

Homework 6. Due October 4

Problem 1. 5 points. A short ultra-relativistic ($1-v/c \ll 1$) bunch with charge of 5 nC is passing through a 0.3 meter long 500 MHz pillbox accelerating cavity operating at the fundamental TM010 with peak accelerating field of 5 MV/m.

(1) Find the change of the cavity voltage $\Delta V/V$ (accelerating field) after the beam passes through it as function of the phase of the beam passing the cavity. What are the maximum and minimum $\Delta V/V$?

(2) How the beam loading $\Delta V/V$ depends on the accelerating field? At what level of accelerating it reaches $\Delta V/V$ 1%?

- (a) Assume that beam does not change velocity in the cavity;
- (b) Hint – use energy conservation law
- (c) Assume that relative change of the voltage $\Delta V/V$ is small, e.g. the beam loading can be treated as a perturbation.

Solution:

First, we need to find the energy gain by each electron in the cavity operating at $E=5$ MV/m using RF phase in the center as the reference:

$$dz \cong ct;$$
$$\Delta E = ecE \int_{-L/2c}^{L/2c} \cos(\omega t + \varphi) dt = eLE \frac{\sin\left(\frac{\omega L}{2c}\right)}{\frac{\omega L}{2c}} \cos \varphi = eV_{RF} \cos \varphi$$
$$FF = \frac{\sin\left(\frac{\omega L}{2c}\right)}{\frac{\omega L}{2c}} = 0.636179.. \quad V_{RF} = LE \cdot FF = 0.9543 \text{ MV}$$

It means that that energy take/given by the beam is

$$\Delta U = qV_{RF} \cos \varphi = \Delta U_o \cos \varphi$$
$$q = 5 \text{ nC} = 5 \cdot 10^{-9} \text{ C}; V_{RF} = 0.9543 \cdot 10^6 \text{ V}$$
$$\Delta U_o = \text{sign}[q] \cdot 4.77 \cdot 10^{-3} \text{ J}$$

Naturally, when energy is taken by electron beam, $q \cos \varphi > 0$, RF voltage in the cavity drops (it is called beam loading) and with beam loses energy, $q \cos \varphi < 0$, RF voltage increases. To know the voltage change we need to know what EM energy is stored in the RF cavity. We should use the your favorite units system (SI)

$$W = \int \left(\epsilon_o \frac{\vec{\mathbf{E}}^2}{2} + \mu_o \frac{\vec{\mathbf{H}}^2}{2} \right) dV = \frac{\epsilon_o}{2} \int \vec{\mathbf{E}}_o^2 dV$$

or GSG

$$W = \frac{1}{8\pi} \int (\vec{\mathbf{E}}^2 + \vec{\mathbf{H}}^2) dV = \frac{1}{8\pi} \int \vec{\mathbf{E}}_o^2 dV$$

and the field pattern we derived for TM₀₁₀ mode

$$\mathbf{E}_o = \hat{z} \cdot E_o \cdot J_o \left(2.405 \frac{r}{a} \right); E_o = 5 \cdot 10^6 \frac{V}{m} \equiv 166.7 \text{ Gs}$$

where radius of the cavity, a , is defined by its frequency:

$$J_o(ka) \equiv J_o \left(\frac{\omega}{c} a \right) = 0 : TM_{010} \rightarrow \frac{\omega}{c} a = 2.405$$

$$a = \frac{2.405 c}{\omega} = \frac{2.405}{2\pi} \frac{c}{f} = 0.2295 \text{ m}$$

and than integrate Bessel function over the radius of the cavity

$$\int J_o^2 r dr d\theta dz = 2\pi L \int_o^a J_o^2(kr) r dr = \frac{2\pi L}{k^2} \int_o^{x_o} J_o^2(x) x dx;$$

$$x_o = 2.404825557695773 \dots: \int_o^{x_o} J_o^2(x) x dx = 0.779325$$

$$\frac{2\pi L}{k^2} \int_o^{x_o} J_o^2(x) x dx = 1.337 \cdot 10^{-2} \text{ m}^3 = 1.337 \cdot 10^4 \text{ cm}^3$$

Than using you favorite units we got identical

$$W = 1.48 \text{ J} = 1.48 \cdot 10^7 \text{ erg}$$

Side note: A smart RF engineer would use known value of R_{sh}/Q

$$\frac{R_{sh}}{Q_0} = \frac{V_{RF}^2}{\omega_0 W} \rightarrow W = \frac{V_{RF}^2}{\omega_0} / \frac{R_{sh}}{Q_0}$$

for pillbox cavity of 196 Ohm (Slide 10, Lecture 11) to get the same:

$$\frac{R_{sh}}{Q_0} = 196; V_{RF} = 9.54E5 \text{ V}; \omega_0 = 3.141593E9 \text{ Hz} \Rightarrow W = 1.48 \text{ J}$$

Finally we should notice that

$$V_{RF} \sim \sqrt{W} \rightarrow \frac{\Delta V_{RF}}{V_{RF}} = \frac{1}{2} \frac{\Delta W}{W}; \Delta W = -\Delta U$$

and maximum voltage drop in our case is

$$\Delta V_{RF} = -V_{RF} \frac{1}{2} \frac{\Delta U}{W} = -1.54 \text{ kV}; \frac{\Delta V_{RF}}{V_{RF}} = -1.6 \cdot 10^{-3} = 0.16\%$$

Beam loading dependence on the accelerating field (RF voltage) is very simple to find from following

$$\Delta U = qV_{RF} \sim E_o; W \sim V_{RF}^2 \sim E_o^2 \Rightarrow \frac{\Delta V_{RF}}{V_{RF}} \sim \frac{1}{V_{RF}} \sim \frac{1}{E_o}$$

e.g. beam loading is inverse proportional to the accelerating field. Thus, to increase beam loading from 0.16% to 1% we should make the accelerating voltage to be $0.16 E_o = 0.8 \text{ MV/m}$.

Problem 2. 5 points. Cavities filled with ferrite material are used for RF system requiring large frequency tuning range. The frequency is controlled by applying external magnetic field, B_{ext} , to the ferrite material and by doing so to change its permeability $\mu(B_{ext})$. A 300 m in circumference AGS synchrotron accelerates polarized protons from total energy of 2.5 GeV to 25 GeV.

- Calculate the range of the beam revolution frequency in AGS;
- Assuming 100% filling by ferrite, what should be ratio of μ_{max} to μ_{min} . Where μ should have maximum value?

Note: RF systems operate on a fixed integer harmonic of the revolution frequency.

Solution:

(a) Rest energy of a proton is 0.93837 GeV. It means that Lorentz factor changes from 2.66 to 26.6 and v/c changes from 0.9269 to 0.9993, e.g. revolution frequency increases 1.0781 fold during acceleration from 927.5 kHz to 999.987 kHz (e.g. 1 MHz!).

(b) Since the frequency of an RF cavity scales the same as speed of light in the media:

$$\omega_{res} = \frac{\omega_o}{\sqrt{\epsilon\mu}} \rightarrow \mu \propto \omega_{res}^{-2}$$

one should reduce μ 1.162-fold to accommodate necessary change in resonant frequency.