PHY 564
Advanced Accelerator Physics
Lecture 27

Nonlinear elements and nonlinear dynamics. Part III

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Let's now introduce one more object, **Lie operator** $:f:$ defined as :

$$:f:g=[f,g];$$
$$:f:^0=g;\ :f:^2 g=[f[f,g]];\ \ :f:^{n+1} g=[f,:f:^n g]. \tag{26-16}$$

together with its powers. Obviously the Lie operator and its power are linear operators

$$:f:^n(a\cdot g+b\cdot h)=(a\cdot :f:^n g+b\cdot :f:^n h) \tag{26-17}$$

since functions the operator acts upon appearing linearly. Similarly, since $:f:$ is a differential operator, the following rule

$$:f:(g\cdot h)=(:f:g)\cdot h+g\cdot(:f:h) \tag{26-18}$$

is trivial to prove. Furthermore, similarly to the ordinary differentiation $:f:^n$ obeys Leibnitz rule

$$:f:^n(g\cdot h)=\sum_{m=0}^{n} C_m^n(:f:^m g)(:f:^{n-m} h);\ C_m^n=\frac{n!}{m!(n-m)!}. \tag{26-19}$$

Finally, the Jacoby identity

$$[f,[g,h]]=[[f,g],h]+[g,[f,h]] \tag{26-20}$$

which I recommend you to prove as an exercise (not a home work!) is equivalent to identity for Lie operators

$$:f:[g,h]=[:f:g,h]+[g,:f:h] \tag{26-21}$$

Now we will convert linear Lie operators into a linear algebra by defining their product (algebraic, not simple multiplication) of Lie operators:

$$\{:f:,:g:\}=:f::g:-:g::f: \tag{26-22}$$

or using Jacoby identity

$$\{:f:,:g:\}h=(:f::g:-:g::f:)h=[:f:,:g:]h=:[f,g]:h \tag{26-23}$$

with $:[f,g]:$ being a compact form of the product of two operators.

Hence we established homomorphism between the Lie algebra of function (Poisson brackets) and Lie operators. Naturally (26-22) turns the set Lie operators into Lie algebra.

We are not done yet with definitions: we define Lie transform as an exponent of the Lie operators:

$$\exp(:f:)=\sum_{n=0}^{\infty}\frac{:f:^{n}}{n!} \tag{26-24}$$

which have unbelievably beautiful properties:

$$\exp(:f:)(gh)=\left(\exp(:f:)g\right)\left(\exp(:f:)h\right) \tag{26-25}$$

which can be prove using Leibnitz rule in manner similar to prove of $\exp(x+y)=\exp(x)\exp(y)$ in mathematical analysis.

Applying it to

$$\begin{gathered}\exp(:f:)x^{n}=\left(\exp(:f:)x\right)^{n};\\ g(X)=\sum_{n=0}^{\infty}g_{n}X^{n}\rightarrow\exp(:f:)g(X)=\sum_{n=0}^{\infty}g_{n}\left(\exp(:f:)X\right)^{n}=g\left(\exp(:f:)X\right).\end{gathered} \tag{26-26}$$

with the later being the most remarkable quality: Lie transformation of a function is a function of Lie transformation of its argument! (We cheated a bit here – we really needed to expand function of multiple variables as $\sum_{k=0}^{\infty}g_{k_1...k_{2n}}x_1^{k_1}\cdot\cdot x_{2n}^{k_{2n}}$ with the same final result). Lastly, the important (and elegant) property we will use later:

$$\exp(:f:)[g,h]=\left[\exp(:f:)g,\exp(:f:)h\right] \tag{26-27}$$

It worth noting that all relations mentioned above without checking are relatively straight forward to prove, but proves are not necessarily compact.

Let's now switch to symplectic maps denoted as:

$$\mathtt{M}: x \to \bar{x}(x,s); \mathtt{M}: X \to \bar{X}(x,s); \tag{26-26}$$

which generate local symplectic matrices

$$\mathbf{M}(s,X) = \left[\frac{\partial \bar{x}_i}{\partial x_j}\right] = \frac{\partial \bar{X}}{\partial X}; \quad \mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S}. \tag{26-29}$$

We discussed the invariants and result of these important features of symplectic maps such as Poincare invariants and will not repeat it. Instead we will focus on connection between Lie algebras and symplectic maps. First, let's show that Lie transformation is symplectic, lets consider

$$\bar{x} = \mathtt{M}x; \mathtt{M} = \exp(:f:); \tag{26-30}$$

than we have

$$\left[\bar{x}_i, \bar{x}_j\right] = \left[\left(\exp(:f:)x\right)_i, \left(\exp(:f:)x\right)_j\right] = \exp(:f:)\left[x_i, x_j\right] = S_{ij} \tag{26-31}$$

which proves the symplecticity of local transformation and the map as a whole.

As we discussed in our class, accelerator physics is interested in particles motion around the reference orbit, e.g. in maps which map origin X=0 into itself. It easy very easy to show that

$$\mathtt{M} = \exp(:f:);\ f = \sum_{k=1}^{2n} a_i x_i; \tag{26-32}$$

generate a displacement of the origin. For example $f = ax$ generates

$$f = ax, a[x,p] = \frac{\partial x}{\partial x}\frac{\partial p}{\partial p} = a;\ \overline{p} = a;\ \ :x:^n [x,p] = 0, n > 0 \tag{26-33}$$

First, we are not interest in such trivial shifts. Second, in general case, we always eliminate shift of the origin by choosing appropriately coefficients in (26-32).

Let's, for a moment, consider a Lie transformation with quadratic terms

$$f_2 = -\frac{1}{2}X^T \mathbf{H} X = -\frac{1}{2}\sum_{i,j=1}^{2n} h_{ij} x_i x_j; \mathbf{H}^T = \mathbf{H}. \tag{26-34}$$

Let's calculate action of $:f_2:$ on x_k:

$$\begin{gathered}
:f_2:x_k = -\frac{1}{2}\sum_{i,j=1}^{2n} h_{ij}\left[x_i x_j, x_k\right]; \\
\left[x_i x_j, x_k\right] = \left[x_i, x_k\right]x_j + x_i\left[x_j, x_k\right] = S_{ik}x_j + S_{jk}x_i \\
-\frac{1}{2}\sum_{i,j=1}^{2n} h_{ij}\left(S_{ik}x_j + S_{jk}x_i\right) = \sum_{i}^{2n}(\mathbf{SH})_{ki}x_i \\
:f_2:x_k = (\mathbf{SH})_{ki}\, x_i \rightarrow :f_2:X = \mathbf{SH}X
\end{gathered} \tag{26-35}$$

to see that it generates a linear matrix transformation.

Then we prove that Lie transformation with second order Hamiltonian polynomial as a generation function

$$:f_2:X = (\mathbf{SH})X;\; :f_2:^n X = (\mathbf{SH})^n X;$$
$$\exp(:f_2:) = \exp(\mathbf{SH}). \tag{26-36}$$

generates linear transformation. Which is equivalent to that generated by s-independent Hamiltonian of linear motion. As we discussed, linear motion is a trivial (when stable!) and is reduced to **n** independent oscillators with their amplifies (actions) and phases.

So far we had shown that Lie transforms are symplectic maps, that linear Lie map generated by second order Hamiltonian generate linear symplectic matrix and, vice versa, we can find such Lie transform for any symplectic matrix (for example using Sylvester formula for ln**M**). The remaining and very potent question remains: if a any analytical symplectic map can be presented in exponential form of a Lie operator? The answer is given by the **factorization theorem**: the keystone for application of the Lie transformation to non-linear Hamiltonian maps.

Factorization theorem: For an analytical symplectic map $\mathtt{M}$ (which transfers the origin in itself) and relation are assumed to be expandable into as power series:

$$\bar{X} = \mathtt{M}X;\;\; \bar{x}_i = M_{ik}x_k + \sum_{\sum_{i=1}^{2n} p_i=2}^{\infty} a_{1...2n} x_1^{p_1}\cdots x_{2n}^{p_{2n}}; \tag{26-37}$$

the map can be written in from of

$$\mathtt{M} = \exp(:f_2:)\exp(:f_3:)\exp(:f_4:)\exp(:f_5:)... \tag{26-38}$$

where f_m are homogeneous polynomials of power m of $\{x_i\}, i = 1,2n$.

Sketch of a proof – which is long- in based on the observation that if f_m and g_k are homogenies polynomials of order m and k, than their Poisson bracket

$$\left[f_m, g_k\right] = p_{m+k-2}$$

is also a homogeneous polynomial of order m+k-2. This is why f_2 generates linear map with linear polynomial X. Hence, f_3 will generate second order term and its exponential will generate all higher orders as well.

Let's apply using the linear map at the origin (X=0) the inverse transformation:

$$\exp(-:f_2:) = \exp(-\mathbf{SH}) \qquad (26\text{-}39)$$

to both sides of (26-37)

$$\exp(-:f_2:)\bar{X} = \exp(-:f_2:)\mathtt{M}X =$$

$$X + \exp(-:f_2:)(\sum_{2+}^{\infty} a_{1\ldots 2n} x_1^{p_1} \cdots x_{2n}^{p_{2n}}, \text{higher orders}) \qquad (26\text{-}40)$$

$$\exp(-:f_2:)\bar{x}_i = x_k + \exp(-:f_2:)\sum_{2+}^{\infty} a_{1\ldots 2n} x_1^{p_1} \cdots x_{2n}^{p_{2n}};$$

Suppose that f_3 is some cubic polynomial

$$\exp(-:f_3:)\exp(-:f_2:)\bar{X} = X - :f_3: X + (\text{higher orders}); \qquad (26\text{-}41)$$

Than (hopefully) we can select coefficients of f_3 to leave only cubic and higher order terms. Than we repeat the procedure for f_4, f_5.....

$$.....\exp(-:f_5:)\exp(-:f_5:)\exp(-:f_3:)\exp(-:f_2:)\bar{X} \to X \qquad (26\text{-}42)$$

with natural conclusion that multiplying (26-42) by (26-38) we get:

$$\bar{X} = \mathtt{M}X. \qquad (26\text{-}43)$$

While logically straightforward, the process (especially for 3D case) becomes cumbersome right away and in real situation (with few exceptions which prove the rule) computers do it much better.

Thus, we concluded that any analytical symplectic map can be presented as a product of linear (Gaussian optics) Lie transformation and product of Lie transformations comprising homogeneous polynomials of increasing power:

$$\mathrm{M} = \overbrace{\exp(:f_2:)}^{Gaussian\,optics} \bullet \overbrace{\exp(:f_3:)\exp(:f_4:)\exp(:f_5:)\ldots}^{Abberations,\,Nonlinear\;effects} \tag{26-44}$$

While looking as a final result, the remaining question is – how we can use it?

While there are hundreds of very important Lie algebraic relations and many-many tricks, one is important for interpretation (normalization) of the non-linear symplectic maps. In linear case we have set the action and angle canonical pairs describing each oscillator:

$$\{\varphi_k, I_k\} \Leftrightarrow \tilde{x}_k = \sqrt{2I_k}\cos(\psi+\varphi_k);\, \tilde{p}_k = -\sqrt{2I_k}\sin(\psi+\varphi_k);\, I_k = \frac{\tilde{x}_k{}^2 + \tilde{p}_k{}^2}{2}; \tag{26-45}$$

where $\{\tilde{x}_k, \tilde{p}_k\}$ are also canonical pairs. We could bring our linear map (matrix) to an oscillator turn using

$$U = [\ldots,\mathrm{Re}\,Y_k;\mathrm{Im}\,Y_k\ldots];\, MY_k = e^{i\mu_k}Y_k \to M\cdot U = UR;\, k=1,..,n$$

$$R = \begin{bmatrix} \ldots & 0 & 0 \\ 0 & R_k & 0 \\ 0 & 0 & \ldots \end{bmatrix};\, R_k = \begin{bmatrix} \cos\mu_k & -\sin\mu_k \\ \sin\mu_k & \cos\mu_k \end{bmatrix};\; U^{-1}\cdot M\cdot U = R = \exp(:\vec{\mu}\cdot\vec{I}:). \tag{26-46}$$

In linear approximation trajectories in $\{\tilde{x}_k, \tilde{p}_k\}$ planes are boring circles with radius $\sqrt{2I_k}$. This representation is called normal form of representation for linear symplectic map.

Normal form treatment

Instead of describing the dynamics in a beam line using an s-dependent Hamiltonian, we can construct a map, for example, in the form of a Lie transformation. Such a map may be constructed by concatenating the maps for individual elements. The beam dynamics (for example, the strengths of different resonances) may then be extracted from the transformation.

To better understand the concept of map (transformation), we take a look at the well-known linear transport matrix for a periodic accelerator (say, a storage ring)

$$\mathrm{M} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}, \ \beta\gamma = 1 + \alpha^2$$

the matrix is symplectic.

Normal form analysis of a linear system involves finding a transformation to variables in which the map appears as a pure rotation.

Consider matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

We find that

$$NMN^{-1}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} = R$$

Becomes a pure rotation in phase space.

The coordinates are "normalized" $\vec{x}_N = N\vec{x}$

And the normalized coordinates transform in one revolution as

$$\vec{x}_N \to NM\vec{x} = NMN^{-1}N\vec{x} = RN\vec{x} = R\vec{x}_N$$

Is simply a rotation in phase space.

Note that since the transformation N is symplectic, the normalized variables are canonical variables.

The treatment of nonlinear dynamics follows the same procedure however more complicated.

We can assume the map can be represented by a Lie transformation and factorized as

$$\mathbf{M} = R e^{:f_3:} e^{:f_4:} \cdots$$

Where f3 is a homogeneous polynomial of order 3 of the phase space coordinates and f4 is a homogeneous polynomial of order 4. The detailed order depends on the truncation.

The linear part of the map can be written in action angle variables as

$$R = e^{:-\mu J:}$$

To simplify this map, i.e., separate the contribution from different orders, we can construct a map M3

$$U = e^{:F_3:} M e^{:-F_3:}$$

Where F3 is a generator that removes resonance driving terms from $e^{:f_3:}$

So we have

$$U = e^{:F_3:} R e^{:f_3:} e^{:f_4:} e^{:-F_3:} = R R^{-1} e^{:F_3:} R e^{:f_3:} e^{:-F_3:} e^{:F_3:} e^{:f_4:} e^{:-F_3:}$$

Using relation $e^{:h:} e^{:g:} e^{:-h:} = e^{:e^{:h:} g:}$

$$U = R e^{:R^{-1} F_3:} e^{:f_3:} e^{:-F_3:} e^{:e^{:F_3:} f_4:}$$

Using Baker-Campbell-Hausdorff formula

$$e^{:A:}e^{:B:} = e^{:C:}, \quad where \quad C = A + B + \frac{1}{2}[A, B] + \cdots$$

The map now becomes

$$U = Re^{:R^{-1}F_3 + f_3 - F_3 + O(4):}e^{:e^{:F_3:}f_4:}$$

We can further reduce it to (non-trivial)

$$U = Re^{:f_3^{(1)}:}e^{:f_4^{(1)}:} = Re^{:R^{-1}F_3 + f_3 - F_3:}e^{:f_4^{(1)}:}$$

Where $f_3^{(1)} = R^{-1}F_3 + f_3 - F_3$ contains all the 3rd order contribution.

Thus the solution is

$$F_3 = \frac{f_3 - f_3^{(1)}}{I - R^{-1}}$$

Since f3 is periodic in the angle variable Φ, we can write

$$f_3 = \sum_m \bar{f}_{3,m}(J) e^{im\phi}$$

We can construct a f3(1) that does not have phase dependence, i.e., we can write it as

$$f_3^{(1)} = \bar{f}_{3,0}(J)$$

Thus now the generation function F3 reads

$$F_3 = \sum_{m \neq 0} \frac{\bar{f}_{3,m}(J) e^{im\phi}}{1 - e^{-im\mu}}$$

Taking Octupole as an example (assume it is the only nonlinear element in the beam line), we can write the map as

$$\mathbf{M} = Re^{:f_4:}$$

where f4 is

$$f_4 = -\frac{1}{24}k_3 l x^4$$

Rewrite it in action-angle variables

$$x = \sqrt{2\beta J}\cos\Phi$$

$$f_4 = -\frac{1}{6}k_3 l\beta^2 J^2\cos^4\Phi = -\frac{1}{48}k_3 l\beta^2 J^2(3+4\cos 2\Phi+\cos 4\Phi)$$

Thus the generation function for normalized map $f_{4,0}$ reads

$$f_{4,0} = -\frac{1}{16}k_3 l\beta^2 J^2$$

And the normalized map becomes (with BCH theorem)

$$\mathbf{M}_4 = Re^{:f_{4,0}:} = e^{:-\mu J-\frac{1}{16}k_3 l\beta^2 J^2:}$$

Thus the mapping of action-angle variables becomes

$$J \to J$$
$$\Phi \to \Phi + \mu + \frac{1}{8} k_3 l \beta^2 J$$

In other words, we see the tune shift with amplitude right away.

Similar to previous case for sextupole, we have

$$\mathrm{M}_4 = R e^{:f_{4,0}:} = e^{:-\mu J - \frac{1}{16} k_3 l \beta^2 J^2:} \doteq e^{:F_4:} M e^{:-F_4:}$$

Last equation is valid if we keep the normalization up to 4th order.

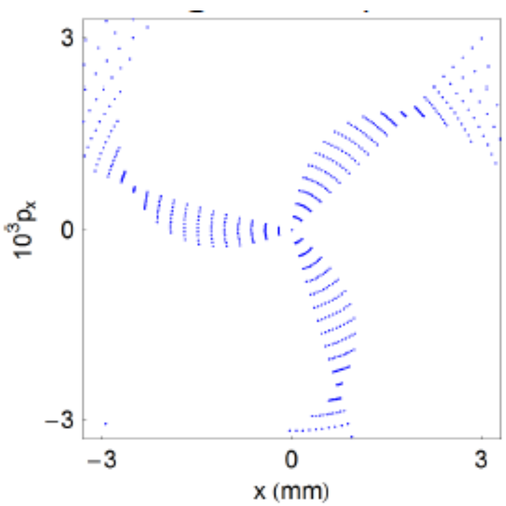
We can obtain the normalization generator F_4 easily

$$F_4 = \sum_{m \neq 0} \frac{f_{4,m}(J) e^{im\phi}}{1 - e^{-im\mu}}$$

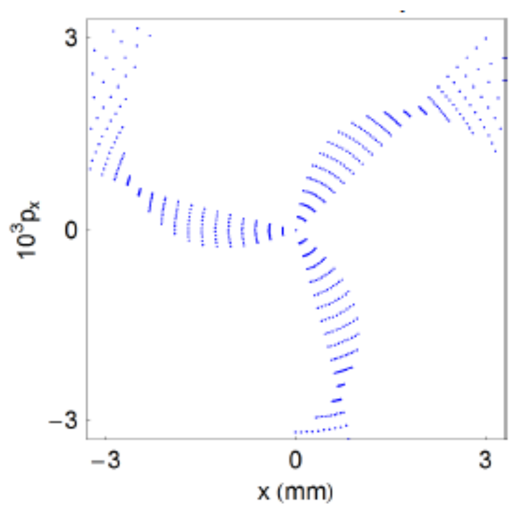
$$F_4 = -\frac{1}{96} k_3 l \beta^2 J^2 \left(\frac{4[\cos 2\Phi - \cos 2(\Phi + \mu)]}{1 - \cos 2\mu} + \frac{\cos 4\Phi - \cos 4(\Phi + \mu)}{1 - \cos 4\mu} \right)$$

The normalized map now contains only action variable (easy to integrate) while all the phase information has been pushed to higher order.

From the generator F_4, we see the octupole drives half integer and quarter integer resonances. We can track the Poincare map using exact map and the normalized map respectively (assum $k_3 l = 4800\ m^{-3}$ and $\beta = 1$ m). Assuming the tune μ is $0.33 \times 2\pi$ far from resonances



exact map

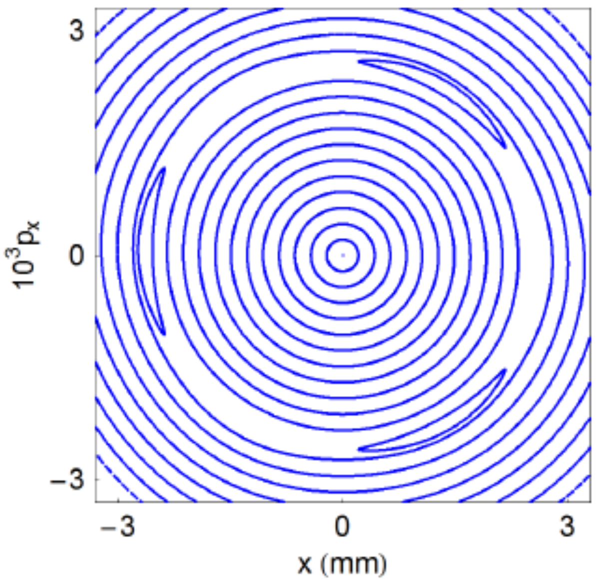


normalized map

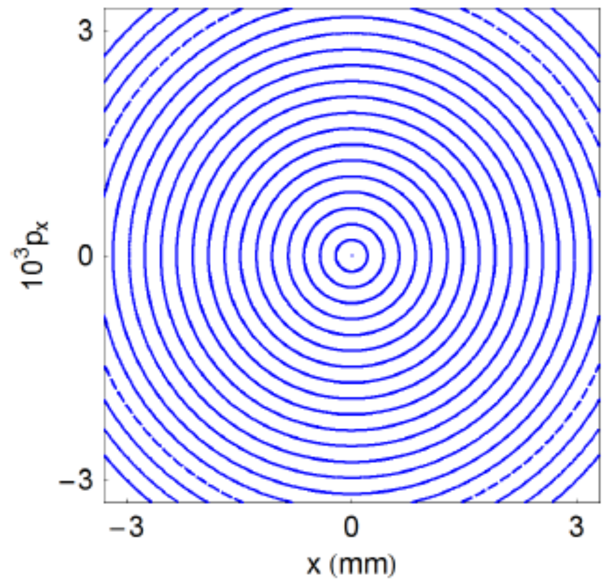
30 turns

Tune shift with amplitude!!

Tracking for longer turns results in different feature where we pay the price of the simplified (normalized) map. Some of the phase information (3rd order resonance island) is lost during this process.



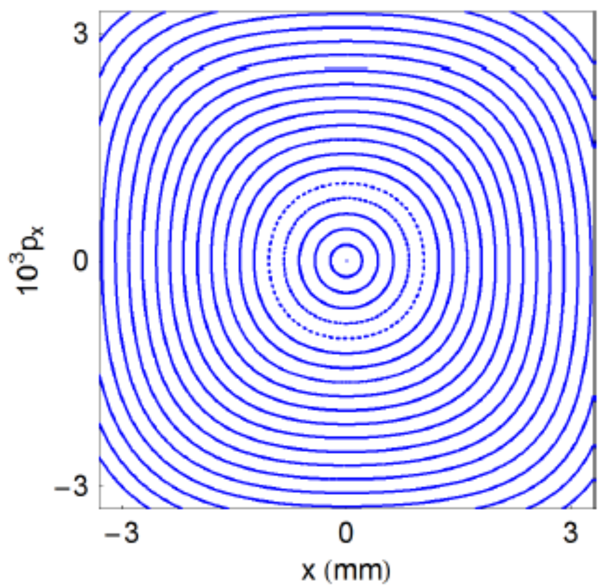
exact map



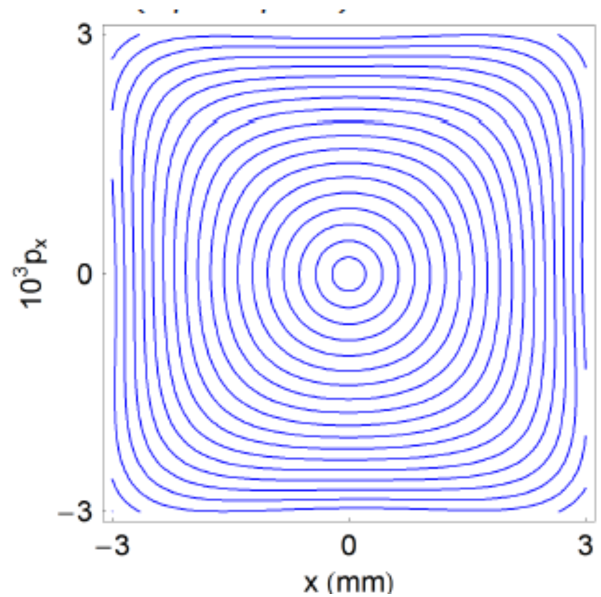
normalized map

2500 turns

Tracking for tunes near 4th order resonance is a bit tricky. Since the $k_3 l$ is positive, the tune shift with amplitude drives the tune up. Thus if the tune μ is $0.252 \times 2\pi$, we barely see resonances. The two tracking results resemble



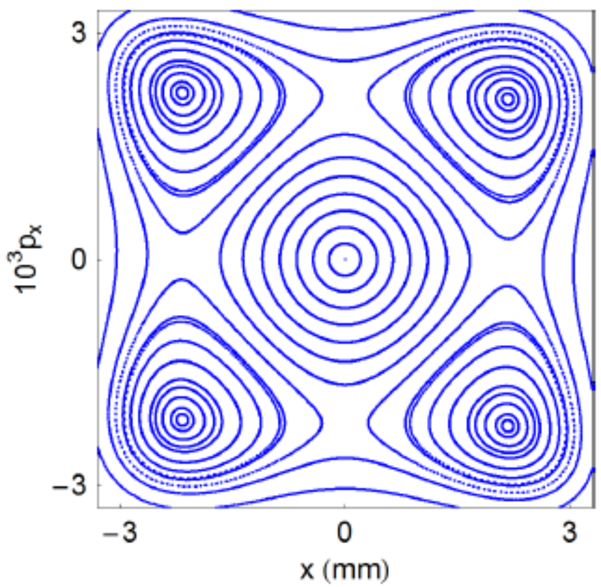
exact map



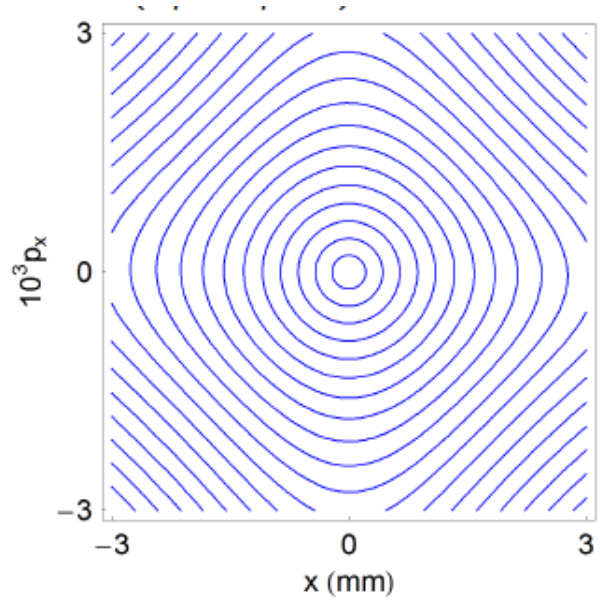
normalized map

2500 turns

For a tune less than quarter integer, i.e., μ is $0.248 \times 2\pi$, we see strong resonances from exact tracking while for the normalized map, we only see a rotation in phase space.



exact map



normalized map

2500 turns

Normal form of a one turn map preserves the information on tune amplitude dependence while loses the key phase information (when close to resonances). Need to retain higher order terms!

Resonance driving terms(RDTs)

We can interpret the Fourier coefficients $\overline{f}_{3,m}(J)$ as resonance strengths. And the generating function diverges when resonance condition mμ=2π is satisfied, meaning such driving term has large effect. Put it into polynomial expression, the generating function can be written as

$$F = \sum_{jklm} f_{jklm} \varsigma_x^{+} \varsigma_x^{-} \varsigma_y^{+} \varsigma_y^{-} = F_3 + F_4 + \cdots$$

where

$$f_{jklm} = \frac{h_{jklm}}{1 - e^{i2\pi[(j-k)\nu_x + (l-m)\nu_y]}}$$

hjklm are called resonance driving terms in many accelerator tracking codes. The entire process of the normal form the one turn map can be visualized as

$$\begin{tikzcd}
\overline{x(n)} \arrow[r, "M(J,\phi)"] \arrow[d, "\Phi"'] & \overline{x(n+1)} \arrow[d, "\Phi = e^{:F:}"] \\
\overline{\varsigma(n)} \arrow[r, "U(J) = e^{:H:}"] & \overline{\varsigma(n+1)}
\end{tikzcd}$$

Resonance driving terms(RDTs)

Incorporating the optics of a lattice, the resonance driving term (RDT) coefficients h_{jklm} (1^{st} order RDT) are usually calculated as

$$h_{jklm} = c\sum_{i=1}^{N} S_2 \beta_{xi}^{(j+k)/2} \beta_{yi}^{(l+m)/2} e^{i[(j-k)\mu_{xi}+(l-m)\mu_{yi}]}$$

It is very sensitive to linear lattice thus a carefully designed linear lattice with proper phase advance per periodic structure benefits greatly in reducing the RDTs (we will talk about a few tactics later).