g\left(\frac{z^2}{l_e^2} + y^2, t\right) dy,$$

$$k_{rf} = 2\pi/\lambda_{rf}$$

$$\rho_N \equiv \frac{N}{2\pi}\frac{l_e}{\sigma_i^2}.$$

Zero diffusion



Ion profile after $t = 2\zeta_0^{-1}$ (left) and $t = 10\zeta_0^{-1}$ (right).

Finite diffusion, numerical solution

$$
\begin{aligned}
\bar{D}(r) &= D(r^2 I_e)/D(0) \\
\bar{\zeta} &= \zeta(r^2 I_e)\zeta_0 \\
\bar{D}_0 &= D(0)/(\zeta_0 I_e) \\
\bar{t} &= t\zeta_0 \\
R(r,\bar{t}) &= \frac{2\pi I_{ion}}{N} F(r^2 I_e, \bar{t}\zeta_0^{-1}) \\
r &= \sqrt{I/I_e}
\end{aligned}
$$

Finite diffusion, numerical solution

$$r\frac{\partial R(r,\bar{t})}{\partial \bar{t}} + \alpha(r)\frac{\partial R(r,\bar{t})}{\partial r} + \beta(r)\frac{\partial^2 R(r,\bar{t})}{\partial r^2} + \gamma(r)R(r,\bar{t}) = 0,$$

$$\alpha(r) = -\frac{r^2}{2}\bar{\zeta}(r) - \frac{\bar{D}_0}{4}\bar{D}(r) - \frac{\bar{D}_0 r}{4}\frac{d\bar{D}(r)}{dr},$$

$$\beta(r) = -\frac{\bar{D}_0 r}{4}\bar{D}(r),$$

$$\gamma(r) = -\frac{r^2}{2}\frac{d\bar{\zeta}(r)}{dr} - r\bar{\zeta}(r).$$

Finite diffusion, numerical solution

$2 \le j < N$:

$$\frac{\beta_j}{\Delta r^2} R_{j-1}^{n+1} + \left(\frac{r_j}{\Delta \bar{t}} - \frac{\alpha_j}{\Delta r} - 2\frac{\beta_j}{\Delta r^2} + \gamma_j \right) R_j^{n+1} + \left(\frac{\alpha_j}{\Delta r} + \frac{\beta_j}{\Delta r^2} \right) R_{j+1}^{n+1} = \frac{r_j}{\Delta \bar{t}} R_j^n$$

$j = 1$:

$$-\frac{\alpha_j}{\Delta r} R_1^{n+1} + \frac{\alpha_j}{\Delta r} R_2^{n+1} = 0$$

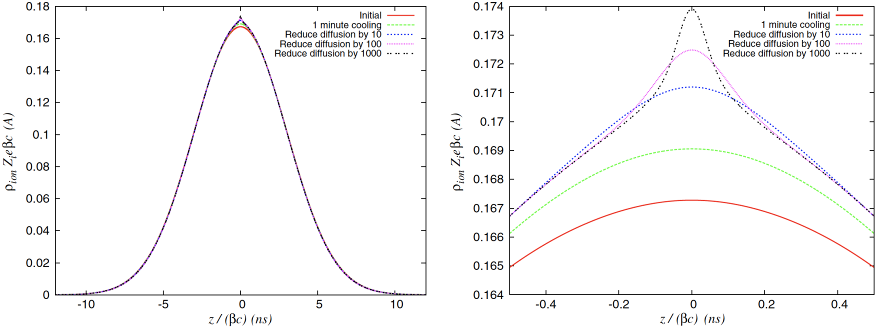
$j = N$:

$$\frac{\beta_N}{\Delta r^2} R_{N-1}^{n+1} + \left(\gamma_N + \frac{r_N}{\Delta t} - \frac{\alpha_N}{\Delta r} - 2\frac{\beta_N}{\Delta r^2} \right) R_N^{n+1} = \frac{r_N}{\Delta \bar{t}} R_N^n$$

Finite diffusion, numerical solution

$$\rho(z,t) = \rho_N \int_{-\infty}^{\infty} R\left(\sqrt{y^2 + \frac{z^2}{l_e^2}}, t\zeta_0\right) dy.$$

FEL-based CeC



Ion profile after 1 min of cooling.