PHY 564
Advanced Accelerator Physics

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Plan to teach you about:

- The fundamental physics and in depth exploration of advanced methods of modern particle accelerators
- Theoretical concept related to the above
- Principle of least actions, relativistic mechanics and E&D, 4D notations
- N-dimensional phase space, Canonical transformations, simplecticity and invariants of motion
- Relativistic beams, Reference orbit and Accelerator Hamiltonian
- Parameterization of linear motion in accelerators, Transport matrices, matrix functions, Sylvester's formula, stability of the motion
- Invariants of motion, Canonical transforms to the action and phase variables, emittance of the beam, perturbation methods. Poincare diagrams
- Standard problems in accelerators: closed orbit, excitation of oscillations, radiation damping and quantum excitation, natural emittance
- Non-linear effects, Lie algebras and symplectic maps
- Vlasov and Fokker-Plank equations, collective instabilities & Landau Damping
- Free electron lasers, cooling techniques
- Spin motion in accelerators
- Types and Components of Accelerators

Learning goals

- Have full understanding of transverse and longitudinal particles dynamics in accelerators
- Being capable of solving problems arising in modern accelerator theory
- Understand modern methods in accelerator physics
- Being capable to fully understand modern accelerator literature

- We can not teach you every trick in the books available in accelerator physics but we plan to provide you with very solid foundation: so you can explore any topic in modern accelerator physics with full confidence

Materials

- Lecture notes presented at class should be used as the main text – it will be available at CASE website: http://case.physics.stonybrook.edu/index.php/PHY564_fall_2017
- Presently there is no textbook, which covers the material of this course.
- Additional material can be found in notes summarizing USPAS lectures: http://www0.bnl.gov/isd/documents/74289.pdf

Additional reading:

- H. Wiedemann, "Particle Accelerator Physics" Springer, 2007
- S. Y. Lee, "Accelerator Physics”, World Scientific, 2011
- L.D. Landau, Classical theory of fields
- ......

Course***

- *Linear algebra, relativistic mechanics and E&M.*
  - This will be a brief but complete rehash of *relativistic mechanics, E&M and linear algebra* material required for this course.
  - *It is a very brief and intense introduction/refresher of these topics. There will be home works during this introduction, but students would require to do a lot of reading at home – it is critical for understanding of the remainder of the material.*
- *Hamiltonian formalism, N-dimensional phase space, Canonical transformations, Simplecticity, Invariants*
  - Canonical transformations and related to it simplecticity of the phase space are important part of beam dynamics in accelerators. We will consider connections between them as well as derive all Poincare invariants (including Liouville theorem). We will use a case of a coupled N-dimensional linear oscillator system for transforming to the action and phase variables. We finish with adiabatic invariants.
- *Relativistic beams, Reference orbit and Accelerator Hamiltonian*
  - We will use least action principle to derive the most general form of accelerator Hamiltonian using curvilinear coordinate system related to the beam trajectory (orbit).
- *Linear beam dynamics*
  - This part of the course will be dedicated to detailed description of linear dynamics of particles in accelerators. You will learn about particles motion in oscillator potential with time-dependent rigidity. You will learn how to calculate matrices of arbitrary element in accelerators. We will use eigen vectors and eigen number to parameterize the particles motion and describe its stability in circular accelerators. Here you find a number of analogies with planetary motion, including oscillation of Earth's moon. You will learn some "standards" of the accelerator physics – betatron tunes and beta-function and their importance in circular accelerators.
- *Longitudinal beam dynamics*
  - Here you will learn about one important approximation widely used in accelerator physics – "slow" longitudinal oscillations, which are have a lot of similarity with pendulum motion. If you were ever wondering why Saturn rings do not collapse into one large ball of rock under gravitational attraction – this where you will learn of the effect so-called negative mass in longitudinal motion of particles when attraction of the particles cause their separation.

*** We will use least action principle as foundation for all topics we cover in this course.

Course cont..

- *Invariants of motion, Canonical transforms to the action and phase variables, emittance of the beam, perturbation methods, perturbative non-linear effects*
  - In this part of the course we will remove "regular and boring" oscillatory part of the particle's motion and focus on how to include weak linear and nonlinear perturbations to the particles motion. We will solve a number of standard accelerator problems: perturbed orbit, effects of focusing errors, "weak effects" such as synchrotron radiation, resonant Hamiltonian, etc. We will re-introduce Poincare diagrams for illustration of the resonances. You will learn how non-linear resonances may affect stability of the particles and about their location on the tune diagram. You will learn about chromatic (energy dependent) effects, use of non-linear elements to compensate them, and about problems created by introducing them.
- *Non-linear effects, Lie algebras and symplectic maps*
  - This part of the course will open you the door into a complex nonlinear beam dynamics. We will introduce you to non-perturbative nonlinear dynamics and fascinating world of non-linear maps, Lie algebras and Lie operators. These are the main tools in the modern non-linear beam dynamics. You will learn about dynamic aperture of accelerators as well as how our modern tools are similar to those used in celestial mechanics.
- *Vlasov and Fokker-Plank equations*
  - This part of the course is dedicated to the developing of tools necessary for studies of collective effects in accelerators. We will introduce distribution function of the particles and its evolution equations: one following conservation of Poincare invariants and the other including stochastic processes.
- *Radiation effects*
  - You will learn how to use the tools we had developed in previous lectures (both the perturbation methods and Fokker-Plank equation) to evaluate effect of synchrotron radiation on the particle's motion in accelerator. You will see how the effect of radiation damping and quantum excitation lead to formation of equilibrium Gaussian distribution of the particles

Course cont..

- *Collective phenomena, Free electron lasers, cooling techniques*
  - Intense beam of charged particles excite E&M fields when propagate through accelerator structures. These fields, in return, act on the particles and can cause variety of instabilities. Some of these instabilities – such as a free-electron lasers (FEL) – can be very useful as powerful coherent X-rays sources. Others (and they are majority) do impose limits on the beam intensities or limit available range of the beam parameters. You will learn techniques involved in studies of collective effects and will use them for some of instabilities, including FEL. The second part of the collective effect will focus on how we can cool beams, which do not have natural cooling mechanism
- *Spin dynamics*
  - Many particles used in accelerators have spin. Beams of such particles with preferred orientation of their spins called polarized. Large number of high energy physics experiments using colliders strongly benefit from colliding polarized beams. You will learn the main aspects of the spin dynamics in the accelerators and about various ways to keep beam polarized. One more "tunes" to worry about - spin tune .
- *Plasma accelerators*
  - We will try to fit into the course a brief introduction into a fascinating world of plasma wake-field accelerators, which are definitely are accelerators of 21st century.
- *Accelerator applications*
  - We will finish the course with a discussion of accelerator applications, among which are accelerators for nuclear and particle physics, X-ray light sources, accelerators for medical uses, etc. You will also learn about future accelerators at the energy and intensity frontiers as well as about new methods of particle acceleration.

Grading

- **Home works :** **40% of the grade**
- **Presentation of a research topic:** **40% of the grade**
- **Class Participation :** **20% of the grade**

- There will be a substantial number of problems. Most of them are aiming for better understanding of material covered during classes.
- **Presentation on a Research Project:** This presentation will be in place of the final exam. You will pick an accelerator project of your interest from a list provided by the instructors. We allow presentations on papers directly related to your research if they are linked to accelerator physics, but you will have to get it approved by the instructors. The presentations will be in a PowerPoint or equivalent a form. We will grade your presentations on: adequate understanding (good physics), adequate preparation (clear way of presentation, Visual Aids - pictures and figures), adequate references (where you find materials). The research project should be fun and we encourage you to choose an original topic and an original way of presentation. Nevertheless, any topic prepared and presented properly will have high grade.

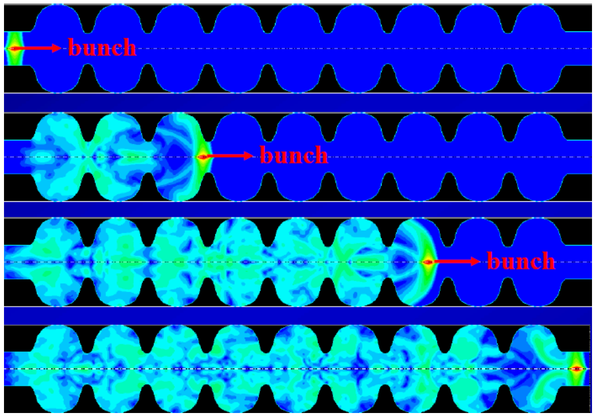
The Rules *or "feed your pets with healthy food"*

- Home works will be distributed in class and also will be posed that CASE website together with the rest of course materials http://case.physics.stonybrook.edu/index.php/PHY564_fall_2017
- You may collaborate with your classmates on the homework's if you are contributing to the solution. You must personally write up the solution of all problems. It would be appropriate and honorable to acknowledge your collaborators by mentioning their names. These acknowledgments will not affect your grades.
- We will greatly appreciate your homeworks being readable. Few explanatory words between equations will save us a lot of time while checking and grading your home-works. Nevertheless, your writing style will not affect your grades.
- Do not forget that simply copying somebody's solutions does not help you and in a long run we will identify it. If we find two or more identical homeworks, they all will get reduced grades. You may ask more advanced students, other faculty, friends, etc. for help or clues, as long as you personally contribute to the solution.
- You may (and are encouraged to) use the library and all available resources to help solve the problems. Use of Mathematica, other software tools and spreadsheets are encouraged. Cite your source, if you found the solution somewhere.
- We will give you one week (except holiday breaks, when it can be a bit longer) to return your home works. *You should return homework* before the deadline. Homework returned after the deadline could be accepted with reduced grading - 15% per day. Otherwise, it will be unfair for your classmates who are doing their job on time. Therefore, you should be on time to keep your grade high. Exceptions are exceptions and do not count on them (*if your dog eats your homework on a regular basis - feed it with something healthy, eating homework is bad for your pet and for you grade*).
- Solutions of each home work problems will be posted at the CASE webstite http://case.physics.stonybrook.edu/index.php/PHY564_fall_2017 after all HWs are returned. We will grade your HWs and return them typically during next class.
- We will have a dedicated office hours each Monday (except holydays) from 2 pm till 4 pm in office D102 (Physics) to discuss HW solutions as well as to address any questions you have regarding your HWs.

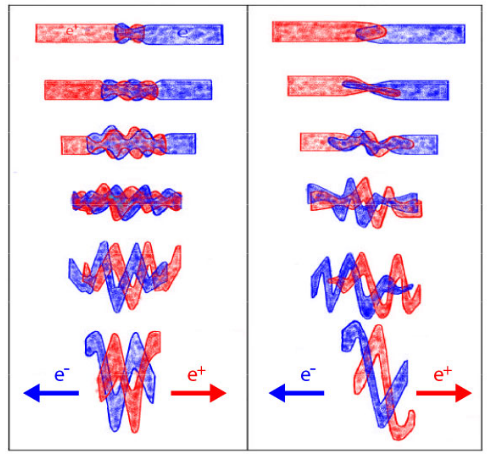
Picture worth 1,000 words ....

Nontrivial examples of accelerator problems

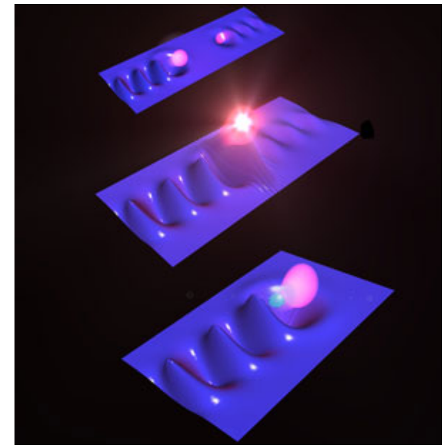
Interaction with self-generated wakefields



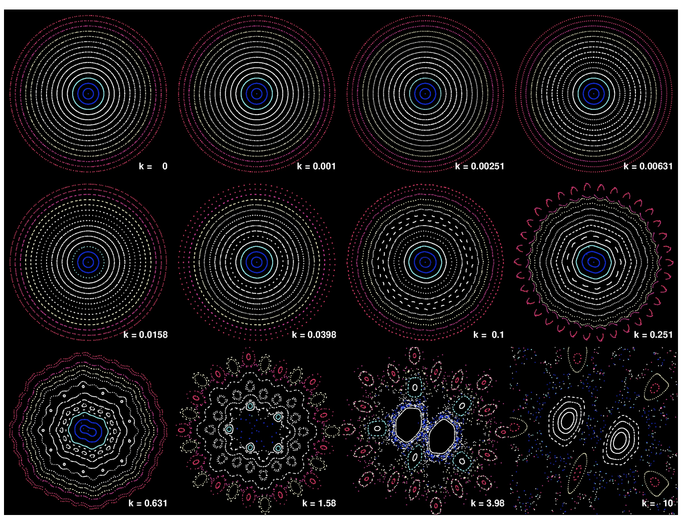
Disruption of colliding beams



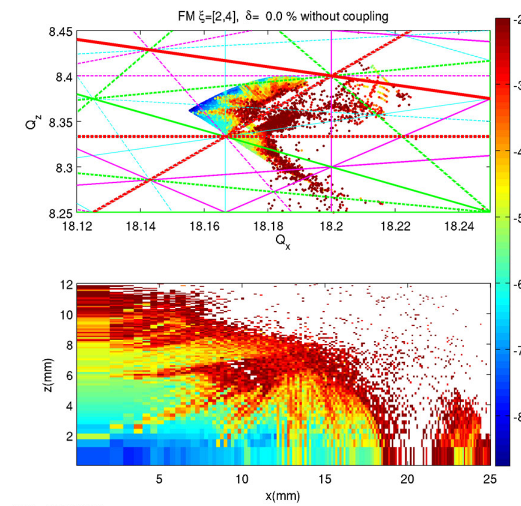
Beams in plasma



Non-linear particle's dynamics



Dynamic aperture in rings



... but equation worth thousands of pictures

- Few things to remember before diving in real AP
  - Mathematics is physicist's best friend
  - Nothing is more natural than relativistic E&M
  - We are creatures of 4-D (or even (4+N)D) world
  - Nothing saves you more time and paper than good definitions and notation: tensors, matrices, maps, operators…name it

- There will be home works for refresher classes, but a lot of home reading – take advantage of this opportunity to build up the foundation.
- Office hours in D102 (unless specified differently) from 2 pm till 4 pm on Mondays

Review of Linear Algebra

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Matrix: definition and properties

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$

Addition: $A + B = C \Leftrightarrow a_{ij} + b_{ij} = c_{ij}$

Multiplied by a constant: $kA = B \Leftrightarrow ka_{ij} = b_{ij}$

Equality: $A = B \Leftrightarrow a_{ij} = b_{ij}$

Multiplication (inner product): $AB = C \Leftrightarrow \sum_k a_{ik} b_{kj} = c_{ij}$

$$(AB)C = A(BC), \qquad A(B+C) = AB + AC$$

In general $AB \neq BA$

Multiplication demands that A has the same number of columns as B has rows.

Matrix: special cases I

- Diagonal matrix:

$$a_{ij} = 0 \quad \text{for} \quad i \neq j$$

$$A = \begin{pmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \ldots & 0 & \ldots & \ldots \\ 0 & 0 & \ldots & a_{nn} \end{pmatrix}$$

If A and B are both diagonal matrix, they are commutative:

$$AB = BA$$

- Identity matrix:

$$AI = IA = A \qquad \text{for} \quad \forall \mathrm{A}$$

$$I = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & 0 & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

$$I_{ij} = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

Matrix: special cases II

- Block diagonal matrix: A and A_i are square matrix.

$$A = \begin{bmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_k \end{bmatrix},$$

$$A = \left[\begin{array}{ccc|c|cc} 1 & 3 & 2 & 0 & 0 & 0 \\ 7 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 6 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & 3 \end{array}\right]$$

- Triangular matrix:

Upper diagonal matrix: elements below diagonal are all zero

Lower diagonal matrix: elements below diagonal are all zero

$$U = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

$$L = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Matrix V: transpose matrix

- A matrix, B, is called the transpose matrix of a matrix A if

$$A_{ij} = B_{ji}$$

The transpose matrix is often denoted as A^T,

i.e. $$A_{ij} = \left(A^T\right)_{ji}$$

- A square matrix A is called an orthognal matrix if
- $$A^T = A^{-1}$$
- A square matrix A is called symmetric matrix if

$A^T = A$ and anti-symmetric if $A^T = -A$

Matrix: trace

- In any square matrix, the sum of the diagonal elements is called the trace.

$$Tr(A) = \sum_i a_{ii}$$

- A useful property: $Tr(AB) = Tr(BA)$

- In general, $Tr(ABC) = Tr(BCA) \neq Tr(BAC)$

- Trace is a linear operator:

$$Tr(A + kB) = Tr(A) + k \cdot Tr(B)$$

Matrix: determinant of a matrix

- For a square matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

- The determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \det(A)$$

is called the determinant of matrix A and is denoted by det(A).

Determinant I

n columns

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

n rows

$$= \sum_{i,j,k} \varepsilon_{ijk\dots} a_{1i} a_{2j} a_{3k} \cdots$$

$\varepsilon_{ijk\cdots}$ is Levi-Civita symbol

$$\varepsilon_{ijk\cdots} = \begin{cases} 1 & \text{if } (i,j,k\dots) \text{ is even permutation of } (1,2,3\dots) \\ -1 & \text{if } (i,j,k\dots) \text{ is odd permutation of } (1,2,3\dots) \\ 0 & \text{if any of the two indices is repeated} \end{cases}$$

$$\varepsilon_{ijk\cdots l\cdots m\cdots} = -\varepsilon_{ijk\cdots m\cdots l\cdots}$$

Determinant II

A determinant of n dimension can be expanded over a column (or a row) into a sum of n determinants of n-1 dimension:

$$D = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nj} & \dots & \dots \end{vmatrix} = \sum_i C_{ij} a_{ij}$$

$C_{ij} = (-1)^{i+j} M_{ij}$ is called the ij[th] cofactor of D.

n-1 columns

$$M_{ij} = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

n-1 rows

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Determinant III

- Multiplied by a constant

$$\begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- The value of a determinant is unchanged if a multiple of one column (row) is added to another column (row)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{vmatrix}$$

- A determinant is equal to zero if any two columns (rows) are proportional

$$\begin{vmatrix} a_{12} & ka_{12} & a_{13} \\ a_{22} & ka_{22} & a_{23} \\ a_{32} & ka_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = 0$$

Linear equation system

- Existence of non-trivial solution of homogeneous equations

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \end{cases}$$

$$x \cdot D \equiv x \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}x + a_{12}y + a_{13}z & a_{12} & a_{13} \\ a_{21}x + a_{22}y + a_{23}z & a_{22} & a_{23} \\ a_{31}x + a_{32}y + a_{33}z & a_{32} & a_{33} \end{vmatrix} = 0$$

Similarly, $y \cdot D = 0$ and $z \cdot D = 0$

- Thus a set of homogenous linear equations have non-trivial solutions only if the determinant of the coefficients, D, vanishes.

Matrix: properties of the determinant of a matrix

- Some properties of the determinant of matrices
  - $\det\left(A^T\right)=\det\left(A\right)$
  - $\det\left(kA\right)=k^n\det\left(A\right)$
  - $\det\left(AB\right)=\det\left(A\right)\det\left(B\right)$

Proof of det(AB)=det(A)det(B):

(1)
$$\sum_{i,j,k\ldots}\varepsilon_{ijk\ldots}a_{\beta i}a_{\alpha j}a_{\gamma k}\cdots$$
$$=\sum_{j,i,k\ldots}\varepsilon_{jik\ldots}a_{\beta j}a_{\alpha i}a_{\gamma k}\cdots$$
$$=\sum_{i,j,k\ldots}\varepsilon_{jik\ldots}a_{\alpha i}a_{\beta j}a_{\gamma k}\cdots$$
$$=-\sum_{i,j,k\ldots}\varepsilon_{ijk\ldots}a_{\alpha i}a_{\beta j}a_{\gamma k}\cdots$$

(2)
$$\sum_{i,j,k\ldots}\varepsilon_{ijk\ldots}a_{\alpha i}a_{\beta j}a_{\gamma k}\cdots$$
$$=\varepsilon_{\alpha\beta\gamma\ldots}\sum_{i,j,k\ldots}\varepsilon_{ijk\ldots}a_{1i}a_{2j}a_{3k}\cdots$$
$$=\varepsilon_{\alpha\beta\gamma\ldots}\left|A\right|$$

(3)
$$\left|AB\right|=\sum_{i,j,k\ldots}\varepsilon_{ijk\ldots}\left(AB\right)_{1i}\left(AB\right)_{2j}\left(AB\right)_{3k}\cdots$$
$$=\sum_{i,j,k\ldots}\sum_{\alpha,\beta,\gamma}\varepsilon_{ijk\ldots}A_{1\alpha}B_{\alpha i}A_{2\beta}B_{\beta j}A_{2\gamma}B_{\gamma j}\cdots$$
$$=\sum_{\alpha,\beta,\gamma}A_{1\alpha}A_{2\beta}A_{2\gamma}\cdots\left\{\sum_{i,j,k\ldots}\varepsilon_{ijk\ldots}B_{\alpha i}B_{\beta j}B_{\gamma j}\cdots\right\}$$
$$=\left|B\right|\sum_{\alpha,\beta,\gamma}\varepsilon_{\alpha\beta\gamma\ldots}A_{1\alpha}A_{2\beta}A_{2\gamma}\cdots$$
$$=\left|A\right|\left|B\right|$$

Matrix IV: inversion

- Inversion of a square matrix A is to find a square matrix B such that

$$AB = BA = I$$

B is called the inverse matrix of A and often denoted by A^{-1}, i.e.

$$AA^{-1} = A^{-1}A = I$$

- One way to find the inverse matrix is by

$$\left(A^{-1}\right)_{ij} = \frac{C_{ji}}{|A|}, \quad \text{where } C_{ji} \text{ is the } ji^{th} \text{ cofactor of } A$$

Matrix VI: similarity transformation and diagonalization

- Two matrix, A and B, are called similar if there exists a invertible matrix P such that

$$B = P^{-1}AP,$$

and the transformation from A to B is called similarity transformation.

- Diagnolization of a matrix, A, is to find a similarity transformation matrix, P, such that $P^{-1}AP$ is a diagonal matrix:

Matrix VII: diagonalization

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix} \quad \text{or} \quad AP = P\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

- If we look at the j[th] column of the second equation, it follows

$$\sum_k a_{ik} p_{kj} = \sum_k p_{ik} \lambda_k \delta_{kj} = \lambda_j p_{ij} \qquad (*)$$

Defining a nx1 matrix (i.e. a column vector) $|P^J\rangle$ such that $|P^j\rangle_i = p_{ij}$ (note: j is fixed)

Equation (*) becomes: $A|P^j\rangle = \lambda_j |P^j\rangle$

Matrix VIII: eigenvalue and eigenvector

- For a matrix A, a vector matrix X is called an eigenvector of A if

$$A \cdot X = \lambda X$$

where λ is called the eigenvalue associated with the eigenvector X.

- The eigenvalues are found by solving the following polynomial equation

$$(A - \lambda I) \cdot X = 0 \Rightarrow \det(A - \lambda I) = 0$$

Defective Matrix

- Not all square matrix can be diagonalized:

$$A = \begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}; \ \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 \\ 3 & -4-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda = -1$$

$$\begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} 3x_1 - 3x_2 = 0 \\ 3x_1 - 3x_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

We end up with only one eigenvector.

- A square matrix that does not have a complete set of eigenvectors is not diagonalizable and is called a defective matrix.

- If a matrix, A, is defective (and hence is not similar to a diagonal matrix), then what is the simplest matrix that A is similar to?

Jordan form matrix

- Definition: a Jordan block with value λ is a square, upper triangular matrix whose entries are all λ on the diagonal, all 1 on the entries immediately above the diagonal, and zero elsewhere:

$$J(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

1D: $[\lambda]$

2D: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

3D: $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

- Definition: a Jordan form matrix is a block diagonal matrix whose blocks are all Jordan blocks

- Theorem: Let A be a nxn matrix. Then there is a Jordan form matrix that is similar to A.

$$\left[\begin{array}{cc|cc|c|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

Home reading

- The linear algebra notation introduced during this lecture will be frequently used during the course
- It is critical that you would read all material of this lecture before next class and, if necessary, refresh your understanding with your favorite text on Linear Algebra
- Let us know if you need a recommendation on what book to read?