PHY 564
Advanced Accelerator Physics
Lectures 16 & 17

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Because the parameterization and the action-angle are very useful tools for solving many standard accelerator problems, we will go through steps in a logical manner: from simple to more complicated cases.

Let's start from a simple 1D oscillator with Hamiltonian:

$$H = \frac{p^2}{2m} + k\frac{x^2}{2};\ \mathbf{H} = \begin{bmatrix} \frac{1}{m} & 0 \\ 0 & k \end{bmatrix}; \mathbf{D} = \begin{bmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{bmatrix} \tag{1}$$

in our standard matrix form. The equations of motion are:

$$X = \begin{bmatrix} x \\ p_x \end{bmatrix}; \frac{dX}{dt} = \mathbf{D}\cdot X \Leftrightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$
$$\lambda = i\omega \to -\lambda^2 = \omega^2 = \det\mathbf{D} = \frac{k}{m} \tag{2}$$

This is something you did studies many times before. The stability criteria is

$$k > 0$$

and general solution is

$$\mathbf{M}(t) = e^{\mathbf{D}t} = e^{i\omega t}\frac{i\omega I + D}{2i\omega} - e^{-i\omega t}\frac{i\omega I - D}{2i\omega} = I\cos\omega t + \frac{D}{\omega}\sin\omega t = \begin{bmatrix} \cos\omega t & \frac{\sin\omega t}{m\omega} \\ -\frac{k}{\omega}\sin\omega t & \cos\omega t \end{bmatrix}$$

$$X(t) = \begin{bmatrix} x \\ p_x \end{bmatrix} = \begin{bmatrix} \cos\omega t & \frac{\sin\omega t}{m\omega} \\ -\frac{k}{\omega}\sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} x_o \\ p_{xo} \end{bmatrix}; \frac{k}{\omega} = m\omega = \sqrt{mk} \tag{3}$$
$$x = x_o\cos\omega t + \frac{\sin\omega t}{m\omega}p_{xo};\quad p_x = p_{xo}\cos\omega t - x_o\frac{k}{\omega}\sin\omega t.$$

We know another form of these equations of motions:

$$x = A\cos(\omega t+\varphi);\quad p_x = -m\omega\cdot A\cdot\sin(\omega t+\varphi) \tag{4}$$

which is a consequence of

$$\mathbf{M}Y = e^{i\omega t}Y;\; Y = \begin{bmatrix} w \\ \dfrac{i}{w} \end{bmatrix}; w = \frac{1}{\sqrt{m\omega}};\; X(t) = a\,\mathrm{Re}\,Ye^{i\omega t+\varphi} = \begin{bmatrix} \dfrac{a}{\sqrt{m\omega}}\cos(\omega t+\varphi) \\ -a\sqrt{m\omega}\sin(\omega t+\varphi) \end{bmatrix}; \tag{5}$$

which coincides with (4) using $A = a\sqrt{m\omega}$. Naturally, we know that amplitude and phase are constant of motion.

Let's make a Canonical transformation to $\left\{\tilde{q}=\varphi, \tilde{p}=\dfrac{a^2}{2}\right\}$ using

$$\begin{aligned} &F(q=x,\tilde{q}=\varphi) = -m\omega\frac{x^2}{2}\tan(\omega t+\varphi); \\ &\frac{\partial F}{\partial x} = -m\omega x\tan(\omega t+\varphi) = -m\omega\frac{a}{\sqrt{m\omega}}\sin(\omega t+\varphi) = p_x; \\ &\tilde{p} = -\frac{\partial F}{\partial \varphi} = m\omega\frac{x^2}{2\cos^2(\omega t+\varphi)} = \frac{a^2}{2} = I \end{aligned} \tag{6}$$

Finally, the reduced Hamiltonian is

$$\tilde{H} = H + \frac{\partial F}{\partial t} = \frac{p^2}{2m} + k\frac{x^2}{2} - m\omega^2 \frac{x^2}{2\cos^2(\omega t + \varphi)} =$$
$$\omega\frac{a^2}{2}\left(\cos^2(\omega t + \varphi) + \sin^2(\omega t + \varphi)\right) - \omega\frac{a^2}{2} = 0 \tag{7}$$

Thus, if

$$H = \frac{p^2}{2m} + k\frac{x^2}{2} + H_1(x,p) \to \tilde{H}(\varphi, I) = H - \left(\frac{p^2}{2m} + k\frac{x^2}{2}\right) = H_1(x(I,\varphi), p(I,\varphi))$$

$$\frac{d\varphi}{dt} = \frac{\partial \tilde{H}}{\partial I}; \frac{dI}{dt} = -\frac{\partial \tilde{H}}{\partial \varphi}.$$

General parameterization of motion of an arbitrary linear Hamiltonian system:

$$H = \frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n} h_{ij}(s)x_i x_j \equiv \frac{1}{2}X^T \cdot \mathbf{H}(s)\cdot X \,,$$

$$X^T = \begin{bmatrix} q^1 & P_1 & ..... & .... & q^n & P_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & ..... & .... & x_{2n-1} & x_{2n} \end{bmatrix}.$$

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X;\ \mathbf{D} = \mathbf{S}\cdot\mathbf{H}(s),\ X(s) = \mathbf{M}\left(s_o|s\right)\cdot X_o. \qquad \backslash \qquad \text{(I)}$$

$$\mathbf{M}' \equiv \frac{d\mathbf{M}}{ds} = \mathbf{D}(s)\cdot\mathbf{M};\ \ \mathbf{M}\left(s_o\right) = \mathbf{I},\ \mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M} = \mathbf{S},\ \ \mathbf{M}^{-1} = -\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}. \qquad \text{(II)}$$

Parameterization of motion: a periodic system with period C $\mathbf{H}(s+C) = \mathbf{H}(s)$

$$\mathbf{T}(s) = \mathbf{M}\left(s|s+C\right),\ \det\left[\mathbf{T} - \lambda_i\cdot\mathbf{I}\right] = 0 \qquad \text{(III)}$$

Stable system $|\lambda_i| = 1.\ \lambda_k \equiv 1/\lambda_{k+n} \equiv \lambda^*{}_{k+n} \equiv e^{i\mu_k};\ \ \mu_k \equiv 2\pi\nu_k,\ \ \{k = 1,...n\}.$ (IV)

We proved that

$$\tilde{Y}_k(s) = Y_k(s)e^{\psi_k(s)}; \quad Y_k(s+C) = Y_k(s); \qquad \psi_k(s+C) = \psi_k(s) + \mu_k$$

$$\tilde{\mathbf{U}} = \left[\ldots \tilde{Y}_k, \tilde{Y}^*_k \ldots \right]; \ \tilde{\mathbf{U}}(s) = \mathbf{U}(s) \cdot \Psi(s), \ \Psi(s) = \begin{pmatrix} e^{i\psi_1(s)} & 0 & & 0 \\ 0 & e^{-i\psi_1(s)} & & 0 \\ & & \ldots & 0 \\ 0 & 0 & 0 & e^{-i\psi_n(s)} \end{pmatrix}$$

$$X = \frac{1}{2}\sum_{k=1}^{n}\left(a_k\tilde{Y}_k + a^*_k\tilde{Y}^*_k\right) = \mathrm{Re}\sum_{k=1}^{n} a_k\tilde{Y}_k = \frac{1}{2}\tilde{\mathbf{U}}A = \frac{1}{2}\mathbf{U}\Psi A = \frac{1}{2}\mathbf{U}\tilde{A}$$

$$a_i = \frac{1}{i}Y_i^{*T}SX; \ \tilde{a}_i \equiv a_i e_i^{i\psi} = \frac{1}{i}Y_i^{*T}SX;$$

$$A = 2\tilde{\mathbf{U}}^{-1} \cdot X = -i\Psi^{-1} \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X; \ \tilde{A} = \Psi A = -i \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X.$$

$$\mathbf{T}(s)=\mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s);\ \Lambda=\begin{pmatrix}\lambda_1 & 0 & & 0\\ 0 & \lambda_1^{\ *} & & 0\\ & & \ldots & 0\\ 0 & 0 & 0 & \lambda_n^{\ *}\end{pmatrix};\quad \mathbf{T}\cdot\mathbf{U}=\mathbf{U}\cdot\Lambda;\ \Lambda=\mathbf{U}^{-1}\cdot\mathbf{T}\cdot\mathbf{U}.... \tag{V}$$

$$\mathbf{U}(s)=\left[Y_1,Y_1^{\ *}....Y_n,Y_n^{\ *}\right];\qquad \mathbf{T}(s)Y_k(s)=\lambda_k Y_k(s)\quad\Leftrightarrow\quad \mathbf{T}(s)Y_k^{\ *}(s)=\lambda_k^{\ *}Y_k^{\ *}(s) \tag{VI}$$

$$\tilde{Y}_k(s_1)=\mathbf{M}\left(s|s_1\right)\tilde{Y}_k(s)\ \Leftrightarrow\ \frac{d}{ds}\tilde{Y}_k=\mathbf{D}\left(s\right)\cdot\tilde{Y}_k;\qquad\qquad \tilde{\mathbf{U}}(s_1)=\mathbf{M}\left(s|s_1\right)\tilde{\mathbf{U}}(s)\ \Leftrightarrow\ \frac{d}{ds}\tilde{\mathbf{U}}=\mathbf{D}\left(s\right)\cdot\tilde{\mathbf{U}} \tag{VII}$$

$$\tilde{\mathbf{U}}(s+C)=\tilde{\mathbf{U}}(s)\cdot\Lambda,\ \tilde{Y}_k(s+C)=\lambda_k\tilde{Y}_k(s)=e^{i\mu_k}\tilde{Y}_k(s) \tag{VIII}$$

$$\tilde{Y}_k(s)=Y_k(s)e^{\psi_k(s)};\quad Y_k(s+C)=Y_k(s);\qquad \psi_k\left(s+C\right)=\psi_k\left(s\right)+\mu_k \tag{IX}$$

$$\tilde{\mathbf{U}}(s)=\mathbf{U}(s)\cdot\Psi(s),\ \Psi(s)=\begin{pmatrix}e^{i\psi_1(s)} & 0 & & 0\\ 0 & e^{-i\psi_1(s)} & & 0\\ & & \ldots & 0\\ 0 & 0 & 0 & e^{-i\psi_n(s)}\end{pmatrix} \tag{X}$$

$$Y_k^{T*} \cdot \mathbf{S} \cdot Y_{j \neq k} = 0; \quad Y_k^{T} \cdot \mathbf{S} \cdot Y_j = 0; \quad Y_k^{T*} \cdot \mathbf{S} \cdot Y_k = 2i \tag{XI}$$

$$\mathbf{U}^T \cdot \mathbf{S} \cdot \mathbf{U} \equiv \tilde{\mathbf{U}}^T \cdot \mathbf{S} \cdot \tilde{\mathbf{U}} = -2i\mathbf{S}, \;\; \mathbf{U}^{-1} = \frac{1}{2i} \mathbf{S} \cdot \mathbf{U}^T \cdot \mathbf{S} \tag{XII}$$

$$X(s) = \frac{1}{2}\sum_{k=1}^{n}\left(a_k \tilde{Y}_k + a_k{}^* \tilde{Y}_k{}^*\right) \equiv \mathrm{Re}\sum_{k=1}^{n} a_k Y_k e^{i\psi_k} \equiv \frac{1}{2}\tilde{\mathbf{U}} \cdot A = \frac{1}{2}\mathbf{U} \cdot \mathbf{\Psi} \cdot A = \frac{1}{2}\mathbf{U} \cdot \tilde{A} \tag{XIII}$$

$$\begin{gathered} a_i = \frac{1}{2i} Y_i^{*T} SX; \; \tilde{a}_i \equiv a_i e^{i\psi}{}_i = \frac{1}{2i} Y_i^{*T} SX; \\ A = 2\tilde{\mathbf{U}}^{-1} \cdot X = -i\mathbf{\Psi}^{-1} \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X; \; \tilde{A} = \Psi A = -i \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X. \end{gathered} \tag{XIV}$$

and finished with fact that this parameterization is equivalent to Canonical transformation to action-angle variables.

$$\left\{ \varphi_k, I_k = \frac{a_k^2}{2} \right\}; k = 1,2..,n$$

Than if $\mathcal{H} = \frac{1}{2} X^T \mathbf{H}(s) X + \mathcal{H}_1(X, s)$

$$\tilde{H}(\varphi_k, I_k, s) = \mathcal{H}_1(X(\varphi_k, I_k, s), s);$$

$$\frac{d\varphi_k}{ds} = \frac{\partial \tilde{H}}{\partial I_k}; \frac{dI_k}{ds} = -\frac{\partial \tilde{H}}{\partial \varphi_k}.$$

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1D – ACCELERATOR

$$\tilde{h} = \frac{p^2}{2} + K_1(s)\frac{y^2}{2};\ \mathbf{H} = \begin{bmatrix} K_1 & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{D} = \mathbf{SH} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix};$$

$$\frac{d}{ds}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} p \\ -K_1 x \end{bmatrix}\ (i.e.\ x' \equiv p).$$

$$Y = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix}; \psi' = \frac{1}{\mathrm{w}^2}; \tilde{Y} = Ye^{i\psi}$$

$$\mathbf{T}(s) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s);\ \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} = \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{pmatrix};$$

$$\mathbf{T} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;$$

$$\mathbf{J} = \begin{bmatrix} -ww' & w^2 \\ -w'^2 - \dfrac{1}{w^2} & ww' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}; \quad \mathbf{J}^2 = -\mathbf{I}$$

Note: for not normalized Hamiltonian $H = \frac{p_y^2}{2p_o} + p_o K_1(s)\frac{y^2}{2}$ and

$$Y = \begin{bmatrix} \dfrac{w}{\sqrt{p_o}} \\ \sqrt{p_o}\left(w' + i / w\right) \end{bmatrix}; \psi' = \frac{1}{w^2}; \tilde{Y} = Ye^{i\psi}$$

$$\mathbf{T} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;$$

$$\mathbf{J} = \begin{bmatrix} -ww' & w^2 \\ -w'^2 - \dfrac{1}{w^2} & ww' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}; \quad \mathbf{J}^2 = -\mathbf{I}; \quad \gamma = \left(1+\alpha^2\right)/\beta$$

$$\cos\mu = Trace(\mathbf{T})/2 = \frac{T_{11}+T_{22}}{2}$$

$$Stability\ if: -1 < Trace(\mathbf{T})/2 < 1$$

$$w^2 \equiv \beta = \frac{T_{12}}{\sin\mu} = \frac{\left|T_{12}\right|}{\sqrt{1-\left(Trace(\mathbf{T})/2\right)^2}}; w = \sqrt{\frac{\left|T_{12}\right|}{\sqrt{1-\left(Trace(\mathbf{T})/2\right)^2}}};$$

$$ww' \equiv -\alpha = \frac{T_{22}-T_{11}}{2\cos\mu} = -\frac{T_{11}-T_{22}}{T_{11}+T_{22}}$$

1D - ACCELERATOR

$$Y = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix}; \tilde{Y} = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix} e^{i\psi}; \mathbf{U} = \begin{bmatrix} \mathrm{w} & \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} & \mathrm{w}' - i/\mathrm{w} \end{bmatrix}; \tilde{\mathbf{U}} = \mathbf{U} \cdot \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}$$

$$\mathrm{w}'' + K_1(\mathrm{s})\mathrm{w} = \frac{1}{\mathrm{w}^3}, \quad \psi' = 1/\mathrm{w}^2; \begin{bmatrix} x \\ x' \end{bmatrix} = \mathrm{Re}\left(a e^{i\varphi} \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix} e^{i\psi} \right);$$

$$x = a \cdot \mathrm{w(s)} \cdot \cos(\psi(s) + \varphi)$$
$$x' = a \cdot \left(\mathrm{w}'(\mathrm{s}) \cdot \cos(\psi(s) + \varphi) - \sin(\psi(s) + \varphi)/\mathrm{w(s)} \right)$$

$$\beta \equiv \mathrm{w}^2 \;\Rightarrow\; \psi' = 1/\beta; \alpha \equiv -\beta' \equiv -\mathrm{w}\,\mathrm{w}', \; \gamma \equiv \frac{1+\alpha^2}{\beta} \text{ - definitions}$$

$$x = a \cdot \sqrt{\beta(\mathrm{s})} \cdot \cos(\psi(s) + \varphi)$$
$$x' = -\frac{a}{\sqrt{\beta(\mathrm{s})}} \cdot \left(\alpha(\mathrm{s}) \cdot \cos(\psi(s) + \varphi) + \sin(\psi(s) + \varphi) \right)$$

Complex amplitude and real amplitude and phase are easy to calculate. Expression for a^2 is called Currant-Snyder invariant.

Thus, the final form of the eigen vector can be rewritten as

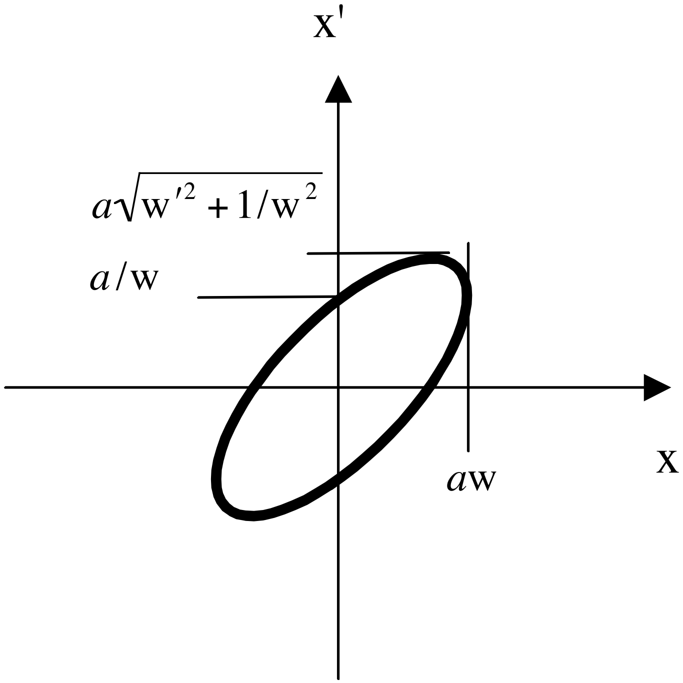
$$Y = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix}; \psi' = \frac{1}{\mathrm{w}^2}; \tilde{Y} = Ye^{i\psi} \tag{34}$$

The parameterization of the linear 1D motion is

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \mathrm{Re}\left(ae^{i\varphi} \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix} e^{i\psi} \right);$$

$$\begin{aligned} x &= a \cdot \mathrm{w(s)} \cdot \cos(\psi(s) + \varphi) \\ x' &= a \cdot \left(\mathrm{w}'(\mathrm{s}) \cdot \cos(\psi(s) + \varphi) - \sin(\psi(s) + \varphi)/\mathrm{w(s)} \right) \end{aligned} \tag{35}$$

where ***a*** and $\boldsymbol{\varphi}$ are the constants of motion.



$$X = \mathrm{Re}\,\tilde{a}Y;$$

$$ae^{i\varphi} = -\mathrm{i}Y^{*T}SX = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' - \mathrm{i}/\mathrm{w} \end{bmatrix}^T \cdot \begin{bmatrix} \mathrm{x}' \\ \text{-x} \end{bmatrix} = \mathrm{x}/\mathrm{w} + \mathrm{i}(\mathrm{w}'\mathrm{x} - \mathrm{w}\mathrm{x}')$$

$$a^2 = \frac{\mathrm{x}^2}{\mathrm{w}^2} + (\mathrm{w}'\mathrm{x} - \mathrm{w}\mathrm{x}')^2 \equiv \frac{\mathrm{x}^2 + (\alpha\mathrm{x} + \beta\mathrm{x}')^2}{\beta} \equiv \gamma\mathrm{x}^2 + 2\alpha\mathrm{x}\mathrm{x}' + \beta\mathrm{x}'^2$$

$$\varphi = \arg\left(\mathrm{x}/\mathrm{w} + \mathrm{i}(\mathrm{w}'\mathrm{x} - \mathrm{w}\mathrm{x}')\right) = \tan^{-1}\frac{\mathrm{w}\mathrm{w}'\mathrm{x} - \mathrm{w}^2\mathrm{x}'}{\mathrm{x}} = -\tan^{-1}\frac{\alpha\mathrm{x} + \beta\mathrm{x}'}{\mathrm{x}};$$

$$\varphi = \sin^{-1}\frac{\mathrm{w}'\mathrm{x} - \mathrm{w}\mathrm{x}'}{\sqrt{\mathrm{x}^2 + \mathrm{w}^2(\mathrm{w}'\mathrm{x} - \mathrm{w}\mathrm{x}')^2}} = -\sin^{-1}\frac{\alpha\mathrm{x} + \beta\mathrm{x}'}{\sqrt{\gamma\mathrm{x}^2 + 2\alpha\mathrm{x}\mathrm{x}' + \beta\mathrm{x}'^2}}$$

Inverse ratios – matrices through parameterization: reverse of eq (VII), where U is propagated by M.

$$\mathbf{M}(s_1|s_2) = \tilde{\mathbf{U}}(s_2)\tilde{\mathbf{U}}^{-1}(s_1) = \frac{i}{2}\tilde{\mathbf{U}}(s_2)\cdot\mathbf{S}\cdot\tilde{\mathbf{U}}^T(s_1)\cdot\mathbf{S} = \frac{i}{2}\mathbf{U}(s_2)\cdot\Psi(s_2)\cdot\mathbf{S}\cdot\Psi^{-1}(s_1)\cdot\mathbf{U}^T(s_1)\cdot\mathbf{S} \qquad \text{(XV)}$$

$$\Psi(s_2)\cdot\mathbf{S}\cdot\Psi^{-1}(s_1) \equiv \Psi(s_2 - s_1)\cdot\mathbf{S}; \quad \mathbf{M}(s_1|s_2) = \frac{i}{2}\mathbf{U}(s_2)\cdot\Psi(s_2 - s_1)\cdot\mathbf{S}\cdot\mathbf{U}^T(s_1)\cdot\mathbf{S}$$

$$\mathbf{T} = \mathbf{U}\Lambda\mathbf{U}^{-1} = \frac{i}{2}\mathbf{U}\Lambda\mathbf{S}\mathbf{U}^T\mathbf{S}$$

Specific case of $\boldsymbol{s_1 = s + C}$

$$\mathbf{T} = \frac{1}{2i}\mathbf{U}\Lambda\mathbf{S}\mathbf{U}^T\mathbf{S} = \frac{1}{2i}\begin{bmatrix} w & w \\ w' + \frac{i}{w} & w' - \frac{i}{w} \end{bmatrix}\cdot\begin{bmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{bmatrix}\cdot$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\cdot\begin{bmatrix} w & w' + \frac{i}{w} \\ w & w' - \frac{i}{w} \end{bmatrix}\cdot\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} =$$

$$\frac{1}{2i}\begin{bmatrix} we^{i\mu} & we^{-i\mu} \\ \left(w' + \frac{i}{w}\right)e^{i\mu} & \left(w' - \frac{i}{w}\right)e^{-i\mu} \end{bmatrix}\cdot\begin{bmatrix} \frac{i}{w} - w' & w \\ \frac{i}{w} + w' & -w \end{bmatrix} =$$

$$\begin{bmatrix} \frac{e^{i\mu} + e^{-i\mu}}{2} - \frac{e^{i\mu} - e^{-i\mu}}{2i}ww' & w^2\frac{e^{i\mu} - e^{-i\mu}}{2i} \\ -\left(w'^2 + \frac{1}{w^2}\right)\frac{e^{i\mu} - e^{-i\mu}}{2} & \frac{e^{i\mu} + e^{-i\mu}}{2} + \frac{e^{i\mu} - e^{-i\mu}}{2i}ww' \end{bmatrix}$$

$$\mathbf{T} = \frac{1}{2i}\mathbf{U}\Lambda\mathbf{S}\mathbf{U}^T\mathbf{S} = \begin{bmatrix} \cos\mu - ww'\cdot\sin\mu & w^2\cdot\sin\mu \\ -\left(w'^2 + \dfrac{1}{w^2}\right)\cdot\sin\mu & \cos\mu + ww'\cdot\sin\mu \end{bmatrix} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;$$

$$\mathbf{J} = \begin{bmatrix} -ww' & w^2 \\ -w'^2 - \dfrac{1}{w^2} & ww' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix};$$

$$\mathbf{J}^2 = \begin{bmatrix} -ww' & w^2 \\ -\left(w'^2 + \dfrac{1}{w^2}\right) & ww' \end{bmatrix}\begin{bmatrix} -ww' & w^2 \\ -\left(w'^2 + \dfrac{1}{w^2}\right) & ww' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}$$

with obvious simplification for one-turn matrix:

$$\mathbf{T} = M(s|s+C) = \begin{bmatrix} t11 & t12 \\ t21 & t22 \end{bmatrix} = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\dfrac{(1+\alpha^2)\sin\mu}{\beta} & \cos\mu - \alpha\sin\mu \end{bmatrix} \mathbf{I}\cos\mu + \mathbf{J}\sin\mu; \ \mathbf{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}$$

from where one can get easily all functions and constants:

$$\mu = \cos^{-1}\left(\frac{t11 + t22}{2}\right) \mathit{with}\ \mathit{sign}(\sin\mu) = \mathit{sign}(t12); \beta = t12/\sin\mu; \ \alpha = \frac{t11 - t22}{2\sin\mu}$$

A little bit more complex is fully coupled 2D case.

$$\mathbf{T}(s) = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}; \ \det[\mathbf{T} - \lambda I] = \lambda^4 - \lambda^3 Tr[T] + \lambda^2 (2 + a) - \lambda Tr[T] + 1 =$$

$$a = Tr[N_1] \cdot Tr[N_4] - Tr[N_2 N_3] - 2\det N_2$$

$$(note \ \det N_2 = \det N_3 = 1 - \det N_1 = 1 - \det N_4)$$

Finding roots:

$$z_k = \lambda_k + \lambda_k^{-1}; \ z_k = \frac{Tr[N_1 + N_1]}{2} \pm \left\{ \frac{Tr^{\ 2}[N_1 - N_1]}{4} + Tr[N_2 N_3] + 2\det N_2 \right\}$$

$$X = \begin{bmatrix} x \\ P_x \\ y \\ P_y \end{bmatrix} = \mathrm{Re}\,\tilde{a}_1 Y_1 + \mathrm{Re}\,\tilde{a}_1 Y_2 = \mathrm{Re}\,a_1 \tilde{Y}_1 + \mathrm{Re}\,\tilde{a}_1 \tilde{Y}_2$$

$$Y_k = R_k + iQ_k; \quad \tilde{Y}_k = \begin{bmatrix} w_{kx} e^{i\psi_{kx}} \\ \left(u_{kx} + iv_{kx}\right) e^{i\psi_{kx}} \\ w_{ky} e^{i\psi_{ky}} \\ \left(u_{ky} + iv_{ky}\right) e^{i\psi_{ky}} \end{bmatrix};$$

$$\psi_{kx}(s+C) = \psi_{kx}(s) + \mu_k; \psi_{ky}(s+C) = \psi_{ky}(s) + \mu_k;$$

$$w_{kx} v_{kx} + w_{ky} v_{ky} = 1;$$

Conditions: there are

$$Y_k^{*T} S Y_k = 2i; \quad Y_1^{*T} S Y_2 = 0; \quad Y_1^T S Y_2 = 0; \quad \theta_k = \psi_{kx} - \psi_{ky}$$

$$a) \quad w_{1x} v_{1x} = w_{2y} v_{2y} = 1 - q \qquad \Rightarrow v_{1x} = \frac{1-q}{w_{1x}}; \quad v_{2y} = \frac{1-q}{w_{2y}}$$

$$b) \quad w_{1y} v_{1y} = w_{2x} v_{2x} = q \quad \Rightarrow v_{2x} = \frac{q}{w_{2x}}; \quad w_{1y} = \frac{q}{w_{1y}}$$

$$c) \quad c = w_{1x} w_{1y} \sin\theta_1 = -w_{2x} w_{2y} \sin\theta_2$$

$$d) \quad d = w_{1x}\left(u_{1y}\sin\theta_1 - v_{1y}\cos\theta_1\right) = -w_{2x}\left(u_{2y}\sin\theta_2 - v_{2y}\cos\theta_2\right)$$

$$e) \quad e = w_{1y}\left(u_{1x}\sin\theta_1 + v_{1x}\cos\theta_1\right) = -w_{2y}\left(u_{2x}\sin\theta_2 + v_{2x}\cos\theta_2\right)$$

Conditions are result of symplecticity. Conditions a) and b) are equivalent to Poincaré's invariants conserving sum of projections on (x-px) and (y-py) planes.

$$Y_1 = \begin{bmatrix} w_{1x} e^{i\varphi_{1x}} \\ \left(u_{1x} + i \dfrac{q}{w_{1x}} \right) e^{i\varphi_{1x}} \\ w_{1y} e^{i\varphi_{1y}} \\ \left(u_{1y} + i \dfrac{1-q}{w_{1y}} \right) e^{i\varphi_{1y}} \end{bmatrix}; \quad Y_2 = \begin{bmatrix} w_{2x} e^{i\varphi_{2x}} \\ \left(u_{2x} + i \dfrac{1-q}{w_{2x}} \right) e^{i\varphi_{2x}} \\ w_{2y} e^{i\varphi_{2y}} \\ \left(u_{2y} + i \dfrac{q}{w_{2y}} \right) e^{i\varphi_{2y}} \end{bmatrix}$$

One should note for completeness that there is another way of parameterization of coupled motion proposed by Edward and Teng parameterization (*D.A.Edwards, L.C.Teng, IEEE Trans. Nucl. Sci.* ***NS-20*** *(1973) 885*), which differs from what we are discussing here.

Solutions of standard accelerator problems

Q: Why we need parameterization? -> A: To comfortably solve typical accelerator problems

Lecture 17

Applications of parameterization to standard problems

Complete parameterization developed in previous lecture can be used to solve most (if not all) of standard problems in accelerator. Incomplete list is given below:

1. Dispersion
2. Orbit distortions
3. AC dipole (periodic excitation)
4. Tune change with quadrupole (magnets) changes
5. Chromaticity
6. Beta-beat
7. Weak coupling
8. Synchro-betatron coupling
9. ……..

We do not plan to go through all these examples while focusing on general methodology and use selected examples to demonstrate power of the symplectic linear parameterization.

Sample I. Let's start from simplest problems such as dispersion and closed orbit. We found a general form of parameterization of linear motion in Hamiltonian system, which is solution of homogeneous linear equations, where ***B*** is constant vector:

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X;\ X = \tilde{\mathbf{U}}(s)\cdot B \qquad (17\text{-}1)$$

A standards problems is a solution of inhomogeneous equations:

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X + F(s); \qquad (17\text{-}2)$$

It can be done analytically by varying the constant ***B:***

$$X = \tilde{\mathbf{U}}(s)B(s) \Rightarrow \tilde{\mathbf{U}}\cdot B' = F(s) \Rightarrow B' = \tilde{\mathbf{U}}^{-1}(s)F(s) \Rightarrow B(s) = B_o + \int_{s_o}^{s} \tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi$$

A general solution is a specific solution of inhomogeneous equation plus arbitrary solution of the homogeneous – result you expect in linear ordinary differential equations (in this case with s-depended coefficients):

$$X(s) = \tilde{\mathbf{U}}(s)A_o + \tilde{\mathbf{U}}(s)\int_{s_o}^{s} \tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi;\ \ \tilde{\mathbf{U}}^{-1} = \frac{i}{2}\mathbf{S}\cdot\tilde{\mathbf{U}}^{T}\cdot\mathbf{S} \qquad (17\text{-}3)$$

For a periodic force (orbit distortions, dispersion function) $F(s+C)=F(s)$ one can fine periodic solution $X(s+C)=X(s)$:

$$\tilde{\mathbf{U}}^{-1}(s)\times\left\{\tilde{\mathbf{U}}(s)A_o+\tilde{\mathbf{U}}(s)\int_{s_o}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi=\tilde{\mathbf{U}}(s+C)A_o+\tilde{\mathbf{U}}(s+C)\int_{s_o}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\right\}$$

$$A_o(\mathbf{I}-\Lambda)=\Lambda\int_{s}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\equiv\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\Rightarrow A_o=(\mathbf{I}-\Lambda)^{-1}\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi \qquad (17\text{-}4)$$

$$X(s)=\tilde{\mathbf{U}}(s)(\mathbf{I}-\Lambda)^{-1}\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi$$

It is easy to see that $X(s+C)=X(s)$ exists if none of the eigen values is not equal 1 – otherwise matrix $(\mathbf{I}-\Lambda)$ would have zero determinant and can not be inverted!

It is called integer resonance - closed orbit does not exist!

Specific examples: Orbit distortions caused by the field errors, transverse dispersion.

When the conditions for the equilibrium particle and the reference trajectory are slightly violated:

$$X^T = \{x, P_1, y, P_3, \tau, \delta\}; F^T = \left\{0, \frac{e}{c}\left(\delta B_y + \frac{E_o}{p_o c}\delta E_x\right), 0, \frac{e}{c}\left(\delta B_x - \frac{E_o}{p_o c}\delta E_y\right), 0, 0\right\}$$

$$K(s) \equiv \frac{1}{\rho(s)} - \frac{e}{p_o c}\left(B_y\big|_{ref} + \frac{E_o}{p_o c}E_x\big|_{ref}\right) - f_x; \quad f_x = \frac{e}{p_o c}\left(\delta B_y + \frac{E_o}{p_o c}\delta E_x\right); \quad . \quad (17\text{-}5)$$

$$\frac{e}{p_o c}B_x\bigg|\left(_{ref} - \frac{E_o}{p_o c}E_y\big|_{ref}\right) = -f_y = \frac{e}{p_o c}\left(\delta B_x - \frac{E_o}{p_o c}\delta E_y\right)$$

Plugging (17-5) into (17-4) will give one the periodic closed orbit for such a case. For finding reduces to

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L(xP_3 - yP_1) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + g_x x\delta + g_y y\delta$$

with

$$F = S\frac{\partial H}{\partial X} = \left\{0, -g_x, 0, -g_y, 0, -\frac{m^2c^2}{p_o^3}\right\}^T. \qquad (17\text{-}6)$$

1D ACCELERATOR

$$X(s)=\int_{s-C}^{s}\begin{bmatrix} \mathrm{w(s)} & \mathrm{w(s)} \\ \mathrm{w'(s)}+i/\mathrm{w(s)} & \mathrm{w'(s)}-i/\mathrm{w(s)} \end{bmatrix}\cdot\begin{bmatrix} \left(\mathrm{w}'(\xi)-i/\mathrm{w}(\xi)\right)e^{i\psi(\mathrm{s})-i\psi(\xi)}\left(1-e^{i\mu}\right)^{-1} & \text{-w}(\xi)e^{i\psi(\mathrm{s})-i\psi(\xi)}\left(1-e^{i\mu}\right)^{-1} \\ \left(\text{-w}'(\xi)+i/\mathrm{w}(\xi)\right)e^{i\psi(\xi)-i\psi(\mathrm{s})}\left(1-e^{i\mu}\right)^{-1} & \mathrm{w}(\xi)e^{i\psi(\xi)-i\psi(\mathrm{s})}\left(1-e^{i\mu}\right)^{-1} \end{bmatrix}\frac{iF(\xi)}{2}d\xi$$

(17-7)

$X^T=\{x,x'\};F^T=\dfrac{e}{p_o c}\delta B_y\{0,1\}$ - orbit; $F^T=K(s)\{0,1\}$ for dispersion, i.e. $F^T=f(s)\{0,1\}$

$$X(s)=\int_{s-C}^{s}\begin{bmatrix} \dfrac{\mathrm{Re}\left(\mathrm{w(s)w}(\xi)e^{i(\psi(\mathrm{s})-\psi(\xi)-\mu/2)}\left(\dfrac{e^{-i\mu/2}-e^{i\mu/2}}{-i}\right)^{-1}\right)}{\ldots\ldots} \end{bmatrix} f(\xi)d\xi$$

i.e. $x(s)=\dfrac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C f(\xi)\mathrm{w}(\xi)\cos(\psi(\mathrm{s})-\psi(\xi)-\mu/2)d\xi$

(17-8)

First example: orbit distortion

$$f_x(s) = -\frac{e\delta B_y(s)}{p_o c}; \quad f_y(s) = \frac{e\delta B_x(s)}{p_o c}$$

$$\delta x(s) = -\frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C \frac{e\delta \mathrm{B_y}(\xi)}{p_o c}\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi$$

$$\delta y(s) = \frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C \frac{e\delta \mathrm{B_x}(\xi)}{p_o c}\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi \qquad (17\text{-}9)$$

but this is not the end of the story for horizontal motion! (what about change of the orbiting time?)

Second example: Dispersion

$$f_x(s) = K_o(s)\pi_l = K_o(s)\pi_\tau/\beta_o;$$

$$x(s) = D(s)\cdot\pi_l = D(s)\cdot\pi_\tau/\beta_o;$$

$$D(s) = -\frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C K_o(\xi)\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi \qquad (17\text{-}10)$$

Sample II: Beta-beat – 1D case

It is simple fact that any solution can be expanded upon the eigen vectors of periodic system (FOD cell repeated again and again is an example). Let ‘s consider that at azimuth $s=s_o$ initial value of “injected” eigen vector **V** being different from the periodic solution **Y**. We expand it as

$$V(s_o) = aY_k(s_o) + bY_k^*(s_o) = \begin{bmatrix} v_o \\ v_o' + \frac{i}{v_o} \end{bmatrix}; Y_k = \begin{bmatrix} w_o \\ w_o' + \frac{i}{w_o} \end{bmatrix}$$

$$a = \frac{1}{2i} \; Y_k^{*T}(s_o) SV(s_o) \; ; \; b = \frac{1}{-2i} \; Y_k^{T}(s_o) SV(s_o) \tag{17-11}$$

$$a = \frac{1}{2i}\left\{ v_o w_o' - w_o v_o' + i\left(\frac{v_o}{w_o} + \frac{w_o}{v_o} \right) \right\}; b = -\frac{1}{2i}\left\{ v_o w_o' - w_o v_o' + i\left(\frac{v_o}{w_o} - \frac{w_o}{v_o} \right) \right\};$$

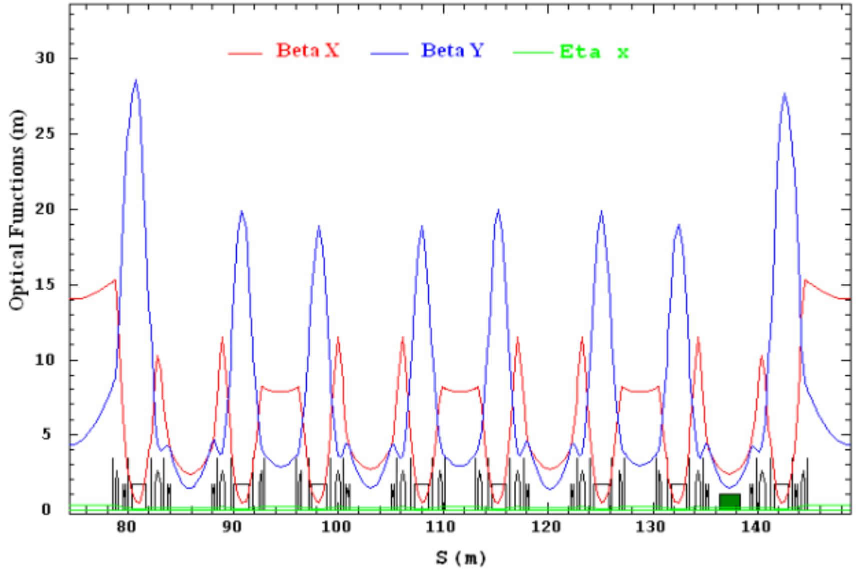
$$\frac{d}{ds}\tilde{Y}(s) = \mathbf{D}(s) \cdot \tilde{Y}(s); \quad \tilde{Y}(s) = Y(s) e^{i\psi(s)}; Y(s+C) = Y(s)$$

It is self-evident that

$$\tilde{V}' = D\tilde{V};\quad \tilde{V}(s) = a\tilde{Y}_k(s) + b\tilde{Y}_k^*(s) = \begin{bmatrix} v \\ v' + \dfrac{i}{v} \end{bmatrix} e^{i\varphi} = Y_k = a\begin{bmatrix} w \\ w' + \dfrac{i}{w} \end{bmatrix} e^{i\psi} + b\begin{bmatrix} w \\ w' - \dfrac{i}{w} \end{bmatrix} e^{-i\psi} \qquad (17\text{-}12)$$

$$|v|^2 = \frac{|w|^2}{4}\left|ae^{i\psi} + be^{-i\psi}\right|^2 = \frac{|w|^2}{4}\left(|a|^2 + |b|^2 - 2\,\mathrm{Re}\left(ab^* e^{2i\psi}\right)\right)$$

i.e. beta-function will beat with double of the betatron phase.



Sample III: Perturbation theory (ala quantum mechanics)

Small variation of the linear Hamiltonian terms (including coupling)

$$\frac{dX}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right)\cdot X; \qquad (17\text{-}13)$$

Assuming that changes are very small we can express the changes in the eigen vectors using basis of (15):

$$\tilde{Y}_{1k} = \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon_j \tilde{Y}_j; \; \frac{d\tilde{Y}_{1k}}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right)\cdot \tilde{Y}_{1k} + o(\varepsilon^2);$$

$$\tilde{Y}_{1k} = \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon_j \tilde{Y}_j; \; \frac{d\tilde{Y}_{1k}}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right)\cdot \tilde{Y}_{1k} + o(\varepsilon^2);$$

$$\frac{d\tilde{Y}_k}{ds} = \mathbf{D}(s)\tilde{Y}_k; \; \frac{d\varepsilon_j}{ds}\tilde{Y}_j = \varepsilon \mathbf{D}_1(s)\tilde{Y}_k; \Rightarrow \varepsilon_j = \varepsilon_{j0} + \frac{\varepsilon}{2i}\int^{s} \tilde{Y}_j^{*T}(\zeta)\mathbf{S}\mathbf{D}_1(\zeta)\tilde{Y}_k(\zeta)\,d\zeta; \qquad (17\text{-}14)$$

$$i\frac{d\delta\phi_k}{ds}\tilde{Y}_k = \varepsilon \mathbf{D}_1(s)\tilde{Y}_k \Rightarrow \delta\phi_k = \frac{\varepsilon}{2}\int^{s} \tilde{Y}_k^{*T}(\zeta)\mathbf{S}\mathbf{D}_1(\zeta)\tilde{Y}_k(\zeta)\,d\zeta;$$

$$\varepsilon_j(s+C) = \varepsilon_j(s)$$

$$\varepsilon_j = \frac{\varepsilon}{2i\left(e^{i(\mu_k-\mu_i)}-1\right)}\int_0^C e^{i\left(\psi_k(s)-\psi_i(\zeta)\right)} Y_j^{*T}(\zeta)\mathbf{S}\mathbf{D}_1(\zeta)Y_k(\zeta)\,d\zeta; \qquad (17\text{-}15)$$

One should be aware of the resonant case $e^{i(\mu_k-\mu_i)}=1$, when one should solve self-consistently the set of (17-14). It is well known case well described in weak coupling resonance case or in the case of parametric resonance.

Sample IV: small variation of the gradient. It can come from errors in quadrupoles or from a deviation of the energy from the reference value. In 1D case (reduced) it is simple addition to the Hamiltonian: (including sextupole term!)

$$
\begin{aligned}
&H_1 = \delta K_1 \frac{z^2}{2};\ z = \{x, y\};\\
&\pi_l = p/p_o - 1 \qquad\qquad (17\text{-}16)\\
&\delta K_{1\ x,y} = \mp\delta\left(\frac{e}{pc}\frac{\partial B_y}{\partial x}\right) = \mp\left(\frac{e}{pc}\delta\frac{\partial B_y}{\partial x}\right) - K_1\pi_l \mp \left(\frac{e}{pc}\frac{\partial^2 B_y}{\partial x^2}D_x\right)\pi_l + o\left(\pi_l{}^2\right)
\end{aligned}
$$

Plugging our parameterization into the residual Hamiltonian we get:

$$
\begin{aligned}
&z = w(s)\sqrt{2I}\cos\left(\psi(s) + \varphi\right)\\
&H_1 = \delta K_1(s)\cdot w^2(s)\cdot I\cdot\cos^2\left(\psi(s) + \varphi\right)
\end{aligned}
\qquad (17\text{-}17)
$$

The easiest way is to average the Hamiltonian (on the phase of fast betatron oscillation – our change is small! And does not effect them strongly) to have a well-know fact that the beta-function is also a Green function (modulo 4π) of the tune response on the variation of the focusing strength.

$$\langle H_1 \rangle = \frac{\langle \delta K_1(s) \cdot w^2(s) \rangle}{2} \cdot I \equiv \frac{\langle \delta K_1(s) \cdot \beta(s) \rangle}{2} \cdot I$$

$$\langle \varphi' \rangle = \frac{\partial \langle H_1 \rangle}{\partial I} = \frac{\langle \delta K_1(s) \cdot \beta(s) \rangle}{2}; \qquad (17\text{-}18)$$

$$\Delta\varphi = \frac{1}{2}\oint \delta K_1(s) \cdot \beta(s) ds; \;\; \Delta Q = \frac{\Delta\varphi}{2\pi} = \frac{1}{4\pi}\oint \delta K_1(s) \cdot \beta(s) ds;$$

Direct way will be to put it into the equations (43) and to find just the same, that <I'>=0 and the above result.

Finally, putting a weak thin lens as a perturbation gives a classical relation:

$$\delta K_1(s) = \frac{1}{f}\delta(s - s_o)$$

$$\Delta Q = \frac{\Delta\varphi}{2\pi} = \frac{1}{4\pi}\frac{\beta_o(s)}{f} \qquad (17\text{-}19)$$

In general case of change in Hamiltonian of linear motion

$$H = \frac{1}{2} X^T \left(\mathbf{H}_o + \mathbf{H}_1\right) X;\ \ X \to \{\varphi_k, I_k\} \to H_1(\varphi_k, I_k, s);$$

$$\Delta\mu_k = \frac{\partial}{\partial I} \int_o^C \left\langle H_1(\varphi_k, I_k, s)\right\rangle_{\varphi_k} ds. \tag{17-20}$$

$$H_1(\varphi_k, I_k, s) = \frac{1}{2} A^T \tilde{U}^T \delta\mathbf{H}_1 \tilde{U} A =$$

$$\frac{1}{8}\left\{\sum_{k=1}^{n} \sqrt{2I_k}\left(Y_k e^{i(\psi_k+\varphi_k)} + Y_k^* e^{i(\psi_k+\varphi_k)}\right)\right\}^T \delta\mathbf{H}_1 \left\{\sum_{k=1}^{n} \sqrt{2I_k}\left(Y_k e^{i(\psi_k+\varphi_k)} + Y_k^* e^{i(\psi_k+\varphi_k)}\right)\right\}$$

$$\left\langle H_1(\varphi_k, I_k, s)\right\rangle_{\varphi} = \frac{1}{2}\sum_{k=1}^{n} I_k \operatorname{Re}\left(Y_k^* \delta\mathbf{H}_1(s) \tilde{Y}_k\right); \frac{d\varphi_k}{ds} = \frac{\partial\langle H_1\rangle}{\partial I_k} = \frac{1}{2}\operatorname{Re}\left(Y_k^* \delta\mathbf{H}_1(s)\tilde{Y}_k\right); \tag{17-22}$$

or

$$\frac{d\varphi_k}{ds} = \frac{\partial H_1}{\partial I_k} = \frac{1}{4}\left(\tilde{Y}_k e^{i(\psi_k+\varphi_k)} + \tilde{Y}_k e^{i(\psi_k+\varphi_k)}\right)\delta\mathbf{H}_1 \left\{\sum_{k=1}^{n}\sqrt{2I_k}\left(\tilde{Y}_k e^{i(\psi_k+\varphi_k)} + \tilde{Y}_k e^{i(\psi_k+\varphi_k)}\right)\right\}$$

$$\left\langle \frac{d\varphi_k}{ds}\right\rangle = \left\langle \frac{\partial H_1}{\partial I_k}\right\rangle = \frac{1}{2}\operatorname{Re}\left(Y_k^* \delta\mathbf{H}_1(s)\tilde{Y}_k\right).$$

with

$$\Delta Q_k = \frac{\Delta\mu_k}{2\pi} = \frac{1}{4\pi}\int_0^c \operatorname{Re}\left(Y_k^*(s)\delta\mathbf{H}_1(s)\tilde{Y}_k(s)\right) ds \tag{17-23}$$

Not too shabby

Just to drive it home: 1D case

$$\Delta Q_k = \frac{\Delta\mu_k}{2\pi} = \frac{1}{4\pi}\int_0^c \mathrm{Re}\left(\begin{bmatrix} w & w' + \frac{i}{w} \end{bmatrix} \begin{bmatrix} \delta K_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ w' + \frac{i}{w} \end{bmatrix} \right) ds = \tag{17-24}$$

$$\frac{1}{4\pi}\int_0^c w^2\, \delta K_1 ds = \frac{1}{4\pi}\int_0^c \beta(s)\delta K_1(s) ds$$