Collective Effects II: Examples of Collective Instabilities

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Outline

- Transverse beam breakup instability (BBU) in linear accelerator
  - Two particle model
  - BNS damping
- Longitudinal Robinson Instability (m=0)
  - Macro particle model
  - Resonator model for cavity impedance
  - Stability condition and growth rate
- Longitudinal microwave instability (optional)
  - Dispersion relation
  - Cold beam
  - Warm beam (Keil-Schnell criteria for stability)

Single pass BBU (Two particle model)

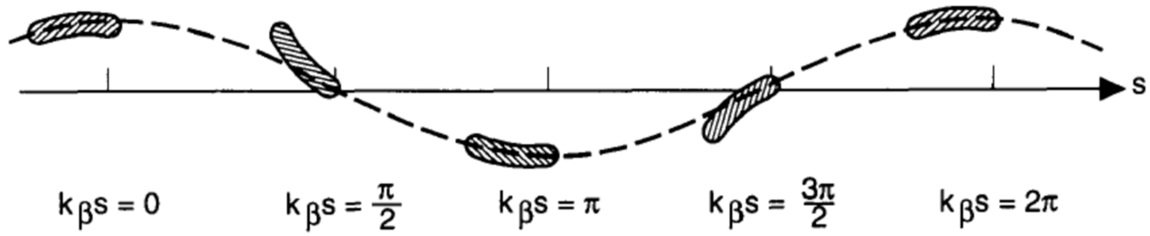


Figure 3.3. Sequence of snapshots of a beam undergoing dipole beam breakup instability in a linac. Values of $k_\beta s$ indicated are modulo 2π. The dashed curves indicate the trajectory of the bunch head.

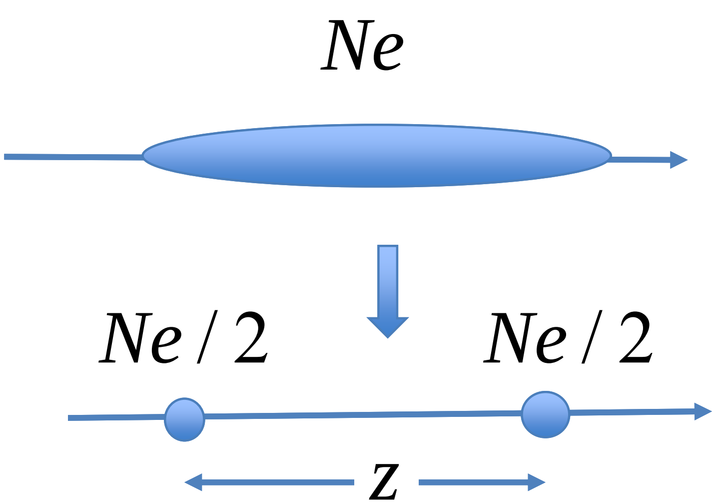
Leading particles $y_1(s) = \hat{y}\cos(k_\beta s)$

Trailing particles

$$y_2(s)'' + k_\beta^2 y_2(s) = \frac{Ne^2 W_1(z)}{2EL} y_1(s)$$

$$= 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{2\gamma L} \hat{y}\cos(k_\beta s)$$

$$r_0 = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{mc^2}$$



*Note: our definition of the transverse wake function follow G. Stupakov's note and has a sign difference from that defined in A. Chao's book, i.e. W1 here is -W1 in Chao's book.

Driving term for particle 2

$$\vec{w}_t(x,y,s) = \frac{c}{qe}\Delta\vec{p}_\perp \qquad q = \frac{Ne}{2}$$

$$m = 1 \quad \vec{w}_t(r',r,\theta,s) = W_1(s)r'\left[\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}\right]$$

$$\Delta p_y = \frac{W_1(s)y_1 e}{c}\frac{Ne}{2}$$

← Transverse momentum change of particle 2 due to wakefield while it goes through the structure

$$\Delta y' = \frac{\Delta p_y}{p_z} \approx \frac{c\Delta p_y}{E} = \frac{W_1(s)y_1 Ne^2}{2E}$$

← Transverse angle change of particle 2 due to wakefield

$$\frac{\Delta y'}{L} = \frac{W_1(s)y_1 Ne^2}{2EL}$$

← Transverse angle changing rate of particle 2 due to wakefield, driving term

Single pass BBU (Two particle model)

For a linear inhomogenous 2nd order differential equation

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = f(t)$$

its solution is given by

$$W(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix}$$

$$x(t) = c_1\phi_1(t) + c_2\phi_2(t) + \int_{t_0}^{t} \frac{\phi_1(\xi)\phi_2(t) - \phi_2(\xi)\phi_1(t)}{W(\xi)} f(\xi)\, d\xi$$

$$\phi_2 = \sin(k_\beta s) \qquad \phi_1 = \cos(k_\beta s)$$

$$y_{2,inh}(s) = 4\pi\varepsilon_0 \frac{Nr_0W_1(z)}{2\gamma L k_\beta}\hat{y}\int_0^s \sin(k_\beta s - k_\beta\xi)\cos(k_\beta\xi)\, d\xi$$

$$= 4\pi\varepsilon_0 \frac{Nr_0W_1(z)}{2\gamma L k_\beta}\hat{y}\frac{1}{2}\left[s\sin(k_\beta s) - \int_{-s/2}^{s/2} \sin(2k_\beta\tilde{\xi})\, d\tilde{\xi} \right]$$

$$\tilde{\xi} \equiv \xi - \frac{s}{2}$$

$$= 4\pi\varepsilon_0 \frac{Nr_0W_1(z)}{4\gamma L k_\beta}\hat{y} s\sin(k_\beta s)$$

$$W(t) = \begin{vmatrix} \cos(k_\beta s) & \sin(k_\beta s) \\ -k_\beta \sin(k_\beta s) & k_\beta\cos(k_\beta s) \end{vmatrix} = k_\beta$$

$$y_2(s)'' + k_\beta^2 y_2(s) = 4\pi\varepsilon_0 \frac{Nr_0W_1(z)}{2\gamma L}\hat{y}\cos(k_\beta s)$$

Single pass BBU (Two particle model)

$$y_2(s) = c_1 \cos(k_\beta s) + c_2 \sin(k_\beta s) + 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4\gamma L k_\beta} \hat{y} s \sin(k_\beta s)$$

Noticing that before going through the structure, particle 2 has the same trajectory as that of Particle 1, i.e.

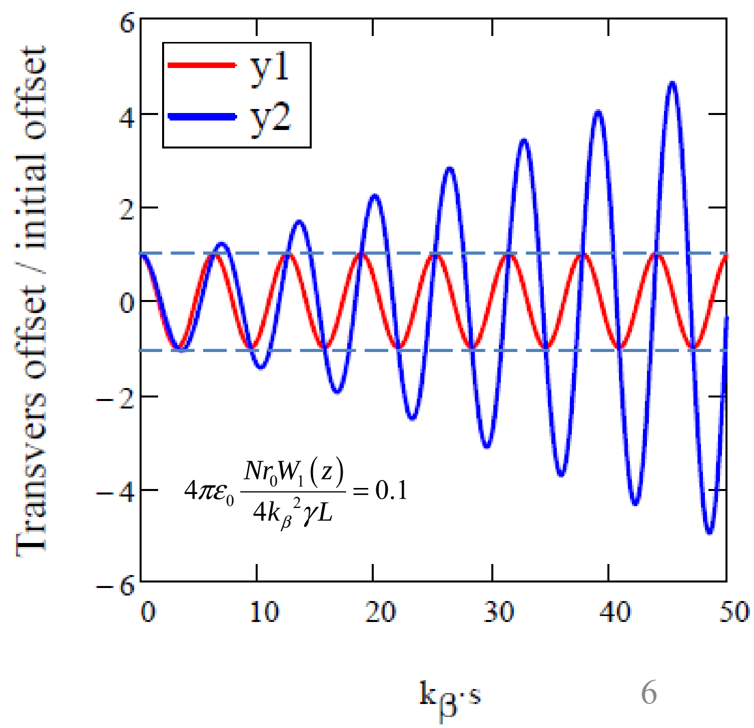
$$y_2(0) = y_1(0) = \hat{y}\cos(0) = \hat{y}$$

$$y'_2(0) = y'_1(0) = -\hat{y} k_\beta \sin(0) = 0$$

We obtain $c_1 = \hat{y}$ and $c_2 = 0$. Thus the solution for particle 2 is

$$y_2(s) = \hat{y}\left[\cos(k_\beta s) + 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4 k_\beta \gamma L} s \sin(k_\beta s)\right]$$

$$y_1(s) = \hat{y}\cos(k_\beta s)$$



Single pass BBU II

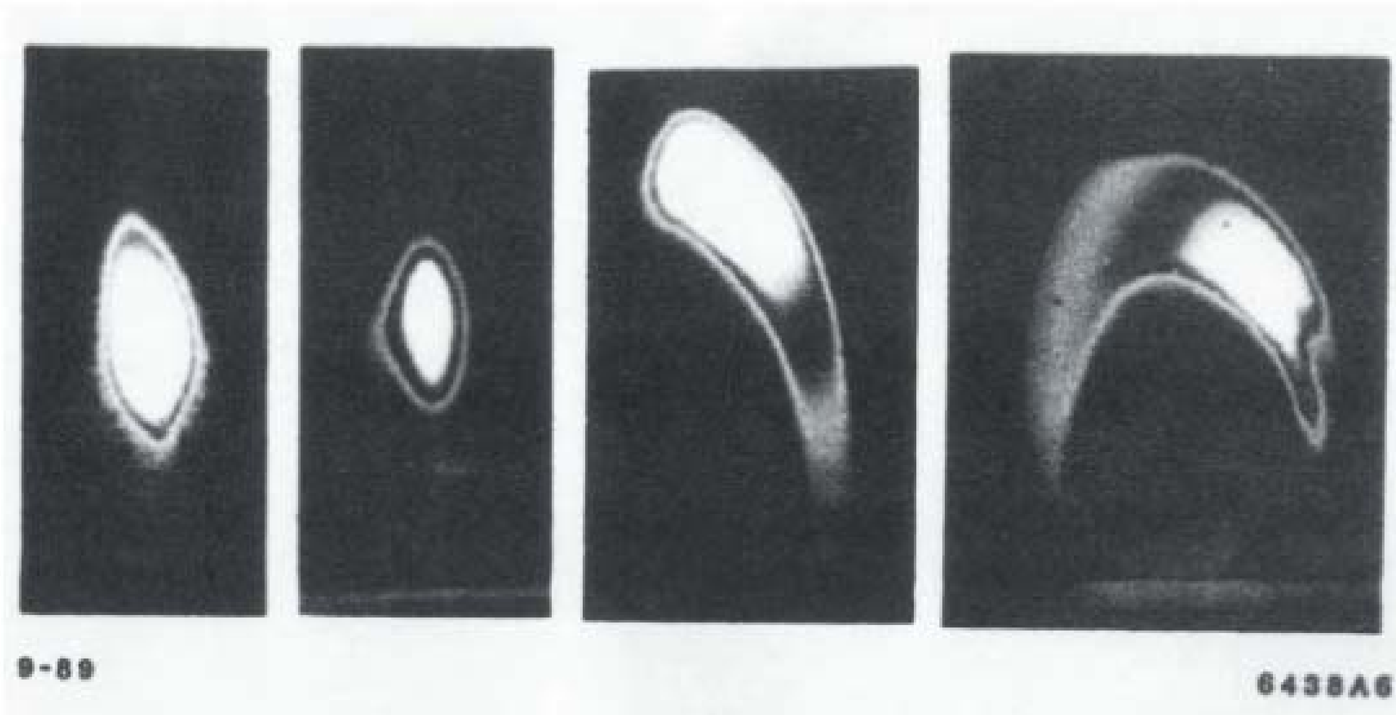


Figure 4.4: Four transverse beam profiles observed at the end of the SLAC linac are shown when the beam was carefully injected, and injected with 0.2, 0.5, and 1 mm offsets. The beam sizes σ_x and σ_y are about 120 μm. (Courtesy John Seeman, 1991)

One possible cure: BNS damping

Introduce focusing variation along the bunch, i.e. head and tail have different focusing strength

$$y_2'' + \left(k_\beta + \Delta k_\beta\right)^2 y_2 = 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{2\gamma L} \hat{y} \cos\left(k_\beta s\right)$$

$$\tilde{k}_\beta \equiv k_\beta + \Delta k_\beta$$

$$\phi_2 = \sin\left(\tilde{k}_\beta s\right) \quad \phi_1 = \cos\left(\tilde{k}_\beta s\right)$$

$$W(t) = \tilde{k}_\beta$$

$$y_{2,inh}(s) = 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{2\gamma L \tilde{k}_\beta} \hat{y} \int_0^s \sin\left(\tilde{k}_\beta s - \tilde{k}_\beta \xi\right) \cos\left(k_\beta \xi\right) d\xi$$

$$= -4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{2\gamma L \tilde{k}_\beta} \hat{y} \frac{1}{2} \left[\int_0^s \sin\left(\Delta k_\beta \left(\xi - \frac{\tilde{k}_\beta}{\Delta k_\beta} s\right)\right) d\xi + \int_0^s \sin\left(\left(\tilde{k}_\beta + k_\beta\right)\left(\xi - \frac{\tilde{k}_\beta}{\tilde{k}_\beta + k_\beta} s\right)\right) d\xi \right]$$

$$= 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{2\gamma L \tilde{k}_\beta} \hat{y} \frac{1}{2} \left[\frac{1}{\Delta k_\beta} + \frac{1}{\tilde{k}_\beta + k_\beta} \right] \left[\cos\left(k_\beta s\right) - \cos\left(\tilde{k}_\beta s\right) \right]$$

$$\approx -4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4\gamma L k_\beta \Delta k_\beta} \hat{y} \left[\cos\left(\tilde{k}_\beta s\right) - \cos\left(k_\beta s\right) \right]$$

← assume $\Delta k_\beta / k_\beta << 1$

$$y_2(s) = \hat{y} \cos\left(\left(k_\beta + \Delta k_\beta\right) s\right) - 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4\gamma L k_\beta \Delta k_\beta} \hat{y} \left[\cos\left(\tilde{k}_\beta s\right) - \cos\left(k_\beta s\right) \right]$$

Condition for complete compensation:

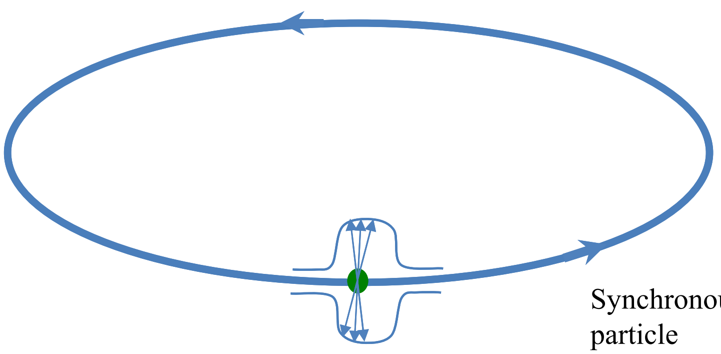
$$4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4\gamma L k_\beta \Delta k_\beta} = 1 \Rightarrow \Delta k_\beta = 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4\gamma L k_\beta} \Rightarrow \quad y_2(s) = \hat{y} \cos\left(k_\beta s\right)$$

Robinson Instability in Circular Machine

- As a charged particle bunch traveling through a cavity, it excites E&M fields, i.e. wakefields. To the leading order in the longitudinal direction, i.e. m=0, particles in the bunch lose some of their energies to the cavity, which is on top of the energy they gain from the acceleration (superposition).

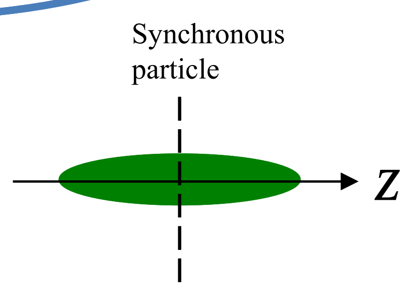
$$\frac{dz_n}{dn} = -\eta C \delta_n$$

Phase slip factor: $\eta \equiv \frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma^2}$



$$\frac{d}{dn}\delta_n = \frac{(2\pi\nu_s)^2}{\eta C} z_n + \frac{eV(z_n)}{E}$$

$$= \frac{(2\pi\nu_s)^2}{\eta C} z_n - \frac{4\pi\varepsilon_0 N r_0}{\gamma} \sum_{k=-\infty}^{n} W_0'(z_n - nC - z_k + kC)$$



n^{th} turn: $z_n - nC$

k^{th} turn: $z_k - kC$

$$r_0 = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{mc^2}$$

z is in the range of bunch length, i.e. z<<C

Robinson Instability II

$$\frac{d^2 z_n}{dn^2} + (2\pi\nu_s)^2 z_n = \frac{4\pi\varepsilon_0 \eta C N r_0}{\gamma} \sum_{k=-\infty}^{n} W_0'(kC - nC + z_n - z_k)$$

$$\approx \frac{4\pi\varepsilon_0 \eta C N r_0}{\gamma} \sum_{k=-\infty}^{n} W_0'(kC - nC)$$

$$+ \frac{4\pi\varepsilon_0 \eta C N r_0}{\gamma} \sum_{k=-\infty}^{n} (z_n - z_k) W_0''(kC - nC)$$

The first term in the RHS can be removed by defining

$$\tilde{z}_n \equiv z_n + \frac{4\pi\varepsilon_0 \eta C N r_0}{\gamma (2\pi\nu_s)^2} \sum_{k=-\infty}^{n} W_0'(kC - nC)$$

Please notice that $\sum_{k=-\infty}^{n} W_0'(kC - nC) = \sum_{\tilde{k}=-\infty}^{0} W_0'(\tilde{k}C)$,

i.e. independent of n $\quad \tilde{k} \equiv k - n$

$$\frac{d^2 \tilde{z}_n}{dn^2} + (2\pi\nu_s)^2 \tilde{z}_n \approx \frac{4\pi\varepsilon_0 \eta C N r_0}{\gamma} \sum_{k=-\infty}^{n} (\tilde{z}_n - \tilde{z}_k) W_0''(kC - nC)$$

Robinson Instability III

Ansatz (test solution): $\tilde{z}_n = A\exp\left(-in\Omega T_0\right)$

$$\frac{d^2\tilde{z}_n}{dn^2} + \left(2\pi\nu_s\right)^2 \tilde{z}_n \approx \frac{4\pi\varepsilon_0\eta CNr_0}{\gamma}\sum_{k=-\infty}^{n}\left(\tilde{z}_n - \tilde{z}_k\right)W_0''\left(kC - nC\right)$$

$$\left(-i\Omega T_0\right)^2 + \left(2\pi\nu_s\right)^2 = \frac{4\pi\varepsilon_0\eta CNr_0}{\gamma}\sum_{k=-\infty}^{n}\left(1-\exp\left(-i\left(k-n\right)\Omega T_0\right)\right)W_0''\left(kC - nC\right)$$

$$= \frac{4\pi\varepsilon_0\eta CNr_0}{\gamma}\sum_{\tilde{k}=-\infty}^{0}\left(1-\exp\left(-i\tilde{k}\Omega T_0\right)\right)W_0''\left(\tilde{k}C\right);$$

$\tilde{k} \equiv k - n$

Causality:

$W'\left(\Delta z > 0\right) = 0$

$z = ct - s$

$$= \frac{4\pi\varepsilon_0\eta CNr_0}{\gamma}\sum_{k=-\infty}^{\infty}\left(1-\exp\left(-ik\Omega T_0\right)\right)W_0''\left(kC\right)$$

$$\Omega^2 - \omega_s^2 = -\frac{4\pi\varepsilon_0\eta Nr_0 c}{\gamma T_0}\sum_{k=-\infty}^{\infty}\left(1-\exp\left(-ik\Omega T_0\right)\right)W_0''\left(kC\right); \qquad \omega_s = \frac{2\pi\nu_s}{T_0}$$

Robinson Instability IV

We will use the following identity (the Poisson Sum Formula)

$$\sum_{l=-\infty}^{\infty} F(lC) = \frac{1}{C}\sum_{p=-\infty}^{\infty} \tilde{F}\left(\frac{2\pi p}{C}\right) \qquad F(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikz}\tilde{F}(k)\,dk \quad \tilde{F}(k) = \int_{-\infty}^{\infty} e^{-ikz}F(z)\,dz$$

$$\Omega - \omega_s \approx -\frac{4\pi\varepsilon_0\eta N r_0 c}{\gamma T_0 2\omega_s}\sum_{k=-\infty}^{\infty}\left(1-\exp(-ik\Omega T_0)\right)W_0''(kC)$$

$$= -\frac{4\pi\varepsilon_0\eta N r_0 c}{\gamma T_0 2\omega_s}\left\{\sum_{k=-\infty}^{\infty} W_0''(kC) - \sum_{k=-\infty}^{\infty} G(kC)\right\} \qquad G(kC) \equiv \exp\left(-i\frac{\Omega}{c}kC\right)W_0''(kC)$$

$$= -\frac{4\pi\varepsilon_0\eta N r_0 c}{\gamma T_0 2\omega_s C}\left\{\sum_{p=-\infty}^{\infty} \tilde{W}_0''\left(\frac{2\pi p}{C}\right) - \sum_{p=-\infty}^{\infty} \tilde{G}\left(\frac{2\pi p}{C}\right)\right\}$$

$$\approx -i\frac{4\pi\varepsilon_0\eta N r_0}{2\gamma T_0^2\omega_s}\sum_{p=-\infty}^{\infty}\left\{p\omega_0 Z_{0,//}(p\omega_0) - (p\omega_0+\omega_s)Z_{0,//}(p\omega_0+\omega_s)\right\}$$

Deriving first term in bracket

$$\tilde{W}_0''\left(\frac{2\pi p}{C}\right) = \int_{-\infty}^{\infty} e^{-i\frac{2\pi p}{C}z} W_0''(z)\,dz$$

$$= \int_{-\infty}^{\infty} e^{-i\frac{2\pi p}{C}z} \frac{d}{dz} W_0'(z)\,dz$$

$$= \int_{-\infty}^{\infty} \frac{d}{dz}\left[e^{-i\frac{2\pi p}{C}z} W_0'(z)\right] dz - \int_{-\infty}^{\infty} W_0'(z) \frac{d}{dz} e^{-i\frac{2\pi p}{C}z} dz$$

$$= i\frac{2\pi p}{C} \int_{-\infty}^{\infty} W_0'(z) e^{-i\frac{2\pi p}{C}z} dz$$

$$W_0'(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\frac{z}{c}} Z_{0,//}(\omega)\,d\omega$$

$$= i\frac{p}{C} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega e^{i\omega\frac{z}{c}} Z_{0,//}(\omega) e^{-i\frac{2\pi p}{C}z} dz$$

$$= i\frac{p}{C} \int_{-\infty}^{\infty} d\omega Z_{0,//}(\omega) \int_{-\infty}^{\infty} e^{iz\left(\frac{\omega}{c}-\frac{2\pi p}{C}\right)} dz$$

$$= i\frac{cp}{C} 2\pi \int_{-\infty}^{\infty} d\omega Z_{0,//}(\omega)\,\delta\left(\omega - \frac{2\pi pc}{C}\right)$$

$$= ip\omega_0 Z_{0,//}(p\omega_0)$$

Deriving second term in bracket

$$\tilde{G}\left(\frac{2\pi p}{C}\right)=\int_{-\infty}^{\infty} e^{-i\frac{2\pi p}{C}z}\exp\left(-i\frac{\Omega}{c}z\right)W_0''(z)\,dz$$

$$=\int_{-\infty}^{\infty} e^{-i\left(\frac{2\pi p}{C}+\frac{\Omega}{c}\right)z}\frac{d}{dz}W_0'(z)\,dz$$

$$=\int_{-\infty}^{\infty} e^{-i\frac{2\pi p'}{C}z}\frac{d}{dz}W_0'(z)\,dz; \qquad p'=p+\frac{\Omega C}{2\pi c}$$

$$=\int_{-\infty}^{\infty}\frac{d}{dz}\left[e^{-i\frac{2\pi p'}{C}z}W_0'(z)\right]dz-\int_{-\infty}^{\infty}W_0'(z)\frac{d}{dz}e^{-i\frac{2\pi p'}{C}z}dz$$

$$=i\frac{2\pi p'}{C}\int_{-\infty}^{\infty}W_0'(z)e^{-i\frac{2\pi p'}{C}z}dz$$

$$=i\frac{p'}{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega e^{i\omega\frac{z}{c}}Z_{0,//}(\omega)e^{-i\frac{2\pi p'}{C}z}dz$$

$$=i\frac{p'}{C}\int_{-\infty}^{\infty}d\omega Z_{0,//}(\omega)\int_{-\infty}^{\infty}e^{iz\left(\frac{\omega}{c}-\frac{2\pi p'}{C}\right)}dz$$

$$=i\frac{cp'}{C}2\pi\int_{-\infty}^{\infty}d\omega Z_{0,//}(\omega)\delta\left(\omega-\frac{2\pi p'c}{C}\right)$$

$$=ip'\omega_0 Z_{0,//}(p'\omega_0)$$

$$=i(p\omega_0+\Omega)Z_{0,//}(p\omega_0+\Omega)$$

$$\approx i(p\omega_0+\omega_s)Z_{0,//}(p\omega_0+\omega_s)$$

$$W_0'(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega\frac{z}{c}}Z_{0,//}(\omega)\,d\omega$$

Robinson Instability IV

We will use the following identity (the Poisson Sum Formula)

$$\sum_{l=-\infty}^{\infty} F(lC) = \frac{1}{C}\sum_{p=-\infty}^{\infty} \tilde{F}\left(\frac{2\pi p}{C}\right) \qquad F(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikz}\tilde{F}(k)\,dk \quad \tilde{F}(k) = \int_{-\infty}^{\infty} e^{-ikz}F(z)\,dz$$

$$\Omega - \omega_s \approx -\frac{4\pi\varepsilon_0 \eta N r_0 c}{\gamma T_0 2\omega_s}\sum_{k=-\infty}^{\infty}\left(1-\exp\left(-ik\Omega T_0\right)\right)W_0''(kC)$$

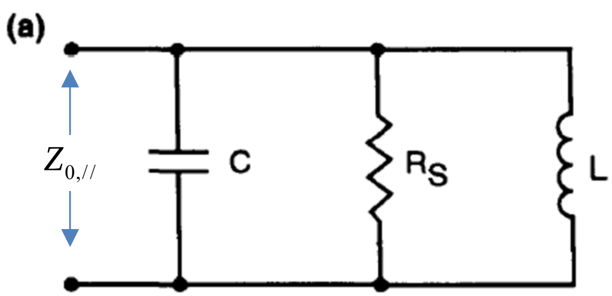
$$\approx -i\frac{4\pi\varepsilon_0 \eta N r_0}{2\gamma T_0^2 \omega_s}\sum_{p=-\infty}^{\infty}\left\{p\omega_0 Z_{0,//}(p\omega_0) - (p\omega_0+\omega_s)Z_{0,//}(p\omega_0+\omega_s)\right\}$$

$\mathrm{Re}\left[Z_{0,//}(p\omega_0)\right]$ is an even function of p.

$$\tau^{-1} = \mathrm{Im}(\Omega) = \frac{4\pi\varepsilon_0 \eta N r_0}{2\gamma T_0^2 \omega_s}\sum_{p=-\infty}^{\infty}(p\omega_0+\omega_s)\,\mathrm{Re}\left\{Z_{0,//}(p\omega_0+\omega_s)\right\}$$

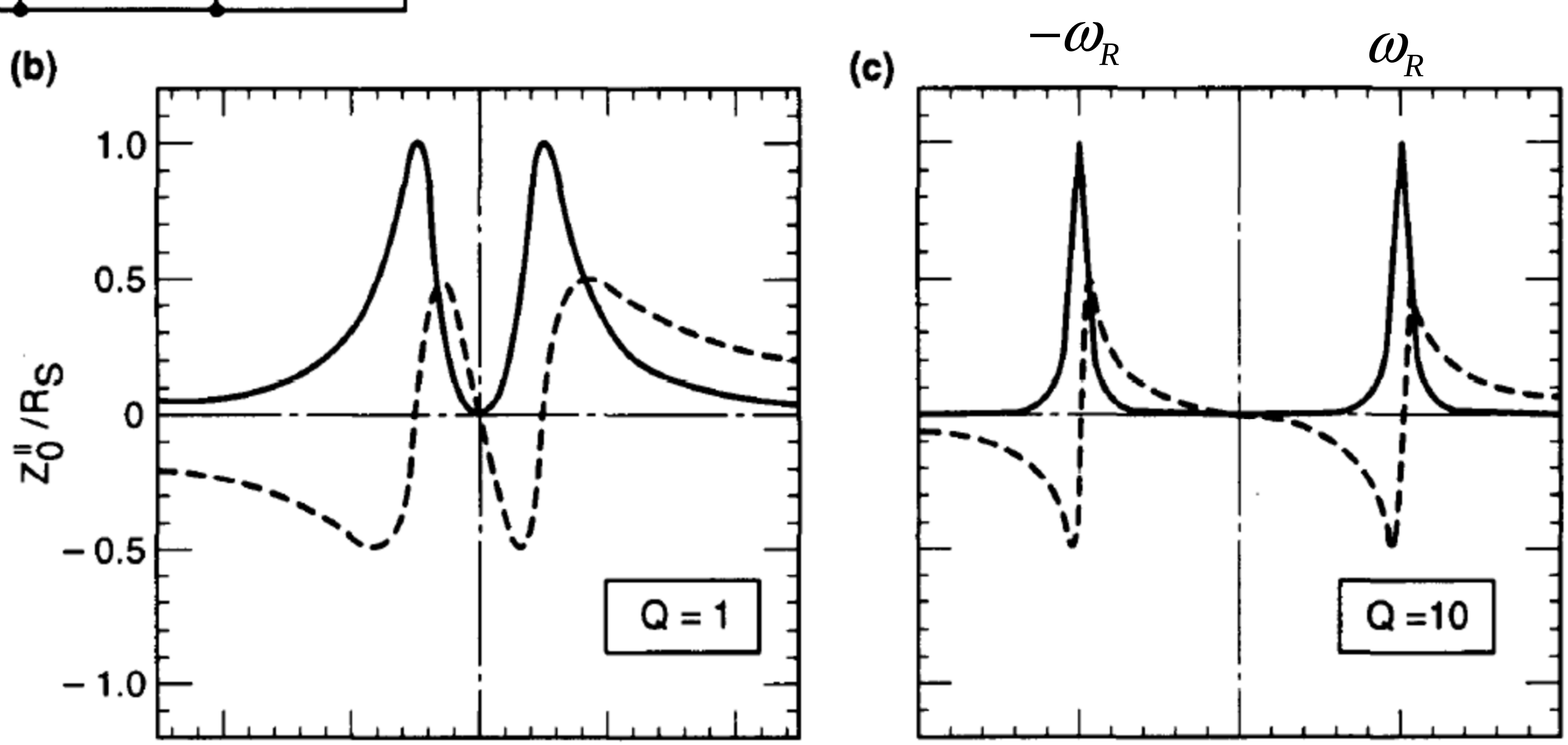
Impedance Model for Fundamental mode of a Cavity

$$\frac{1}{Z_{0,//}} = \frac{1}{R_s} + \frac{i}{\omega L} - i\omega C \quad \Rightarrow \quad Z_{0,//} = \frac{R_s}{1 + iQ\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)}$$



For RF cavity, $\omega_R / Q << \omega_0$, and hence only the two term in the summation over p matters, i.e.

$$p = \pm h \approx \pm \frac{\omega_R}{\omega_0}$$



Robinson Instability V

$$\tau^{-1} = \frac{4\pi\varepsilon_0 \eta N r_0}{2\gamma T_0^2 \omega_s}\left\{(h\omega_0+\omega_s)\mathrm{Re}\left[Z_{0,//}(h\omega_0+\omega_s)\right]+(-h\omega_0+\omega_s)\mathrm{Re}\left[Z_{0,//}(h\omega_0-\omega_s)\right]\right\}$$

$$\approx \frac{4\pi\varepsilon_0 \eta N r_0 h\omega_0}{2\gamma T_0^2 \omega_s}\left\{\mathrm{Re}\left[Z_{0,//}(h\omega_0+\omega_s)\right]-\mathrm{Re}\left[Z_{0,//}(h\omega_0-\omega_s)\right]\right\} \longleftarrow \quad Z_{0,//} = \frac{R_s}{1+iQ\left(\frac{\omega_R}{\omega}-\frac{\omega}{\omega_R}\right)}$$

For ω_s and $\Delta\omega << \frac{\omega_R}{2Q} << \omega_0$, the growth rate is

$$\tau^{-1} \approx 4\pi\varepsilon_0 \frac{4 N r_0 \eta R_s Q^2}{\pi\gamma T_0 h}\Delta\omega$$

$$\Delta\omega \equiv \omega_R - h\omega_0$$

$$\mathrm{Re}\left[Z_{0,//}(h\omega_0+\omega_s)\right]-\mathrm{Re}\left[Z_{0,//}(h\omega_0-\omega_s)\right] = 16\frac{Q^2 R_s}{h^2\omega_0^2}\omega_s\Delta\omega$$

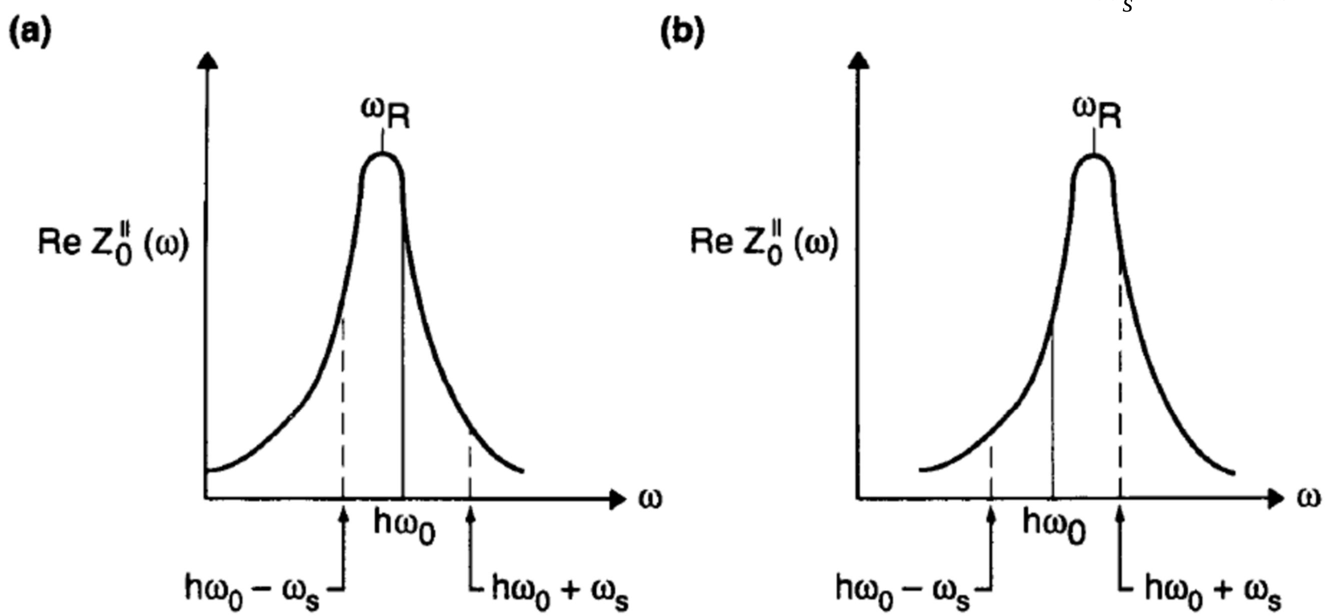


Figure 4.4. Illustration of the Robinson stability criterion. The rf fundamental mode is detuned so that ω_R is (a) slightly below $h\omega_0$ and (b) slightly above $h\omega_0$. (a) is Robinson damped above transition and antidamped below transition. (b) is antidamped above transition and damped below transition.

Derivation of the subtraction of impedance

$$Z_{0,//} = \frac{R_s}{1 + iQ\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)} \qquad \Delta\omega \equiv \omega_R - h\omega_0$$

$$\mathrm{Re}\left[Z_{0,//}\left(h\omega_0 + \omega_s\right)\right] = \mathrm{Re}\left\{\frac{R_s}{1 + iQ\left(\frac{\omega_R}{h\omega_0 + \omega_s} - \frac{h\omega_0 + \omega_s}{\omega_R}\right)}\right\}$$

$$= \frac{R_s}{1 + 4Q^2\left(\frac{\left(\Delta\omega - \omega_s\right)}{h\omega_0}\right)^2}$$

$$= R_s\left(1 - 4Q^2\frac{\left(\Delta\omega - \omega_s\right)^2}{h^2\omega_0^{\ 2}}\right)$$

$$\mathrm{Re}\left[Z_{0,//}\left(h\omega_0 - \omega_s\right)\right] = \mathrm{Re}\left\{\frac{R_s}{1 + iQ\left(\frac{\omega_R}{h\omega_0 + \omega_s} - \frac{h\omega_0 - \omega_s}{\omega_R}\right)}\right\}$$

$$= \frac{R_s}{1 + 4Q^2\left(\frac{\left(\Delta\omega + \omega_s\right)}{h\omega_0}\right)^2}$$

$$= R_s\left(1 - 4Q^2\frac{\left(\Delta\omega + \omega_s\right)^2}{h^2\omega_0^{\ 2}}\right)$$

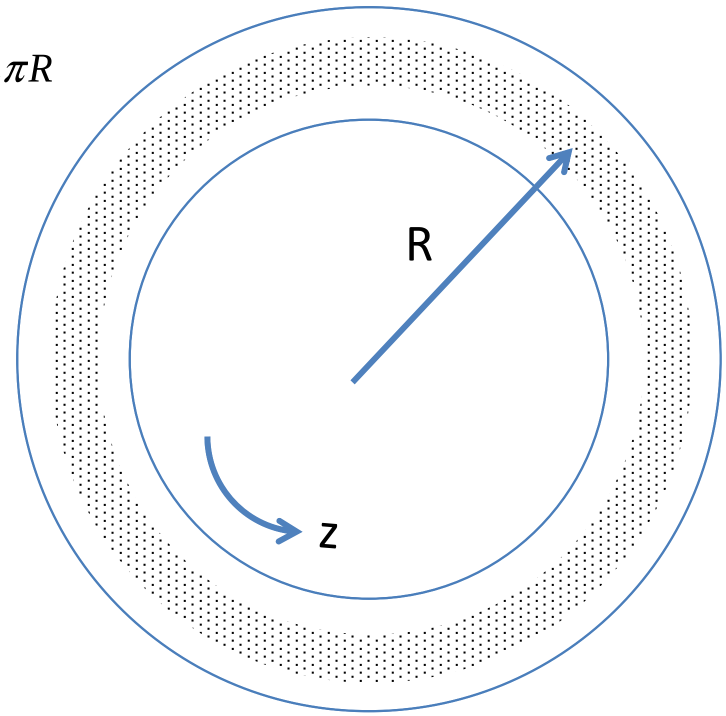
$$\mathrm{Re}\left[Z_{0,//}\left(h\omega_0 + \omega_s\right)\right] - \mathrm{Re}\left[Z_{0,//}\left(h\omega_0 - \omega_s\right)\right] = 16\frac{Q^2R_s}{h^2\omega_0^{\ 2}}\omega_s\Delta\omega$$

Longitudinal Microwave Instability

Unperturbed phase space density:

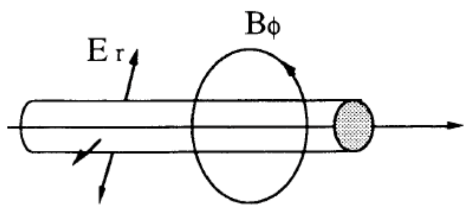
$$\psi_0(z,\Delta E) = \psi_0(\Delta E) = \frac{N}{C_0} f_0(\Delta E) \qquad \rho_0(z) = \rho_0 = \frac{N}{C_0}$$

$$C_0 = 2\pi R$$



DC current does not excite wake

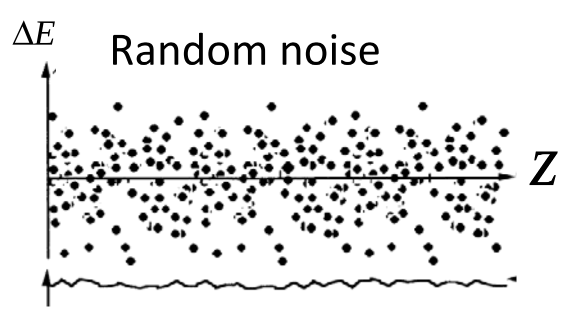
$$V_{//}(z_0) = \int_{z_0}^{\infty} \lambda(z_1) w_{//}(z_1 - z_0) dz_1$$

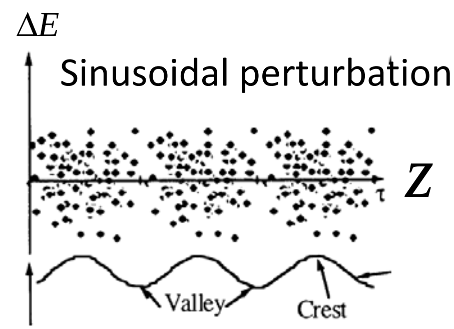


$$= \rho_0 \int_{z_0}^{\infty} W_0^{'}(z_1 - z_0) dz_1 = -\rho_0 W_0(0) = 0$$

Consider perturbation in phase space density: n-th azimuthal mode

$$\psi_1(z,\Delta E,0) = \hat{\psi}_1(\Delta E) e^{inz/R}$$





Ansatz: $\psi_1(z,\Delta E,t) = \hat{\psi}_1(\Delta E) e^{inz/R - i\Omega t}$

*Note that if a perturbation is static,

$$\psi_1{*}(z,\Delta E,t) = \hat{\psi}_1{*}(\Delta E) e^{in(z - v_0 t)/R} = \hat{\psi}_1{*}(\Delta E) e^{inz/R - i\Omega^* t} \Rightarrow \Omega^* = nv_0 / R = n2\pi v_0 / C = n\omega_0$$

Longitudinal Microwave Instability

$$1 = \frac{ieI_0 Z_{//}(\Omega)}{T_0} \int_{-\infty}^{\infty} \frac{f_0'(\Delta E)}{\Omega - \omega(\Delta E) n} d\Delta E$$

*Phase slip factor: $\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$

$$\omega(\Delta E) = \omega_0 + \Delta\omega(\Delta E) = \omega_0 - \eta\omega_0 \frac{\Delta p_z}{p_{0,z}} = \omega_0 - \frac{\eta\omega_0}{\beta^2} \frac{\Delta E}{E_0}$$

*Imaginary part of Ω tell us whether the system is stable

$$\psi_1(z, \Delta E, t) = \hat{\psi}_1(\Delta E) e^{inz/R - i\Omega t}$$

Cold Beam: $f_0(\Delta E) = \delta(\Delta E)$

$$1 = \frac{ieI_0 Z_{//}(\Omega)}{T_0} \frac{\eta n \omega_0}{E_0 \beta^2} \int_{-\infty}^{\infty} \frac{f_0(\Delta E)}{\left(\Omega - n\omega_0 + \frac{\eta n \omega_0}{E_0 \beta^2} \Delta E\right)^2} d\Delta E$$

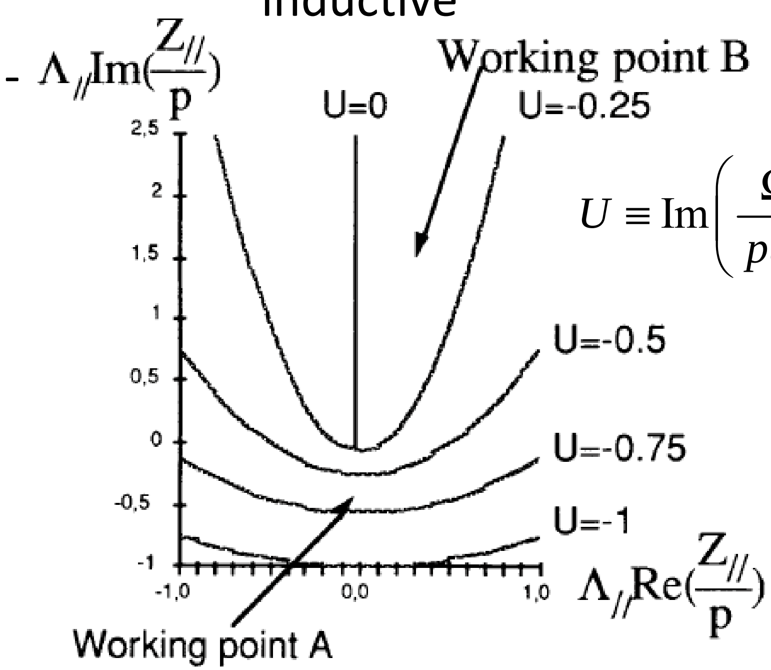
$$\Rightarrow \quad \Omega = n\omega_0 \pm \omega_0 \sqrt{\frac{ieI_0 \eta n Z_{//}(\Omega)}{2\pi E_0 \beta^2}} \approx n\omega_0 \pm \omega_0 \sqrt{\frac{ieI_0 \eta n Z_{//}(n\omega_0)}{2\pi E_0 \beta^2}}$$

Perturbative approach assuming $\frac{|\Omega - n\omega_0|}{n\omega_0} << 1$

Longitudinal Microwave Instabilities

Cold beam continued:
(assuming $\eta > 0$)

inductive



$$U \equiv \mathrm{Im}\left(\frac{\Omega}{p\omega_0}\right)$$

capacitive

$$\Omega \approx n\omega_0 \pm \omega_0 \sqrt{\frac{ieI_0\eta n Z_{//}(n\omega_0)}{2\pi E_0 \beta^2}}$$

For cold beam, the only case for stable beam is the machine impedance is pure inductive, i.e. $\mathrm{Im}(Z_{//}) < 0$, for $\eta > 0$, and capacitive, i.e. $\mathrm{Im}(Z_{//}) > 0$ for $\eta < 0$.

Taken from 'Accelerator Physics' by S.Y. Lee

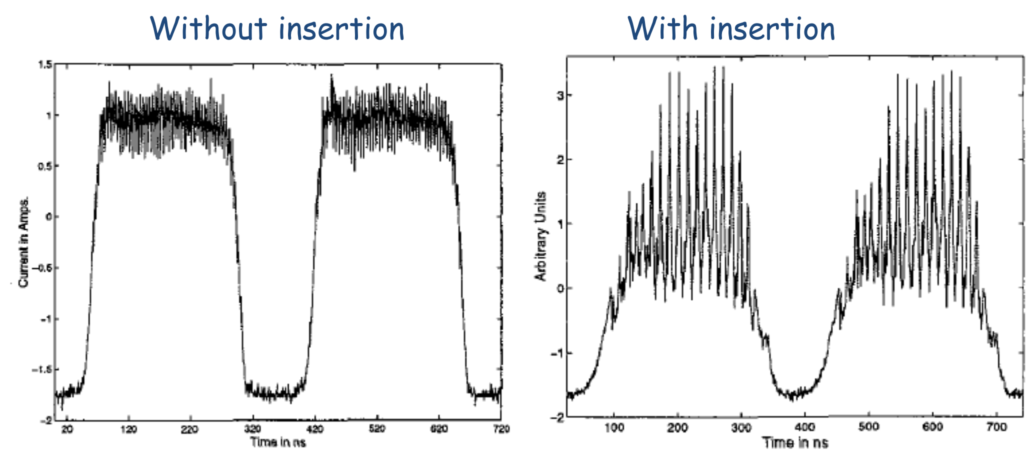


Figure 3.36: The longitudinal beam profiles observed at PSR the bunched coasting beam in the presence of inductive inserts, where three 1-m long ferrite ring cavities were installed in the PSR ring. [Courtesy of R. Macek, LANL]

Longitudinal Microwave Instabilities

Warm Beam: $$f_0(\Delta E) = \frac{1}{\sqrt{2\pi}\sigma_E} \exp\left(-\frac{\Delta E^2}{2\sigma_E^2}\right)$$

Dispersion relation for warm beam

$$1 = \frac{ieI_0 Z_{//}(n\omega_0)}{T_0} \frac{1}{\sqrt{2\pi}\sigma_E^2} \int_{-\infty}^{\infty} \frac{-\frac{\Delta E}{\sigma_E} \exp\left(-\frac{\Delta E^2}{2\sigma_E^2}\right)}{\Omega - \omega(\Delta E) n} d\Delta E = i\frac{1}{2}\left\{\frac{eI_0\left[Z_{//}(n\omega_0)/n\right]E_0\beta^2}{2\pi\eta\sigma_E^2}\right\} J_G(\tilde{\Omega})$$

$$= i\frac{2\ln(2)}{\pi}\left\{\frac{eI_0\left[Z_{//}(n\omega_0)/n\right]E_0\beta^2}{\eta\sigma_{E,FWHM}^2}\right\} J_G(\tilde{\Omega})$$

$$J_G(\tilde{\Omega}) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{-x\exp\left(-\frac{x^2}{2}\right)}{\left[\tilde{\Omega}\frac{E_0\beta^2}{\eta n\omega_0\sigma_E} + x\right]} dx$$

$$U' - iV' \equiv \frac{eI_0\left[Z_{//}(n\omega_0)/n\right]E_0\beta^2}{\eta\sigma_{E,FWHM}^2}$$

$$U' \sim \mathrm{Re}\left(Z_{//}(n\omega_0)\right)$$

$$V' \sim -\mathrm{Im}\left(Z_{//}(n\omega_0)\right)$$

$$\tilde{\Omega} = \mathrm{Re}(\tilde{\Omega}) + \mathrm{Im}(\tilde{\Omega}) \equiv \Omega - \omega_0 n$$

$$U' - iV' = \frac{-i\pi}{2\ln(2)J_G\left(\mathrm{Re}\,\tilde{\Omega} + i\,\mathrm{Im}\,\tilde{\Omega}\right)}$$

- The dispersion relation is solved by numerical plotting the contours for various $\mathrm{Im}\,\tilde{\Omega}$ in the complex impedance plane.

$$\psi_1(z,\Delta E,t) = \hat{\psi}_1(\Delta E)e^{inz/R - i\Omega t}$$

Longitudinal Microwave instability

Gaussian with various growth rate, $\mathrm{Im}(\tilde{\Omega})$

Contours with $\mathrm{Im}(\tilde{\Omega})=0$ for various energy distribution

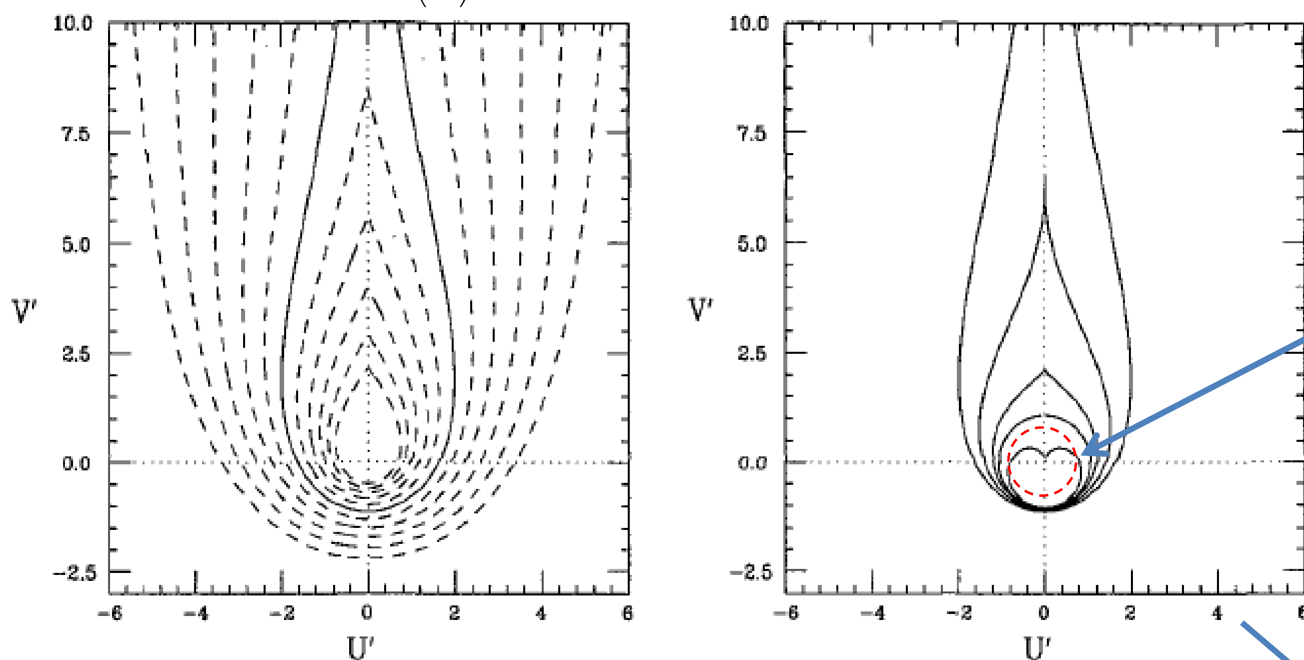


Figure 3.34: Left: The solid line shows the parameters V' vs U' for a Gaussian beam distribution at a zero growth rate. Dashed lines inside the threshold curve are stable. They correspond to $-\mathrm{Im}\,\Omega/(\sqrt{2\ln 2}\,\omega_0\eta\sigma_\delta) = -0.1, -0.2, -0.3, -0.4$, and -0.5. Dashed lines outside the threshold curve have growth rates $-\mathrm{Im}\,\Omega/(\sqrt{2\ln 2}\,\omega_0\eta\sigma_\delta) = 0.1, 0.2, 0.3, 0.4$, and 0.5 respectively. Right: The threshold V' vs U' parameters for various beam distributions.

Simplified estimation for stability condition:

Keil-Schnell criterion

$$\left|Z_{//}(n\omega_0)/n\right| \le \frac{2\pi|\eta|\sigma_E^2}{E_0\beta^2 e I_0}F$$

F depends on distributiion and for Gaussian energy distribution, it is 1.

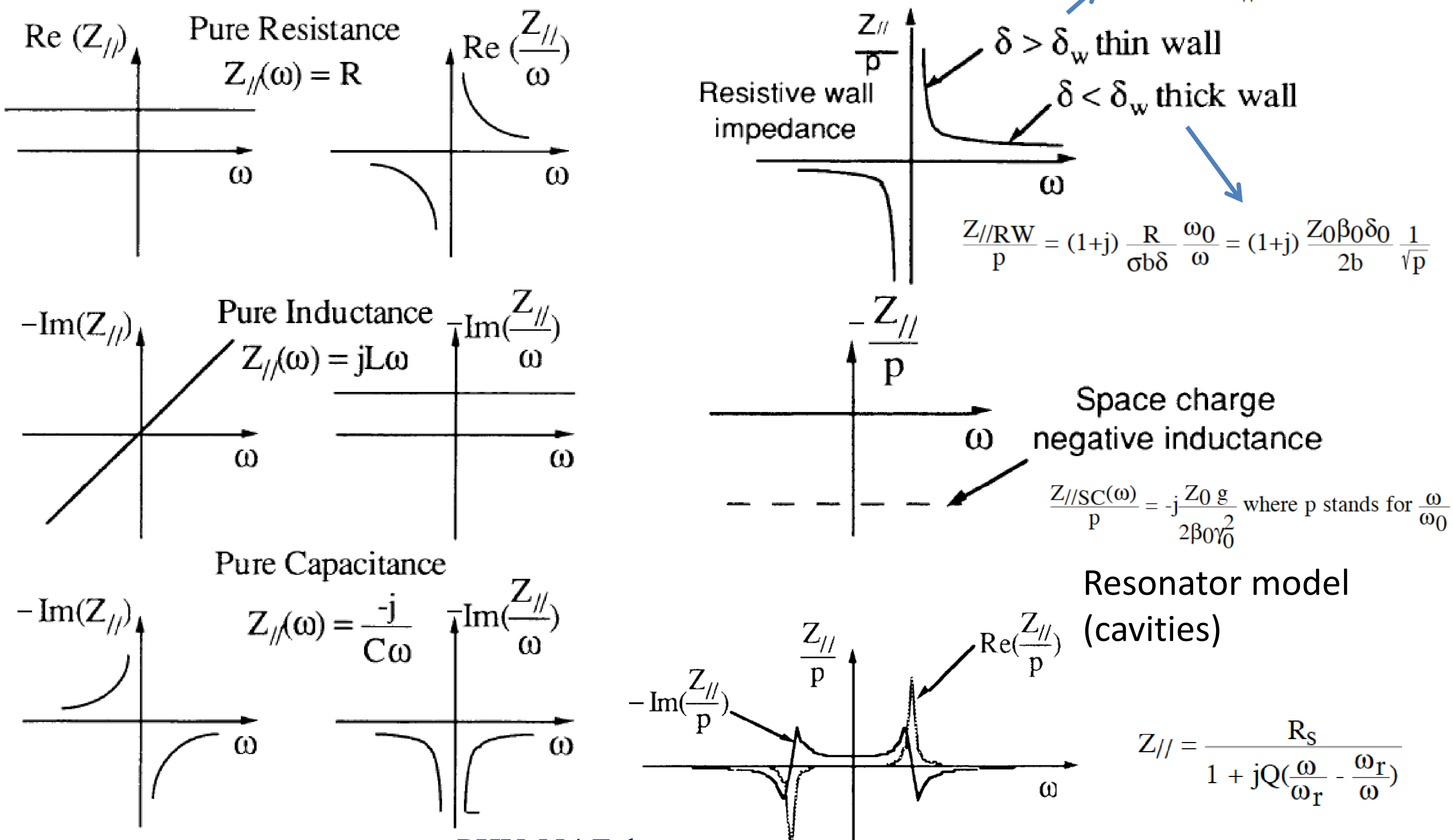
from inside outward, for the normalized distribution functions $\Psi_0(x) = 3(1-x^2)/4$, $8(1-x^2)^{3/2}/3\pi$, $15(1-x^2)^2/16$, $315(1-x^2)^4/32$, and $(1/\sqrt{2\pi})\exp(-x^2/2)$. All dis-

Typical Longitudinal Impedance

$j = -i$

Taken from 'Coasting beam longitudinal coherent instabilities' by J.L. Laclare

$$\frac{Z_{//RW}}{p} = \frac{R}{\sigma b \delta_W}\frac{1}{p}$$



Many pictures and derivations used in the slides are taken from the following references:

[1] ‘Accelerator Physics’ by S.Y. Lee;

[2] ‘Physics of Collective Beam Instabilities in High Energy Accelerators’ by A. Chao;

[3] ‘Coasting beam longitudinal coherent instabilities’ by J.L. Laclare

What we learned today

- In linear accelerator, single bunch transverse beam break up instability can develop if the bunches are not carefully injected and machine transverse wake function / impedance is large. Such an instability can be compensated by introducing focusing variation along the bunch, i.e. BNS damping.
- In circular machine, the leading order (m=0) longitudinal wakefield in the cavity can cause Robinson instability. The cavity resonant frequency should be detuned away from exact harmonics of the revolution frequency to avoid such instability: above transition, the resonant frequency should be slightly below $h\omega_0$; and below transition the resonant frequency should be slightly above $h\omega_0$.
- (optional) We also showed the dispersion relation for longitudinal microwave instability in a coasting beam. For cold beam, the beam is always unstable unless the impedances is pure inductive above transition or pure capacitive below transition. For warm beam, Landau damping make beam stable if the beam energy spread is sufficiently large. The stability condition can be estimated from Keil-Schnell criteria.

Backup Slides

Longitudinal Microwave Instability

But the system is not likely to be static and we need to solve Vlasov equation self-consistently to know the answer for Ω and hence $\psi_1(s,\Delta E,t)$

$$\frac{\partial}{\partial t}\psi_1(z,\Delta E,t)+\frac{dz}{dt}\cdot\frac{\partial}{\partial z}\psi_1(z,\Delta E,t)+\frac{d\Delta E}{dt}\cdot\frac{\partial}{\partial \Delta E}\psi_0(\Delta E)=0$$

where $$\frac{dz}{dt}=v(\Delta E) \qquad (1)$$

And $\frac{d\Delta E}{dt}$ is obtained by calculating the longitudinal wake potential $\frac{d\Delta E(z,t)}{dt}=-\frac{c\Delta p_z(z,t)}{T_0}$

$$c\Delta p_z(z,t)=-eQ_eV_{//}(z,t)=-e^2v_0\int_{-\infty}^{t}\rho_1(z,t_1)w_{//}(t-t_1)dt_1=-e^2v_0\int_0^{\infty}\rho_1(z,t-\tau)w_{//}(\tau)d\tau$$

$\rho_1 v_0 dt$ gives particle number in the slice (t,t+dt).

Longitudinal Microwave Instability

Hence, we obtain $$\frac{d\Delta E(z,t)}{dt} = -\frac{c\Delta p_z(z,t)}{T_0} = -\frac{e^2 v}{T_0}\int_0^{\infty}\rho_1(z,t-\tau)w_{//}(\tau)d\tau$$

where $T_0 = \frac{C_0}{v_0}$ is the revolution period. Using the test solution

$$\psi_1(z,\Delta E,t) = \hat{\psi}_1(\Delta E)e^{inz/R - i\Omega t}$$

and the following relations

$$w_{//}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_{//}(\omega)e^{-i\omega\tau}d\omega$$

$$\rho_1(z,t) = \int_{-\infty}^{\infty}\psi_1(z,\Delta E,t)d\Delta E = \hat{\rho}_1 e^{inz/R - i\Omega t} \qquad \hat{\rho}_1 \equiv \int_{-\infty}^{\infty}\hat{\psi}_1(\Delta E)d\Delta E$$

we can write the energy kick in term of longitudinal impedance

$$\frac{d\Delta E(z,t)}{dt} = -\hat{\rho}_1\frac{e^2 v_0}{2\pi T_0}e^{inz/R - i\Omega t}\int_{-\infty}^{\infty}d\omega Z_{//}(\omega)\int_{-\infty}^{\infty}e^{i(\Omega-\omega)\tau}d\tau = -\hat{\rho}_1\frac{e^2 v_0}{T_0}e^{inz/R - i\Omega t}Z_{//}(\Omega) \qquad (2)$$

Longitudinal Microwave Instability

Inserting eq. (1) and (2) into Vlasov equation, we obtain

$$-i\Omega\psi_1(z,\Delta E,t)+v(\Delta E)\cdot\frac{in}{R}\psi_1(z,\Delta E,t)-\hat{\rho}_1\frac{e^2v_0}{T_0}e^{inz/R-i\Omega t}Z_{//}(\Omega)\cdot\frac{\partial}{\partial\Delta E}\psi_0(\Delta E)=0$$

, which can be rewritten as

$$\psi_1(z,\Delta E,t)=\frac{ie^2v_0Z_{//}(\Omega)}{T_0}\frac{\hat{\rho}_1e^{inz/R-i\Omega t}}{\Omega-\omega(\Delta E)n}\frac{d\psi_0(\Delta E)}{d\Delta E}\qquad\omega(\Delta E)=\frac{v(\Delta E)}{R}$$

Integrating above equation over energy, i.e. $\int_{-\infty}^{\infty}d\Delta E\to$, yields

Dispersion relation:

$$1=\frac{ieI_0Z_{//}(\Omega)}{T_0}\int_{-\infty}^{\infty}\frac{f_0'(\Delta E)}{\Omega-\omega(\Delta E)n}d\Delta E$$

$$\psi_0(\Delta E)=\frac{N}{C_0}f_0(\Delta E)$$

$$I_0=eN/T_0$$