PHY 554

Fundamentals of Accelerator Physics

Lecture 14: Beam Dynamics in an Electron Storage Ring Part 1

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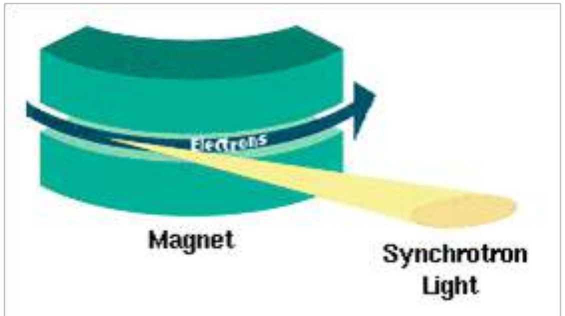
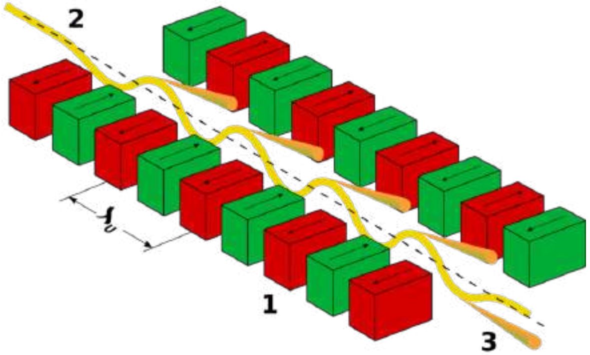
Why electron/positron storage rings are different?

SYNCHROTRON RADIATION

- Energy emitted to infinity or wall.
  - Form: Electromagnetic wave
  - Source: accelerating charged particles
  - Direction: Along the tangent of the beam trajectory





Synchrotron Radiation Power

The power and its distribution can be calculated from the 'retarded potential' – *look again into the dedicated lecture on SR*

$$P_{SR} = \frac{2e^2}{3c^3}\gamma^6\left[\left|\dot{\vec{\beta}}\right|^2 - \left[\vec{\beta}\times\dot{\vec{\beta}}\right]^2\right]$$

$$\frac{dE_{rad}}{dt} = \frac{2}{3}\frac{e^4\gamma^2}{m^2c^3}\left\{\left(\vec{E}-[\vec{\beta}\times\vec{B}]\right)^2-\left(\vec{\beta}\cdot\vec{E}\right)^2\right\};\ \vec{\beta}=\vec{\mathrm{v}}/c$$

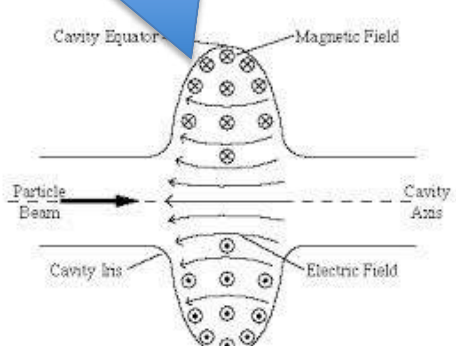
SGS

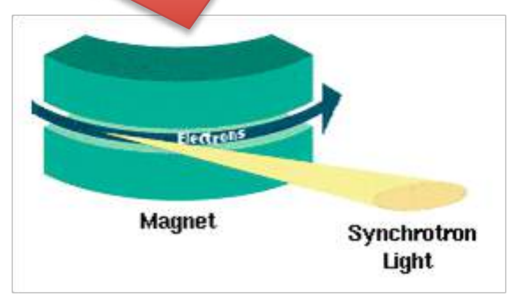
$$P_{SR} = \frac{e^2}{6\pi\varepsilon_o c^3}\gamma^6\left[\left|\dot{\vec{\beta}}\right|^2 - \left[\vec{\beta}\times\dot{\vec{\beta}}\right]^2\right]$$

SI

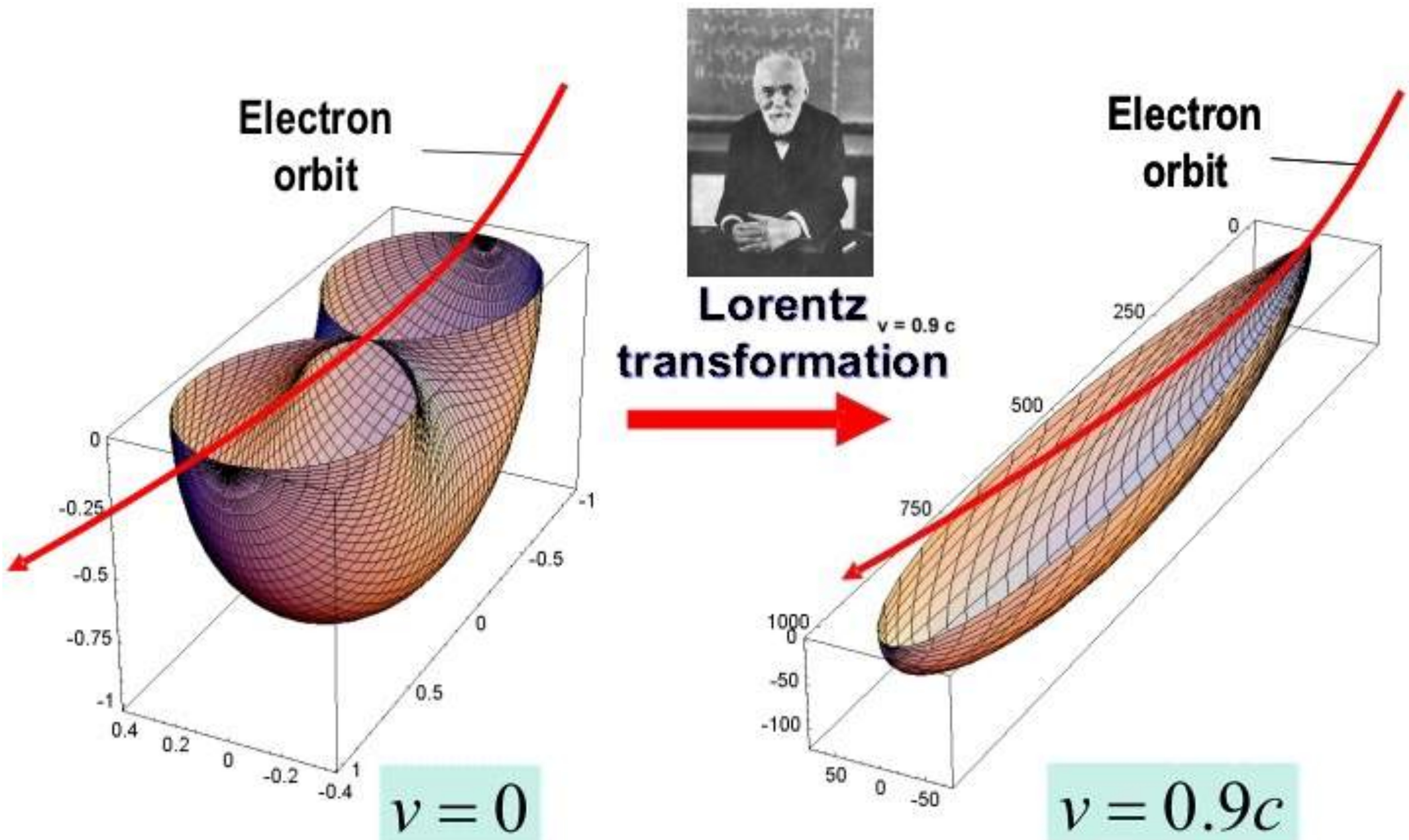
Radiation due to Acceleration (Negligible)

Radiation due to Bending (Dominating)





Radiation Angular Distribution



Opening angle in lab frame: $\theta \sim 1/\gamma;\ \gamma >> 1$

SR in storage ring

- The power of SR radiation in a dipole magnet

$$P_{SR} = \frac{2e^2}{3c^3}\gamma^6\left[\left|\dot{\vec{\beta}}\right|^2 - \left[\vec{\beta}\times\dot{\vec{\beta}}\right]^2\right]$$

$$\frac{dE_{rad}}{dt} = \frac{2}{3}\frac{e^4\gamma^2}{m^2c^3}\left\{\left(\vec{E} - [\vec{\beta}\times\vec{B}]\right)^2 - \left(\vec{\beta}\cdot\vec{E}\right)^2\right\};\ eB = \gamma\frac{mc^2}{\rho}$$

$$P_{SR} = \frac{2e^2c}{3\rho^2}\gamma^4 \qquad\qquad P_{SR} = \frac{e^2c}{6\pi\varepsilon_o\rho^2}\gamma^4$$

SGS　　　　SI

Energy Loss in e-ring

- In one turn, the energy loss is

SGS

$$\Delta E_{SR} = -\oint_C P\frac{ds}{c} = -\frac{2e^2}{3}\gamma^4\oint_C \frac{ds}{\rho^2} \equiv -\frac{2e^2}{3}\gamma^4 \cdot I_2$$

SI

$$\Delta E_{SR} = -\frac{e^2\gamma^4}{6\pi\varepsilon_o}\cdot I_2$$

$$\boxed{I_2 = \oint_C \frac{ds}{\rho^2} \equiv \oint_C K_o^{\,2}(s)\,ds}$$

The 2[nd] radiation integral I_2.

- In a iso-magnetic ring:

$$I_2 = \oint_C K_o^{\,2}(s)\,ds = 2\pi K_o$$

$$\Delta E_{SR} = -\frac{4\pi e^2}{3\rho}\gamma^4$$

$$\Delta E_{SR} = -\frac{e^2\gamma^4}{3\varepsilon_o\rho}$$

Energy losses, practical units

- For electrons and positrons:

$$\Delta E_{SR}\left[keV\right] = -88.46\frac{E\left[GeV\right]^4}{\rho\left[m\right]}$$

- For Protons:

$$\Delta E_{SR}\left[keV\right] = -6.03\frac{E\left[TeV\right]^4}{\rho\left[m\right]}$$

- Typically, the energy loss per turn is much less than the beam energy and should be restored by RF cavity.

What we get so far

- The SR energy loss per turn and power have strong energy dependence.

$$\Delta E_{SR}\left[keV\right] = -\boxed{C_\gamma \frac{E^4}{2\pi} I_2}$$

The 2nd radiation integral I_2

$$I_2 = \oint_C \frac{ds}{\rho^2} \equiv \oint_C K_o^{\,2}(s)\,ds$$

$$C_\gamma = \frac{4\pi}{3}\frac{r_c}{m^3c^6} = 8.85\times10^{-5}\left[\frac{m}{GeV^3}\right]$$

$$P_{SR}\left[keV\right] = \frac{cC_\gamma}{2\pi}\frac{E^4}{\rho^2}$$

SGS $\quad r_c = \dfrac{e^2}{mc^2}$

SI $\quad r_c = \dfrac{e^2}{4\pi\varepsilon_o mc^2}$

SR loss dependences

Radiation power of individual particle depends on its energy and its trajectory (position)

$$P_{loss} = \frac{2e^2c}{3}\frac{\gamma^4}{\rho(s,x,y)^2};\gamma = \frac{\mathbf{E}}{mc^2};\frac{1}{\rho} = \frac{eB(s,x,y)}{pc} = \frac{eB(s,x,y)}{\beta\mathbf{E}};r_c = \frac{e^2}{mc^2}$$

Dependence on energy of ultra relativistic particle: $\gamma \gg 1$

$$P_{loss} = c\frac{2r_c^{\ 2}B^2}{3\beta^2}\gamma^2;\gamma = \gamma_o + \Delta\gamma;\ \frac{\Delta\gamma}{\gamma_o} = \frac{\Delta\mathbf{E}}{\mathbf{E}} = \delta\beta_o^{\ 2};\delta = \frac{p-p_0}{p_0};$$

$$P_{loss} = \frac{2r_c^{\ 2}cB^2\gamma_o^{\ 2}}{3}\left(\beta^{-2}+2\delta\right)+O\left(\delta^2\right) = \frac{2r_c^{\ 2}cB^2\gamma_o^{\ 2}}{3}\left(1+2\delta\right)+O\left(\delta^2,\gamma_o^{\ -2}\right)$$

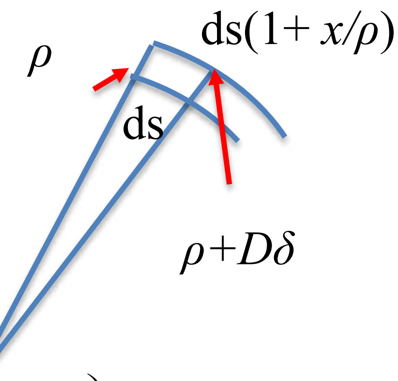
Dependence on particle's position

$$B_\perp^{\ 2} = B_x^{\ 2}+B_y^{\ 2};B_y = B_o(s)+\frac{\partial B_y}{\partial x}x;\ B_x = \frac{\partial B_y}{\partial x}y$$

$$B_\perp^{\ 2} = B_o^{\ 2}(s)\left(1+\frac{2}{B_o(s)}\frac{\partial B_y}{\partial x}x\right)+O\left(x^2,y^2\right)$$

SR loss dependences: continued

$$P_{loss} = P_{SR}\left(1 + 2\delta + \frac{2}{B_o(s)}\frac{\partial B_y}{\partial x}x\right); P_{SR}(s) = c\frac{2r_c^{\ 2}B_o(s)^2\gamma_o^{\ 2}}{3\beta_o^{\ 2}} \equiv \frac{2e^2c}{3\rho_o^{\ 2}}\gamma_o^{\ 4}$$



$$-\Delta E_{loss} = \oint P_{loss}(s)dt;\ dt = \frac{dl}{\mathrm{v}} = \frac{ds}{\mathrm{v}_o}\left(1 + \frac{x}{\rho_o(s)}\right) + dO\left(x^2, y^2, \delta^2\right)$$

All together

$$\Delta E_{loss} = -\frac{2e^2}{3}\gamma_o^{\ 4}\oint\frac{ds}{\rho_o(s)^2}\left(1 + 2\delta + \left(\frac{1}{\rho_o(s)} + \frac{2}{B_o(s)}\frac{\partial B_y}{\partial x}\right)\left(D(s)\delta + x_\beta\right)\right)$$

From Lecture 4:

$$K_x = \frac{1}{\rho_o^{\ 2}} + \frac{1}{B_o\rho_o}\frac{\partial B_y}{\partial x} \Rightarrow \frac{1}{\rho_o} + \frac{2}{B_o}\frac{\partial B_y}{\partial x} = \rho_o\left(2K_x - K_o^{\ 2}\right) \qquad K_o(s) \equiv \frac{1}{\rho(s)}$$

$$x = x_\beta + D\cdot\delta;\ \langle x_\beta\rangle = 0 \qquad I_2 = \oint_C \frac{ds}{\rho^2} \equiv \oint_C K_o^{\ 2}(s)ds$$

$$\Delta E_{loss} = -\frac{2e^2\gamma_o^{\ 4}}{3}I_2\left(1 + \left(2 + \frac{I_4}{I_2}\right)\delta\right);\ I_4 = \oint dsK_o(s)\left(2K_x(s) - K_o(s)^2\right)D(s)$$

The longitudinal motion, revisit

$$\Delta E_{loss} = \Delta E_{SR}\left(1+\left(2+\frac{I_4}{I_2}\right)\delta\right);\ \Delta E_{SR} = -\frac{2e^2\gamma_o{}^4}{3}I_2;$$

$$\dot{\delta} \equiv \frac{d\delta}{dn} = \frac{eV_{rf}}{\mathbf{E}_o\beta_o^2}\sin\phi + \Delta E_{loss}$$

$$eV_{rf}\sin\phi_s = -\Delta E_{SR};\ \phi_s = \arcsin\left(\frac{-\Delta E_{SR}}{eV_{rf}}\right)$$

$$\dot{\phi} = \frac{d\phi}{dn} = 2\pi h\eta\cdot\delta$$

$$\dot{\delta} \equiv \frac{d\delta}{dn} = \frac{eV_{rf}}{\mathbf{E}_o\beta_o^2}\left(\sin\phi - \sin\phi_s\right) + \frac{\Delta E_{SR}}{\mathbf{E}_o\beta_o^2}\left(2+\frac{I_4}{I_2}\right)\delta;$$

$$\dot{\phi} = \frac{d\phi}{dn} = 2\pi h\eta\cdot\delta$$

Now we have second term (dissipative!) in the longitudinal oscillations

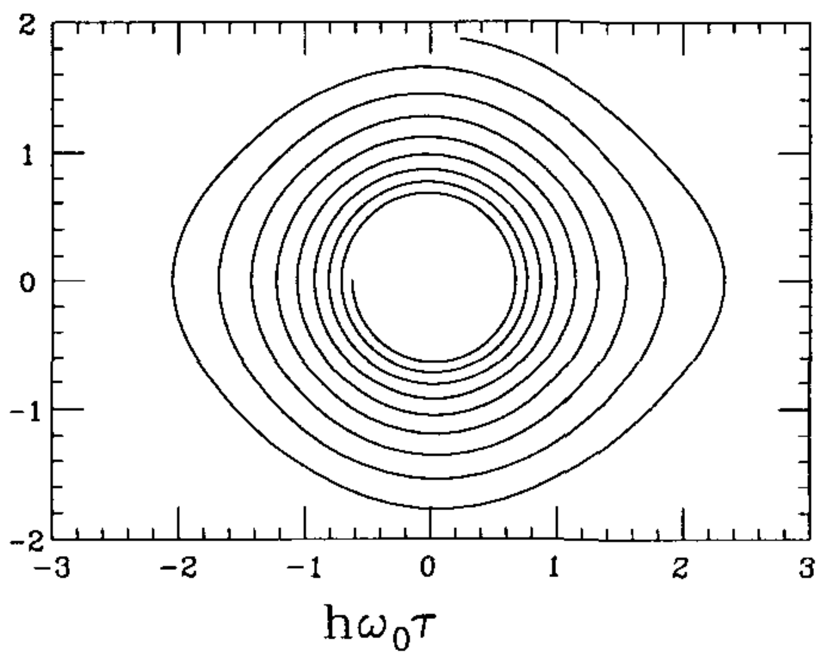
$$\ddot{\delta} \equiv 2\pi h\eta\cdot\frac{eV_{rf}}{\mathbf{E}_o\beta_o^2}\cos\phi\cdot\delta + \frac{\Delta E_{SR}}{\mathbf{E}_o\beta_o^2}\left(2+\frac{I_4}{I_2}\right)\dot{\delta}$$

The longitudinal motion, continued

$$\dot{\delta} \equiv \frac{d\delta}{dn} = \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2}\left(\sin\phi - \sin\phi_s\right) + \frac{\Delta E_{SR}}{\mathbf{E}_o \beta_o^2}\left(2 + \frac{I_4}{I_2}\right)\delta;$$

$$\dot{\phi} = \frac{d\phi}{dn} = 2\pi h \eta \cdot \delta$$

Oscillations decay when $2 + \frac{I_4}{I_2} > 0 \;\; or \;\; I_4 > -2I_2$



Damped Motion

For small oscillations, we can linearize 2nd order equation

$$\ddot{\delta} \equiv 2\pi h\eta \cdot \frac{eV_{rf}}{\mathbf{E}_o\beta_o^2}\cos\phi_s \cdot \delta + \frac{\Delta E_{SR}}{\mathbf{E}_o\beta_o^2}\left(2 + \frac{I_4}{I_2}\right)\dot{\delta};$$

Rewrite it as

$$\ddot{\delta} = -\Omega_s{}^2 \cdot \delta - 2\xi_s\dot{\delta}; \quad \xi_s = -\frac{\Delta E_{SR}}{\mathbf{E}_o\beta_o^2}\left(1 + \frac{I_4}{2I_2}\right) = \frac{2e^2}{3\mathbf{E}_o\beta_o^2}\gamma_o^4\left(I_2 + \frac{I_4}{2}\right)$$

Resulting damping oscillations

$$\delta = a_o e^{-\xi_s n}\cos\left(\Omega^*{}_s n + \varphi_o\right); \Omega^*{}_s = \sqrt{\Omega_s{}^2 - \xi_s{}^2}$$

Usually damping is very slow and

$$\xi_s{}^2 << \Omega_s{}^2;\ \Omega^*{}_s \cong \Omega_s \rightarrow \delta = a_o e^{-\xi_s n}\cos\left(\Omega_s n + \varphi_o\right)$$

Damping Partition Number

$$\delta E_{loss} = -\frac{2e^2\gamma_o{}^4}{3}\left(2I_2 + I_4\right)\delta; = \frac{4e^2\gamma_o{}^4}{3} I_2\left(2 + \bar{D}\right);$$

$$\bar{D} = \frac{I_4}{I_2} - called\ \ partition\ number$$

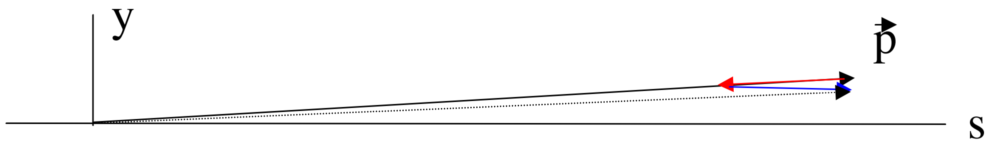
$$\xi_s = -\frac{\Delta E_{SR}}{\mathbf{E}_o\beta_o^2}\left(1 + \frac{\bar{D}}{2}\right) = \frac{2e^2}{3\mathbf{E}_o\beta_o^2}\gamma_o^4 I_2\left(1 + \frac{\bar{D}}{2}\right)$$

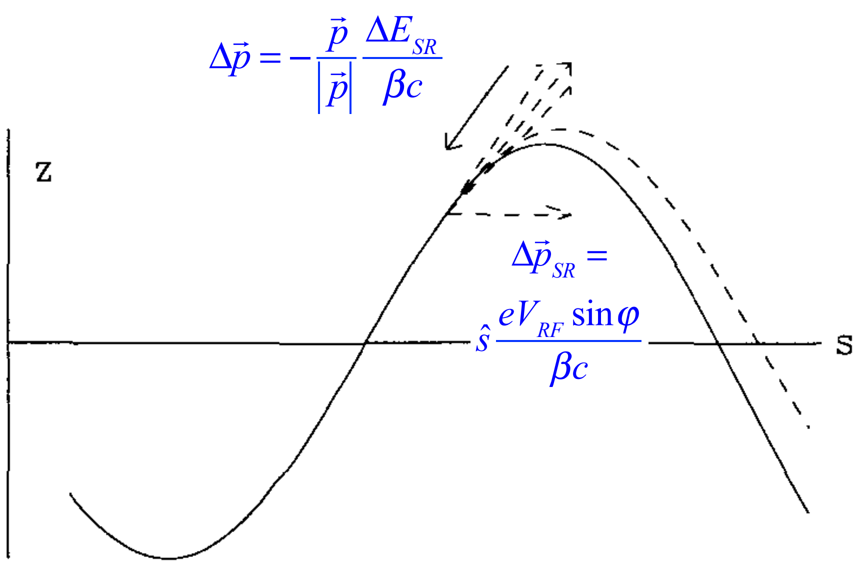
For separate function, iso-magnet ring (constant field in dipoles):

$$\bar{D} = \frac{\alpha_c C}{2\pi\rho_o} << 1 \qquad\qquad \xi_s \cong -\frac{\Delta E_{SR}}{\mathbf{E}_o\beta_o^2}$$

Transverse Damping (Vertical)

- The particle loose it's momentum in the very narrow cone ~1/γ along the direction of motion and in the RF system regains the lost portion momentum in s direction.



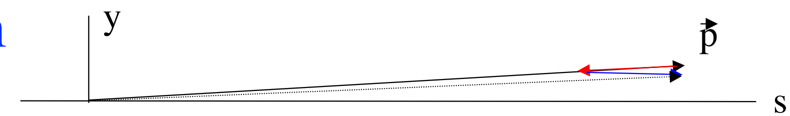


If we jump to the result: the damping rate in vertical plain is ~ half of that in longitudinal plane.

$$\alpha_y \cong -\frac{\Delta E_{SR}}{2\mathbf{E}_o}$$

Vertical Damping - details

- Let's consider case – which was not violated yet in real accelerator - when damping is much slower than betatron oscillations.
- Let's add radiation reaction in y direction



$$y'' + K_y(s)\, y = -\frac{P_{SR}(s)}{E_o c}\cdot y' \qquad dy' = -y'\frac{P_{SR}}{p\beta c^2};\ \beta \cong 1$$

$$y = y_\beta(s)\, e^{-\xi_y(s)};\ y_\beta(s) = a_o w_y(s)\cos\left(\psi(s)+\varphi_o\right);\ y_\beta'' + \left(K_y(s) + \xi_y''(s) - \xi_y'^2\right) y_\beta = 0$$

$$y' = y_\beta' e^{-\xi_y(s)} - \xi_y' y_\beta e^{-\xi_y(s)};\ y'' = y_\beta'' e^{-\xi_y(s)} - 2\xi_y' y_\beta' e^{-\xi_y(s)} + \left(\xi_y'^2 - \xi_y''\right) y_\beta e^{-\xi_y(s)}$$

$$-2\xi_y' y' = -\frac{P_{SR}(s)}{E_o c}\cdot y'$$

And getting the damping of vertical oscillations per turn

$$\xi_y' = \frac{1}{2}\frac{P_{SR}(s)}{\mathbf{E}_o c} \Rightarrow \xi_y \equiv \xi_y(C) = \frac{1}{2\mathbf{E}_o}\oint P_{SR}(s)\frac{ds}{c} = -\frac{1}{2}\frac{\Delta E_{SR}}{\mathbf{E}_o}$$

The damping rate in vertical plain is half of that in longitudinal plane for iso-magnetic lattice

Transverse Damping (Horizontal)

- Horizontal momentum has the same mechanism of losses as the vertical momentum - the particle loose it's momentum in the very narrow cone ~1/γ along the direction of motion and in the RF system regains the lost portion momentum in s direction.

$$dx' = -\frac{P_{SR}(s)}{E_o c} \cdot x' ds$$

- This case is more complicated because we need to take into account coupling with longitudinal motion.

$$x'' + K_x(s)x = \frac{\delta}{\rho(s)} - \frac{P_{SR}(s)}{E_o c} \cdot x'$$

$$x_\beta = x - D \cdot \delta \qquad D'' + K_x D = \frac{1}{\rho}$$

Horizontal Damping

- Combining this equations

$$x_\beta'' + K_x(s)x_\beta = -\frac{P_{SR}(s)}{\mathrm{E}_o c}\cdot x' - 2D'\delta' - D\cdot\delta'';$$

$$\delta' = \frac{d}{ds}\delta = -\frac{P_{SR}(s)}{\mathrm{E}_o c}\left(1 + 2\delta + \rho_o\left(2K_x - K_o{}^2\right)\left(D\delta + x_\beta\right)\right) = f(s,\delta) - \frac{P_{SR}(s)}{\mathrm{E}_o c}\rho_o\left(2K_x - K_o{}^2\right)x_\beta$$

$$\delta'' = f_1(s,\delta) - f_1(s)x_\beta - \frac{P_{SR}(s)}{\mathrm{E}_o c}\rho_o\left(2K_x - K_o{}^2\right)x_\beta'$$

- Removing δ (we already took care of it) we get collecting term in RHS

$$x_\beta'' + K_x(s)x_\beta = -\frac{P_{SR}(s)}{\mathrm{E}_o c}D\left(1 - \rho_o\left(2K_x - K_o{}^2\right)\right)\cdot x_\beta' + g(s)x_\beta;\;\; g(s) \sim \frac{P_{SR}(s)}{\mathrm{E}_o c}$$

Horizontal Damping

$$x_\beta'' + K_x(s)x_\beta = -\frac{P_{SR}(s)}{\mathrm{E}_o c}\left(1-\rho_o\left(2K_x - K_o^{\ 2}\right)D\right)\cdot x_\beta' + g(s)x_\beta;\ \ g(s) \sim \frac{P_{SR}(s)}{\mathrm{E}_o c};\ |g| <<< |K_x| \rightarrow$$

- Term proportional to x_β slightly phase of betatron oscillations, but the damping is result of the term in front of x_β' .
- We will use the same method as for vertical oscillations

$$x_\beta = x_o(s)e^{-\xi_x(s)};\quad x_o(s) = a_o w_x(s)\cos\left(\psi_x(s)+\varphi_o\right);$$

$$x_o'' + \left(K_x(s) - g(s) - \xi_x'^2 + \xi_x''\right)x_o = 0;$$

$$x_\beta' = \left(x_o' - \xi_x' x_o\right)e^{-\xi_x(s)}; x_\beta'' = \left\{x_o'' - 2\xi_x' x_\beta' + \left(\xi_x'^2 - \xi_x''\right)x_o\right\}e^{-\xi_x(s)}$$

$$\xi_x' = \frac{P_{SR}}{2\mathrm{E}_o c}\left(1-\rho_o\left(2K_x - K_o^{\ 2}\right)D\right)$$

$$\xi_x = \xi_x(C) = -\frac{2e^2}{3}\frac{\gamma_o^{\ 4}}{\mathbf{E}_o}\oint\frac{ds}{\rho_o(s)^2}\left(1-\rho_o\left(2K_x - K_o^{\ 2}\right)D\right) = \frac{2e^2}{3\mathbf{E}_o\beta_o^2}\gamma_o^4\left(I_2 - I_4\right)$$

$$\xi_x = -\frac{\Delta E_{SR}}{2\mathbf{E}_o}\left(1-\frac{I_4}{I_2}\right) \equiv -\frac{\Delta E_{SR}}{2\mathbf{E}_o}\left(1-\bar{D}\right)$$

Scaling with the energy

- Note here that all damping decrements are proportional to $\Delta E_{SR}/E \sim E^3 \sim \gamma^3$

Sum of decrements theorem

$$\boxed{\Delta E_{SR} = -\frac{2e^2}{3\mathbf{E}_o \beta_o^2}\gamma_o^4 I_2;}$$

$$\boxed{I_2 = \oint_C \frac{ds}{\rho^2} \equiv \oint_C K_o^{\,2}(s)\,ds}$$

$$\bar{D} = \frac{I_4}{I_2};\; I_4 = \oint ds \cdot DK_o\left(2K_x - K_o^{\,2}\right)$$

$$\boxed{\begin{gathered} \xi_{SR} = -\frac{\Delta E_{SR}}{2\mathbf{E}_o};\quad \xi_{x,y,s} = \zeta_{x,y,s} \cdot \xi_{SR}; \\ \zeta_x = 1 - \bar{D} \qquad \zeta_y = 1 \qquad \zeta_s = 2 + \bar{D} \\ \zeta_x + \zeta_y + \zeta_s = 4 \end{gathered}}$$

We derived decrement for storage rings without x-y coupling and plane orbits, but the sum of decrements theorem is fundamental and is valid for arbitrary storage ring

What we have learned

- SR can cause damping of betatron and synchrotron oscillations: roughly speaking, it take particle to radiate twice of its energy for amplitude of oscillations reduce e-fold in transverse and e^2-fold in longitudinal directions
- Damping decrements can be distributed between horizonal and longitudinal degrees of freedom
- Sum of dimensionless decrements is always equal 4
- *In next lecture we will add quantum fluctuations of synchrotron radiation, which prevent beam from collapsing*