20.2 Solutions of Exercises of Chapter 5: Relativistic Cyclotron

5.1

Modeling Thomas AVF Cyclotron

(a) A field map of a 360^o AVF cyclotron dipole.

A Fortran program, geneAVFMap.f, given in Tab. 20.31, constructs the required map of a field distribution $B_Z(r,\theta)$. A polar mesh is retained (Fig. 20.1), rather than Cartesian, consistently with cyclotron magnet symmetry. The program can be copy-pasted, compiled (*gfortran -o geneAVFMap geneAVFMap.f* will provide the executable) and run. The field map is logged under the name geneAVFMap.out for use by TOSCA keyword.

Note the following:

(i) The field map azimuthal extent (set to 360^o in geneAVFMap, Tab. 20.31) can be changed, for instance to simulate a 90 deg sector instead, with a sequence of four of them thus simulating the complete ring.

(ii) Assuming mid-plane symmetry (a dipole magnet, with symmetric poles with respect to the median plane), the field in the (O;X,Y) plane is axial. The field is taken radially constant in the first part of this exercise: $k = 0$, thus $\mathcal{R}(R) = 1$.

(iii) The origin of the azimuthal angle in $B_Z(R,\theta) = B_0[\,1 + f\sin(4(\theta - \theta_i))\,]$ is taken at $\theta_i = \pi/2$, leading to (Tab. 20.31)

$$B_Z(R,\theta) = B_0[\,1 + f\cos(4\theta)\,]$$

With this cosine dependence, and θ covering $0 \longrightarrow 2\pi$, the entrance and exit faces of a 360 deg field map will be a location of maximum field (hill ridge), thus, owing to the $2\pi/4$ cylindrical symmetry of the field, the closed orbit is normal to these entrance and exit faces, so yielding TE=TS=0 as KPOS arguments under TOSCA. The same property holds in the case a 90 deg field map is used (θ covering $0 \rightarrow \pi/2$): take entrance and exit faces of the field map along hill ridges, *i.e.* normal to the closed orbits.

As an indication of expected outcomes of this field map computation, the top and bottom parts of the file generated by geneAVFMap, in the required format for TOSCA[MOD=22.1] to swallow, are given in Tab. 20.31. Figure 5.4 displays the mid-plane field so obtained (it uses the gnuplot script given at the bottom of Tab. 20.31)

(b) Concentric trajectories.

The input data file to raytrace trajectories with different rigidities (four, here) is given in Tab. 20.32. The computation is performed in two steps

(1) in a first step, FIT finds the periodic coordinates for a given rigidity; note that for this first step a 90 deg field map is INCLUDed (obtained with $AT = 90$ deg in the Fortran, Tab. 20.31; and used in subsequent exercises);

(2) upon completion of FIT, a second step computes the closed trajectory over 360deg; note: this double-step is one way to (i) reduce the volume of zgoubi.plt

Table 20.31 A Fortran program, geneAVFMap.f, which generates a 360^o mid-plane field map. This angle as well as the field amplitude (B_0 =5 kG at R_0 = 50 cm, here) and its modulation (f = 0.2, here) can be changed to any other values, a field index (ak, set to zero here) can be accounted for. The field map produced is logged in geneAVFMap.out, or under different names for the purpose of the exercise, depending upon f, k, or AT values, *e.g.*, geneAVFMap_90deg_f2_k0.out, geneAVFMap_360deg_f9_k0.out...

```
      implicit double precision (a-h,o-z)
      parameter (pi=4.d0*atan(1.d0), BY=0.d0, BX=0.d0, Z=0.d0, dZ=0.d0)

      open(unit=2,file='geneAVFMap.out')

C------------ Hypotheses :
      AT = 360.d0  /180.d0*pi         ! Angular extent of field map. Can be changed 360, 60 deg, etc.).
      f = .2d0                                                           ! azimuthal modulation factor.
      B0 = 5.d0 ;R0 = 12.9248888074d0             ! field at R0 (kG); 200keV radius (cm), B(R0)=B0=5kG.
      ak = 0.d0                                                          ! Field index, defined at R0.
C
      Rmi=1.d0; Rma=76.d0; RM=50.d0  ! cm. Radial extent of field map; reference radius to define mesh.
      dR = 0.5d0 ; NR = NINT((Rma - Rmi)/dR)+1   ! R-distance between nodes in mesh. Number of R-nodes.
C                                         RdA=RM*dA is the distance between two nodes along R=RM arc,
      RdA = 1.d0   !      given angle increment dA (dA is the "Delta theta" quantity in the main text).
      NX= NINT(RM*AT /RdA) +1 ; RdA= RM*AT / DBLE(NX -1)  ! exact mesh step at RM, corresponding to NX.
      dA = RdA / RM ; A1 = 0.d0 ; A2 = AT                                ! corresponding delta_angle.
C----------------------------------------------
      write(2,*) Rmi,dR,dA/pi*180.d0,dZ,
     >'     !  Rmi/cm, dR/cm, dA/deg, dZ/cm'
      write(2,*) '# Field map generated using geneAVFMap.f '
      write(2,fmt='(a)') '# AT/rd,  AT/deg, Rmi/cm, Rma/cm, RM/cm,'
     >//' NR, dR/cm, NX, R*dA/cm, dA/rd : '
      write(2,fmt='(a,1p,5(e16.8,1x),2(i3,1x,e16.8,1x),e16.8)')
     >'# ',AT, AT/pi*180.d0,Rmi, Rma, RM, NR, dR, NX, RdA, dA
      write(2,*) '# For TOSCA: ',NX,NR,' 1 22.1 1.  !IZ=1 -> 2D ; '
     >//'MOD=22 -> polar map ; .MOD2=.1 -> one map file'
      write(2,*) '# R*cosA (A:0->360), Z==0, R*sinA, BY, BZ, BX '
      write(2,*) '# cm                cm    cm      kG  kG  kG '
      write(2,*) '# '
      do jr = 1, NR
        R = Rmi + dble(jr-1)*dR
        do ix = 1, NX
          A = A1 + dble(ix-1)*dA
          BR = (1.D0 + ak * x/R0)
          BZ = B0 * BR * (1.d0+f*sin(4.d0*A +pi/2.d0))
          X = R * sin(A)  ; Y = R * cos(A)
          write(2,fmt='(1p,6(e16.8),2(1x,i0),3(1x,e16.8))')
     >    Y,Z,X,BY,BZ,BX,ix,jr,A,R,atan(X/Y)
        enddo
      enddo
      stop ' Job complete ! Field map stored in geneAVFMap.out'
      end
```

Top and bottom sections of the field map file geneAVFMap.out with modulation factor f=0.2 *and radial index k*=0*. The file starts with an* 8*-line header, of which the first line is effectively used by zgoubi, the following* 7 *being just comments*:

```
    1.0000000000000000       0.50000000000000000        1.1464968152866242        0.0000000000000000          !  Rmi/cm, dR/cm, dA/deg, dZ/cm
  # Field map generated using geneAVFMap.f
 # AT/rd,  AT/deg, Rmi/cm, Rma/cm, RM/cm, NR, dR/cm, NX, R*dA/cm, dA/rd :
 #   6.28318531E+00   3.60000000E+02   1.00000000E+00   7.60000000E+01   5.00000000E+01 151   5.00000000E-01 315   1.00050721E+00   2.00101443E-02
  # For TOSCA:          315          151  1 22.1 1.  !IZ=1 -> 2D ; MOD=22 -> polar map ; .MOD2=.1 -> one map file
  # R*cosA (A:0->360), Z==0, R*sinA, BY, BZ, BX
  # cm                cm    cm      kG  kG  kG
  #
   1.00000000E+00  0.00000000E+00  0.00000000E+00  0.00000000E+00  5.00000000E+00  0.00000000E+00 1 1
   9.99799804E-01  0.00000000E+00  2.00088090E-02  0.00000000E+00  5.07995514E+00  0.00000000E+00 2 1
   9.99199295E-01  0.00000000E+00  4.00096065E-02  0.00000000E+00  5.15939832E+00  0.00000000E+00 3 1
   9.98198715E-01  0.00000000E+00  5.99943846E-02  0.00000000E+00  5.23782087E+00  0.00000000E+00 4 1
   9.96798463E-01  0.00000000E+00  7.99551413E-02  0.00000000E+00  5.31472063E+00  0.00000000E+00 5 1
   ................................
   7.57566832E+01  0.00000000E+00 -6.07659074E+00  0.00000000E+00  4.68527937E+00  0.00000000E+00 311 151
   7.58631023E+01  0.00000000E+00 -4.55957323E+00  0.00000000E+00  4.76217913E+00  0.00000000E+00 312 151
   7.59391464E+01  0.00000000E+00 -3.04073010E+00  0.00000000E+00  4.84060168E+00  0.00000000E+00 313 151
   7.59847851E+01  0.00000000E+00 -1.52066948E+00  0.00000000E+00  4.92004486E+00  0.00000000E+00 314 151
   7.60000000E+01  0.00000000E+00 -1.86146313E-14  0.00000000E+00  5.00000000E+00  0.00000000E+00 315 151
```

---

gnuplot script to obtain Fig. 5.4:

```
set xtics mirror ; set ytics mirror; cm2m = 0.01
set xlabel "X [m]"; set ylabel "Y [m]"; set zlabel "B_Z [kG]"; set hidden3d ; set view 49, 221
splot "geneAVFMap.out" u ($ 1 *cm2m):($3 *cm2m):($5) w l lc palette notit; pause 1
```

file as it is only written to at the second step, once the closed orbit has been found by FIT, and (ii) impose the latter; however, relaxing on these constraints this can be accomplished in a single step, this is the case in exercise 20.2. This process is repeated for additional rigidities (*i.e.*, additional orbits at different energies) using REBELOTE.

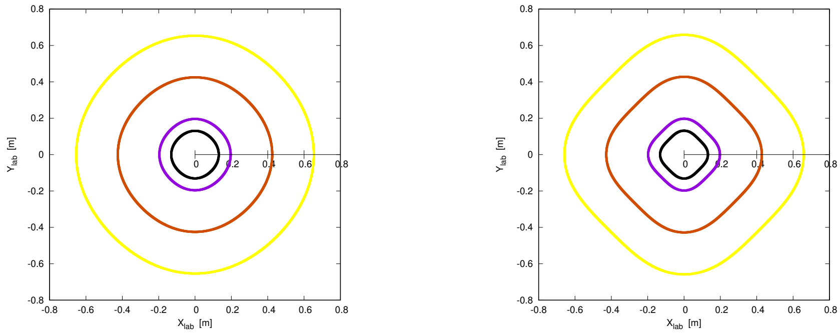
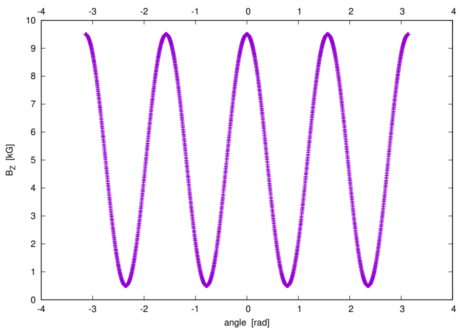


Fig. 20.34 Scalloping closed orbits in the 4-period AVF cyclotron with modulation factor $f = 0.2$ (left) and $f = 0.9$ (right)

Fig. 20.35 Four-periodic field $B_Z(\theta)$ along the closed orbits, case of a modulation factor $f = 0.9$. The field is the same for all four orbits as the field index is zero. The average value of the field along a closed orbit is $\frac{2}{\pi}\int_{\Delta\theta=\pi/2} B_Z(R,\theta)\,d\theta = 5\,$kG



The following keywords are found in the input data file:

(i) OBJET: define a reference rigidity (arbitrary, taken to be 64.624444 kG cm, here, corresponding to 200 keV protons), and define initial coordinates of a single ion (initial radius Y_0 in the field map frame, other space coordinates zero; relative rigidity D=$B\rho/BORO$),

(ii) TOSCA: read the field map and raytrace; IL=2 flag causes log of particle data in zgoubi.plt, after each integration step Δs,

(iii) FAISCEAU: log local particle coordinates in zgoubi.res,

(iv) SYSTEM: run two gnuplot scripts once raytracing is completed, a first one to plot the trajectories, a second one to plot the field along trajectories,

Table 20.32 Simulation input data file FieldMapAVFMag.inc: raytrace a series of ions with different rigidities, spanning 200 keV to 5 MeV. This file also defines the optical segment #S_AVFMag_90d to #E_AVFMag_90d, for use in subsequent exercises

```
FieldMapAVFMag.inc                                          ! Title line: required at top of file.
!                                                            Additional header lines require a !.
'MARKER'   FieldMapAVFMag_S                                           ! Just for edition purposes.
'OBJET'
64.62444403717985                   ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
1 1
12.9248888 0.  0. 0. 0.  1. 'm'      ! Closed orbit coordinates at BR=64.6244440 kG.cm for constant B.
1
'MARKER'   #S_AVFMag_90d
'TOSCA'
0 0                                ! IL=2 to log step-by-step coordinates, spin, etc., in zgoubi.plt.
1. 1. 1. 1.  ! Normalization coefficients applied to field, and X, Y Z coordinate values read from map.
HEADER_8                                              ! The field map file starts with an 8-line header.
80 151 1 22.1 1.               ! IZ=1 for 2D map; MOD=22 for polar frame; .MOD2=.1: only one map file.
geneAVFMap_90deg_f2_k0.out                         ! Or geneAVFMap_90deg_f9_k0.out for f=0.9 modulation.
0 0 0 0                ! Possible boundaries within the field map, to start/stop stepwise integration.
2
.2 ! cm                                                                     ! Integration step size.
2                                                                      ! Magnet positionning option.
0. 0. 0. 0.                                                                   ! Magnet positionning.
'MARKER'   #E_AVFMag_90d
'FAISCEAU'                                                             ! Particle coordinates, here.
'FIT'
1
2  30  0 [5.,100.]               ! Vary Y0 at OBJET, to allow fulfilling the following two constraint:
2  1e-15
3.1 1 2 5 0. 1. 0                                      ! request same radius after a period (90 deg);
3   1 3 5 0. 1. 0                                     ! request orbit angle after a period to be nul.
'FAISCEAU' CHECK   ! Allows quick check of particle coordinates, in zgoubi.res: final should = initial.

'TOSCA'
0 2                                ! IL=2 to log step-by-step coordinates, spin, etc., in zgoubi.plt.
1. 1. 1. 1.  ! Normalization coefficients applied to field, and X, Y Z coordinate values read from map.
HEADER_8                                              ! The field map file starts with an 8-line header.
315 151 1 22.1 1.              ! IZ=1 for 2D map; MOD=22 for polar frame; .MOD2=.1: only one map file.
geneAVFMap_360deg_f2_k0.out    ! Could also be 4*geneAVFMap_90deg_f2_k0.out (changing IX=315 to IX=90)!
0 0 0 0                ! Possible boundaries within the field map, to start/stop stepwise integration.
2
.2 ! cm                                                                     ! Integration step size.
2                                                                      ! Magnet positionning option.
0. 0. 0. 0.                                                                   ! Magnet positionning.
'FAISCEAU'
'REBELOTE'                                                                 ! Repeat what precedes,
3  0.1  0 1                                                                              ! 3 times.
1                             ! Change the value of parameter 35 in OBJET prior to repeating), i.e.,
OBJET 35  1.5:5                                        !  the relative rigidity D=1, 1.5, 3.25, 5.
'SYSTEM'
2
gnuplot < ./gnuplot_Zplt_traj.gnu &                          ! Plot B(R), as read fron zgoubi.plt.
gnuplot < ./gnuplot_Zplt_field.gnu &                         ! Plot B(R), as read fron zgoubi.plt.
'MARKER'   FieldMapAVFMag_E                                         ! Just for edition purposes.
'END'
```

---

gnuplot script to obtain Fig. 20.34:

```
# gnuplot_Zplt_traj.gnu
set xtics; set ytics; set xlabel 'X_{lab}  [m]'; set ylabel 'Y_{lab}  [m]'
set size ratio -1; set polar; cm2m=1e-2; unset colorbox
plot  for [FITnum=1:4] 'zgoubi.plt' u ($49==FITnum? $22 :1/0):($10*cm2m):($49) w p pt 4 ps .4 lc palette notit ; pause 1
```

---

gnuplot script to obtain Fig. 20.35:

```
# gnuplot_Zplt_field.gnu
set xtics nomirror; set x2tics; set ytics; set xlabel 'angle  [rad]'; set ylabel 'B_Z  [kG]'
plot  for [FITnum=1:4] 'zgoubi.plt' u ($49==FITnum ? $22 :1/0):($25) w p  notit; pause 1
```

(v) MARKER: define two “LABEL_1” type labels, for use in INCLUDE statements in subsequent exercises.

Four closed orbits resulting from the data file in Tab. 20.32 (for respective relative rigidities D=1, 1.5, 3.25, 5, spanning about 70 cm radially) are displayed in Fig. 20.34. Inspecting zgoubi.res one finds the following particle coordinates as logged by “’FAISCEAU’ CHECK” (Tab. 20.32), with initial values (left hand side) equal to final values from the FIT procedure (right hand side), as expected:

```
      8  Keyword, label(s) :  FAISCEAU  CHECK
                                          TRACE DU FAISCEAU
                                        (follows element #      7)
                                              2 TRAJECTOIRES
                        OBJET                                                FAISCEAU
           D       Y(cm)    T(mr)    Z(cm)    P(mr)     S(cm)    D-1       Y(cm)    T(mr)   Z(cm)   P(mr)    S(cm)
   m  1   1.0000  13.071   0.000    0.000    0.000     0.0000   0.0000    13.071   0.000   0.000   0.000    2.027537E+01
   m  1   1.5000  19.607   0.000    0.000    0.000     0.0000   0.5000    19.607   0.000   0.000   0.000    3.041305E+01
   m  1   3.2500  42.481   0.000    0.000    0.000     0.0000   2.2500    42.481   0.000   0.000   0.000    6.589494E+01
   m  1   5.0000  65.355   0.000    0.000    0.000     0.0000   4.0000    65.355   0.000   0.000   0.000    1.013768E+02
```

The scalloping (orbit oscillation around the reference circle) is small, as can be seen by comparison, below, with the closed orbit radius in the case of constant field. The latter is obtained with a similar computation using a field map generated with $f = 0$; it can also be obtained from $R = p / qB$ with B=5 kG with $p = q \times D \times B\rho_{\mathrm{ref}}$ and $B\rho_{\mathrm{ref}} = 64.6244440$ kG cm the reference rigidity, under OBJET, yielding:

```
      8  Keyword, label(s) :  FAISCEAU  CHECK
                                          TRACE DU FAISCEAU
                                        (follows element #      7)
                                              2 TRAJECTOIRES
                        OBJET                                                FAISCEAU
           D       Y(cm)    T(mr)    Z(cm)    P(mr)     S(cm)    D-1       Y(cm)    T(mr)   Z(cm)   P(mr)    S(cm)
   m  1   1.0000  12.925   0.000    0.000    0.000     0.0000   0.0000    12.925  -0.000   0.000   0.000    2.030237E+01
   m  1   1.5000  19.387   0.000    0.000    0.000     0.0000   0.5000    19.387  -0.000   0.000   0.000    3.045355E+01
   m  1   3.2500  42.006   0.000    0.000    0.000     0.0000   2.2500    42.006   0.000   0.000   0.000    6.598270E+01
   m  1   5.0000  64.624   0.000    0.000    0.000     0.0000   4.0000    64.624  -0.000   0.000   0.000    1.015118E+02
```

Figure 20.34 also displays an iteration of this closed orbits computation, yet for the case of a modulation factor f=0.9 (thus using different field maps, named *e.g.* geneAVFMap_90deg_f9_k0.out and geneAVFMap_360deg_f9_k0.out, for substitution to the $f = 0.2$ field map names in Tab. 20.32); the scalloping is increased due to deeper modulation. Inspecting "’FAISCEAU’ CHECK" in zgoubi.res one then finds the following particle coordinates for the 4 different rigidities:

```
      8  Keyword, label(s) :  FAISCEAU  CHECK
                                          TRACE DU FAISCEAU
                                        (follows element #      7)
                                              2 TRAJECTOIRES
                        OBJET                                                FAISCEAU
           D       Y(cm)    T(mr)    Z(cm)    P(mr)     S(cm)    D-1       Y(cm)    T(mr)   Z(cm)   P(mr)    S(cm)
   m  1   1.0000  13.165   0.000    0.000    0.000     0.0000   0.0000    13.165   0.000   0.000   0.000    1.978076E+01
   m  1   1.5000  19.747   0.000    0.000    0.000     0.0000   0.5000    19.747   0.000   0.000   0.000    2.967114E+01
   m  1   3.2500  42.786   0.000    0.000    0.000     0.0000   2.2500    42.786   0.000   0.000   0.000    6.428748E+01
   m  1   5.0000  65.825   0.000    0.000    0.000     0.0000   4.0000    65.825   0.000   0.000   0.000    9.890382E+01
```

The magnetic field along these orbits is displayed in Fig. 20.35, it is the same for all four orbits as the field index is zero, here.

(c) Numerical convergence.

Numerical convergence of the stepwise integration is tested using the same input data file as in (b) (Tab. 20.32, f=0.2 and k=0), the integration step size only, Δs, needs be changed, and the resulting change in accuracy translates in a change of closed orbit coordinates as found by FIT. Two values of Δs are tried (in addition to $\Delta s = 0.2$ cm in the previous computations, *cf.* Tab. 20.32). They yield the following outcomes of FAISCEAU (at its occurrence at the bottom of Tab. 20.32, prior to REBELOTE), for closed orbits at the four different relative rigidities D=1, 1.5, 3.25, 5:

◇ Case of $\Delta s = 1$ cm

```
        8  Keyword, label(s) :  FAISCEAU    CHECK
                                            TRACE DU FAISCEAU
                                          (follows element #      7)
                                               2 TRAJECTOIRES
                         OBJET                                                  FAISCEAU
         D        Y(cm)     T(mr)    Z(cm)    P(mr)     S(cm)    D-1       Y(cm)    T(mr)  Z(cm)  P(mr)
m  1   1.0000   13.099    0.000    0.000    0.000     0.0000   0.0000    13.099  -0.018  0.000  0.000
m  1   1.5000   19.610    0.000    0.000    0.000     0.0000   0.5000    19.610   0.000  0.000  0.000
m  1   3.2500   42.481    0.000    0.000    0.000     0.0000   2.2500    42.481   0.000  0.000  0.000
m  1   5.0000   65.355    0.000    0.000    0.000     0.0000   4.0000    65.355   0.000  0.000  0.000
```

◇ Case of $\Delta s = 5\,\mathrm{cm}$

```
      8  Keyword, label(s) :  FAISCEAU  CHECK
                                            TRACE DU FAISCEAU
                                          (follows element #      7)
                                                 2 TRAJECTOIRES
                         OBJET                                                  FAISCEAU
         D        Y(cm)     T(mr)    Z(cm)    P(mr)     S(cm)   D-1       Y(cm)    T(mr)  Z(cm)  P(mr)
m  1   1.0000   14.177    0.000    0.000    0.000     0.0000  0.0000    14.183  -0.011  0.000  0.000
m  1   1.5000   20.761    0.000    0.000    0.000     0.0000  0.5000    20.757   0.142  0.000  0.000
m  1   3.2500   42.624    0.000    0.000    0.000     0.0000  2.2500    42.628   0.000  0.000  0.000
m  1   5.0000   65.355    0.000    0.000    0.000     0.0000  4.0000    65.354  -0.000  0.000  0.000
```

The change of closed orbit coordinates is substantial for the lowest energy trajectory, smaller circumference $C \approx 2\pi \times 13 \approx 80\,\mathrm{cm}$, covered in only 16 steps in the case $\Delta s = 5\,\mathrm{cm}$. Given the strong curvature, the high order derivatives of the field vector take great values so jeopardizing the convergence of the position and velocity vector Taylor series [1, Eq. 1.2.4]. The $\Delta s = 5\,\mathrm{cm}$ case features in addition poor convergence of the FIT procedure, unable to zero the closed orbit angle in the small radius cases, an effect of the field interpolation from a mesh.

(d) Dependence of wave numbers on energy and radius.

A scan of the wave numbers over a relative rigidity interval $D = \frac{B\rho}{BORO} : 1 \rightarrow 5$ is performed using the input data file given in Tab. 20.33 (BORO is the reference rigidity, under OBJET, D is the sixth coordinate of the reference particle as defined under OBJET[KOBJ=5]). Wave numbers are computed using MATRIX.

OBJET[KOBJ=5] generates 13 particles with paraxial radial and axial coordinates, and rigidity sampling, for the computation of transport matrix and wave numbers by MATRIX. REBELOTE repeats this matrix computation sequence, for a series of different rigidities. It is preceded by FIT which finds the closed orbit, this is necessary as, (i) a different rigidity means different orbital radius, (ii) MATRIX computes transport coefficients with respect to particle 1, which requires the latter to be placed on the reference orbit, prior to MATRIX computation.

Inspection of the execution listing (zgoubi.res) shows the structure of a FIT at the end of the FIT procedure, with the status of the variable (one variable only, here) in a top block, followed by the status of the constraints in a bottom block. Here is an excerpt of the FIT section in zgoubi.res, at the last iteration by REBELOTE (case of relative rigidity D=5.00639):

```
        6  Keyword, label(s) :  FIT                                                              IPASS= 16

 STATUS OF VARIABLES  (Iteration #  200 /    199 max.)
LMNT VAR PARAM  MINIMUM    INITIAL         FINAL         MAXIMUM     STEP      NAME   LBL1               LBL2
   2   1    30    5.00        65.9       65.908886       100.       0.00      OBJET    -                   -

 STATUS OF CONSTRAINTS (Target penalty =   1.0000E-08)
TYPE  I   J LMNT#     DESIRED          WEIGHT         REACHED         KI2     NAME   LBL1               LBL2
  3   1   2     5    0.000000E+00    1.000E+00    1.074504E-04    1.00E+00  MARKER    -                   -
  3   1   3     5    0.000000E+00    1.000E+00    1.537897E-06    2.05E-04  MARKER    -                   -
 Fit reached penalty value   1.1548E-08
```

Details regarding FIT[2] input, algorithms, and outcomes, are found in [1].

Table 20.33 Simulation input data file: raytrace a set of 13 particles (defined by OBJET[KOBJ=5]) for a particular reference rigidity, to perform a MATRIX computation. FIT is used to find the closed orbit, prior to MATRIX. Iteration for a series of 35 additional rigidities (relative rigidity D: 1.1→5.00639, in 35 steps) is performed by REBELOTE. This input file INCLUDEs the segment [#S_AVFMag_360d:#E_AVFMag_360d] of file FieldMapAVFMag.inc (Tab. 20.32)

```
Uniform field sector.
'MARKER'   FieldMapAVFQs_S                                        ! Just for edition purposes.
'OBJET'
64.62444403717985                     ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
5                                                  ! Define 13 particles for MATRIX computation.
.001 .01 .001 .01 .001 .00001                              ! Sampling of the initial coordinates.
12.9248888 0.  0. 0. 0.  1. 'm'     ! Closed orbit coordinates at BR=64.6244440 kG.cm for constant B.
'INCLUDE'
1
4 *FieldMapAVFMag.inc[#S_AVFMag_90d:#E_AVFMag_90d]
'FIT'
1                                                                             ! One variable.
2  30  0 [5.,100.]                         ! Vary Y0 at OBJET, to allow for the following constraint:
2  1E-8   199         ! Two constraints. Required penalty is 1e-8. Maximum number of iterations is 199.
3.1 1 2 5 0. 1. 0                                  ! request same radius after a period (90 deg);
3   1 3 5 0. 1. 0                             ! request orbital angle after a 1/4-turn to be zero.
'FAISCEAU' CHECK   ! Allows quick check of particle coordinates, in zgoubi.res: final should = initial.
'MATRIX'
1 11  PRINT           ! PRINT: log computation outcome data in zgoubi.MATRIX.out, for further plotting.
'REBELOTE'                                                                ! Repeat what precedes,
35  0.1  0 1                                                                        ! 15 times.
1                        ! Change the value of parameter 35 in OBJET, namely, to relative rigidity from
OBJET 35  1.1:5.00639           ! D=1.1 by increment(5.00639-1.1)/35, prior to repeating the sequence.
'SYSTEM'
1
gnuplot < ./gnuplot_MATRIX_Qxy.gnu                                       ! Plot the wave number scan.
'MARKER'   FieldMapAVFQs_E                                               ! Just for edition purposes.
'END'
```

---

gnuplot script to obtain Fig. 20.36:

```
# gnuplot_MATRIX_Qxy.gnu
set xlab "kin. E [MeV]"; set x2lab "R [cm]"; set ylab "{/Symbol n}_R, ({/Symbol n}_R^2+{/Symbol n}_y^2)^{1/2}"
set y2label "{/Symbol n}_y"; set xtics nomirror; set x2tics; set ytics nomirror; set y2tics nomirror
BORO = 64.62444403717985; am = 938.27203e6; c = 2.99792458e8; BrhoRef = BORO *1e-3; eV2MeV = 1e-6
plot "zgoubi.MATRIX.out" u ($59):(1-$56) axes x2y1 w p pt 5 ps 0 notit , \
"zgoubi.MATRIX.out" u ((sqrt(($47*BrhoRef*c)**2 + am*am)-am)*eV2MeV):(1-$56) w lp pt 5 lt 1 lw .5 lc rgb "red" \
tit "{/Symbol n}_R", "zgoubi.MATRIX.out" u ((sqrt(($47*BrhoRef*c)**2+am*am)-am)*eV2MeV):($57) axes x1y2 w lp pt 6 lt 3 \
lw .5 lc rgb "blue" tit "{/Symbol n}_y\n" , "zgoubi.MATRIX.out" u ((sqrt(($47*BrhoRef*c)**2 + am*am)-am)*eV2MeV)\
:(sqrt((1-$56)**2+$57**2)) w lp pt 7 lt 1 lw .5 lc rgb "black" t "({/Symbol n}_R^2+{/Symbol n}_y^2)^{1/2} \n"; pause 1
```

Further inspection of the execution listing shows the outcome of a MATRIX command, under the form of two 6×6 blocks, a top one which is the transport matrix $[T_{\rm ij}]$ (see Sect. 19.3.1) from start to end of the optical sequence, and a bottom one, "beam matrix" drawn from the periodicity hypothesis which allows to write (see Sect. 19.3.3) $[T_{\rm ij}] = I\cos(\mu) + J\sin(\mu)$. Here is an excerpt of the MATRIX section in zgoubi.res, at the last iteration by REBELOTE (relative rigidity D=5.00639):

```
          8  Keyword, label(s) :  MATRIX                                                     IPASS= 16
      Reference particle (#     1), path length :   396.12091     cm  relative momentum :    5.00639

                   TRANSFER  MATRIX  ORDRE  1  (MKSA units)

           0.945800        0.282906       0.00000         0.00000         0.00000        2.913089E-02
          -0.393308        0.940498       0.00000         0.00000         0.00000        0.232240
           0.00000         0.00000       -0.621747       -0.754564        0.00000        0.00000
           0.00000         0.00000        0.812954       -0.621754        0.00000        0.00000
           0.254138        4.384335E-0    0.00000         0.00000         1.00000        3.80443
           0.00000         0.00000        0.00000         0.00000         0.00000        1.00000

         Beam  matrix  (beta/-alpha/-alpha/gamma) and  periodic  dispersion  (MKSA units)

           0.848142    -0.007948     0.000000     0.000000     0.000000     0.593089
          -0.007948     1.179123     0.000000     0.000000     0.000000     0.009938
           0.000000     0.000000     0.963418     0.000005     0.000000     0.000000
           0.000000     0.000000     0.000005     1.037971     0.000000     0.000000
           0.000000     0.000000     0.000000     0.000000     0.000000     0.000000
           0.000000     0.000000     0.000000     0.000000     0.000000     0.000000
```

```
                              wave numbers
                   NU_Y =  0.54102699E-01     NU_Z =  0.64321084
```

The radial wave number ν versus $1-\nu$ indetermination (see exercise **??**) can be raised considering that $k = 0$ so that $\nu_R \approx \sqrt{1+k} \approx 1$, thus, actually, $NU_Y = 1 - 0.054102699 = 0.945897301$.

MATRIX includes a PRINT command (Tab. 20.33), which causes the transport coefficients to be stacked in zgoubi.MATRIX.out as REBELOTE iterates; reading from the latter (gnuplot script given at the bottom of Tab. 20.33) yields Fig. 20.36. Results appear reasonably close to theoretical approximations $\nu_R \approx \sqrt{1+k} = 1$, $\nu_y \approx F = 0.6364$ and $(\nu_R^2 + \nu_y^2)^{1/2} \approx (1+F^2)^{1/2} = 1.185$ (Eq. 5.6). The smaller the orbit scalopping (modulation $f \longrightarrow 0$), the better the agreement (see Tab. 20.34).

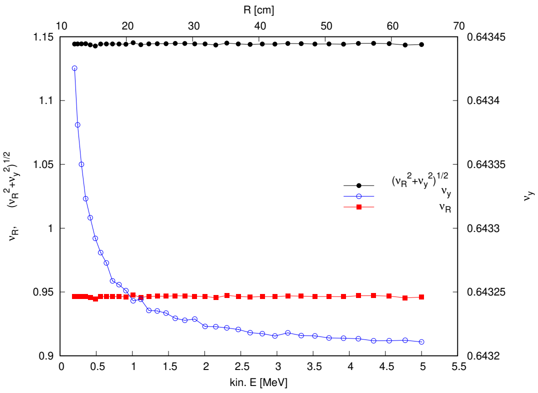


Fig. 20.36 A scan of the wave numbers as a function of radius and proton energy in the cyclotron, with f=0.9 and k=0 here. Fluctuations stem from the use of a field map - performing the scan using DIPOLE analytical field model instead, would yield smooth curves

(e) Flutter.

The axial wave number writes (Eq. 5.8) $\nu_y \approx \sqrt{-k+F^2} = F$. The flutter is given by $F = \left(\frac{<(\mathcal{F}-<\mathcal{F}>)^2>}{<\mathcal{F}>^2}\right)^{1/2}$ (Eq. 5.5). The field modulation used here expresses as $\mathcal{F} = 1 + f\,\cos N\theta$, and f=0.9. From this, one gets

$$< \mathcal{F} >= \frac{2}{\pi}\int_0^{\pi/2} (1 + f \cos N\theta)\, d\theta = 1$$

$$< \mathcal{F}^2 >= \frac{2}{\pi}\int_0^{\pi/2} (1 + f \cos N\theta)^2\, d\theta = 1 + \frac{f^2}{2}$$

$$F = \left(\frac{< \mathcal{F}^2 > - < \mathcal{F} >^2}{< \mathcal{F} >^2}\right)^{1/2} = \frac{f}{\sqrt{2}} = 0.6364$$

$$\text{theoretical } \nu_y = F = 0.6364$$

To assess wave numbers for different values of f, a series of field maps is to be computed, as in (a), one for each f value. The outcomes of both numerical

Table 20.34 Wave number values in the case k=0, depending upon the field modulation f, from numerical raytracing ("ray-tr." column) and from Eqs. 5.6, 5.7, namely, $\nu_R = 1$ and $\nu_y = F = f/\sqrt{2}$

	wave numbers					
	radial, ν_R		axial, ν_y		$(\nu_R^2+\nu_y^2)^{1/2}$	
f	ray-tr.	Eq. 5.6	ray-tr.	$f/\sqrt{2}$	ray-tr.	$(1+f^2/2)^{1/2}$
0.05	0.9999	1	0.0365	0.03535	1.0006	1.0006
0.1	0.9993	1	0.0730	0.0707	1.0020	1.0025
0.2	0.997	1	0.1459	0.1414	1.0076	1.0100
0.3	0.994	1	0.2185	0.2121	1.0177	1.0223
0.6	0.975	1	0.4338	0.4243	1.0671	1.0863
0.9	0.945	1	0.6433	0.6364	1.1432	1.1853

integration as in (d), and theoretical calculation as above, for different values of the field modulation factor f, are summarized in Tab. 20.34. Discrepancy grows with greater modulation, as Eq. 5.6 is a weak-modulation approximation [2, Sect. 3].

5.2

Designing an Isochronous AVF Cyclotron

(a) R-dependent field index.

A field index k(R) proper to ensure R-independent revolution period has to result in (Eq. 5.13)

$$B(R) = \gamma B_0 = \frac{B_0}{\sqrt{1-(R/R_\infty)^2}} \qquad \text{with} \quad B_0 = \frac{M\omega_{\rm rev}}{c^2} = \frac{M}{c^2}\frac{\omega_{\rm rf}}{h} \tag{20.10}$$

For consistency with similar simulations in the Classical Cyclotron Chapter 4, the following hypotheses are considered:

(i) injection energy $E_{\rm inj} = 200\,$keV,

(ii) average radius $R_{\rm inj} = 0.129248888\,$m at that energy,

(iii) average field $B_{\rm inj} = B(R = R_{\rm inj}) = 0.5\,$T.

From this one gets $\omega_{\rm rev}$, the same at all R assuming isochronism, thus in particular

$$\omega_{\rm rev} = \frac{c^2 B_{\rm inj}}{M\gamma_{\rm inj}} = 2\pi\times 7.62096882\times 10^6\ \text{rad/s} \qquad \text{wherein} \quad M\gamma_{\rm inj} = M + 200\times 10^3$$

with $M = 938.27208\times 10^6\ eV/c^2$, proton rest mass. In this exercise h=1 is assumed, thus (Eq. 5.12)

$$R_\infty = \frac{c}{\omega_{\rm rf}} = \frac{2.99792458\times 10^8}{7.62096882\times 10^6} = 6.2608118\,\text{m}$$

Using Eq. 20.10 the value $B_0 \equiv B(R=0)$ results, namely, $B_0 = B_{\rm inj}\sqrt{1-(R_{\rm inj}/R_\infty)^2} =$ 4.9989344, so, finally,

$$B(R) = \frac{B_0}{\sqrt{1-(R/R_\infty)^2}} = \frac{4.9989344}{\sqrt{1-(R/6.2608118)^2}}$$

The Fortran program geneAVFMapIsochro.f given in Tab. 20.35 constructs the map for the $B(R,\theta)$ field distribution. It is derived from the Fortran program of exercise 5.1 (Tab. 20.31) by accounting for the isochronism field dependence properties above. In that file, the modulation factor f can be changed, as well as the field index k and the angular extent of the field map, AT. The resulting field distribution over 360 deg is essentially as in Fig. 5.4 as the radial dependence of the field is weak: $B(R) = \gamma B_0$ whereas $\gamma \approx 1$, varying from 1.00000128 to 1.00745 over $R:\ 10 \to 76\,$cm.

For the purpose of comparisons, four field maps are created and resorted to. Three only differ by the value of the modulation coefficient (Tab. 20.35): $f = 0,\ 0.2$, and 0.9, an additional one is a “classical cyclotron” case (“Bcst” index, for constant $B(R,\theta)$). In the latter case in addition

- `BR=1` is substituted to `BR=1/sqrt(1-(R/Rinfty)**2)` and

- `B0=T2kG/2` is substituted to `B0=T2kG*Bp2k*sqrt(1-(Rp2k/Rinfty)**2)`.

In the following these field maps are handled under the following respective names:

geneAVFMap_360deg_f0_isochro.out, geneAVFMap_360deg_f.2_isochro.out, geneAVFMap_360deg_f.9_isochro.out and geneAVFMap_360deg_Bcst.out.

The input data file to raytrace ion orbits is given in Tab. 20.37. The FIT procedure finds the closed orbit for the particle defined by OBJET, REBELOTE repeats for a series of additional rigidities in the range 1.1 to 5×BORO (BORO=64.624444037 kG cm).

The exercise has been done for modulation factors f=0, 0.2, or 0.9, and as well for a constant field $B_Z(R,\theta) = 5$ kG, as described above. The latter simulation shows a great difference in the R dependence of the revolution time, compared to the two isochronous cases. The sole cases $f = 0.2$ and $f = 0.9$ simulate an AVF cyclotron, yielding stable axial motion; in the other two cases (f=0 and constant B) there is no axial focusing, axial motion is unstable.

The resulting sets of closed trajectories are displayed in Fig. 20.37, the R dependence of the revolution period in the four cases is given in Fig. 20.38. The revolution period on the injection orbit for each of the four cases is given in Tab. 20.36.

(b) Wave numbers.

The energy dependence of wave numbers can be obtained by applying the procedure of exercise 5.1-d.

Table 20.35 A Fortran program, geneAVFMapIsochro.f, which generates an $AT = 360^o$ mid-plane field map of an isochronous cyclotron. AT as well as the field amplitude (B_0 =5 kG, here) and its modulation ($f = 0.2$, here) can be changed, a field index ($ak = 0$, here) can be accounted for. The field map produced is logged in geneAVFMapIsochro.out, it may be saved under a different name for the purpose of the exercise, depending upon f, k, or AT values

```
C geneAVFMapIsochro.f program
      implicit double precision (a-h,o-z)
      parameter (c=2.99792458d8, am=938.27208d6, T2kG=10.d0)
      parameter (pi=4.d0*atan(1.d0), BY=0.d0, BX=0.d0, Z=0.d0, dZ=0.d0)

      open(unit=2,file='geneAVFMapIsochro.out')
C------------ Hypotheses :
      AT = 360.d0  /180.d0*pi         ! Angular extent of field map. Can be changed 360, 60 deg, etc.).
      f =.9d0                                                          ! azimuthal modulation factor.
      Bp2k =0.5d0  !For consistency with other exercises, assume a 0.5T average field at 200keV energy,
      Rp2k = 0.129248888074 ; Ep2k = 200.d3    ! Rp2k: 200keV average radius. Reference kinetic energy.
      ah = 1.d0                                                                  ! Harmonic number.
      Rmi=1.d0; Rma=76.d0; RM=50.d0  ! cm. Radial extent of field map; reference radius to define mesh.
      dR = 0.5d0 ; NR = NINT((Rma - Rmi)/dR)+1   ! R-distance between nodes in mesh. Number of R-nodes.
      RdA = 1.d0        !  RdA=RM*dA= distance between two nodes along R=RM arc, dA is angle increment.
      NX= NINT(RM*AT /RdA) +1 ; RdA= RM*AT / DBLE(NX -1)  ! exact mesh step at RM, corresponding to NX.
      dA = RdA / RM ; A1 = 0.d0 ; A2 = AT                               ! corresponding delta_angle.
      gma = (Ep2k+am)/am ;  omgrv = Bp2k*c**2 / (gma * am)
      omgrf = ah * omgrv ; Rinfty = c/omgrf
      B0 = T2kG * Bp2k * sqrt(1-(Rp2k/Rinfty)**2)                          ! Field at R=Rp2k (kG).
C      B0=  T2kG *  0.5d0                                                         ! case B=Cst.
      Rinfty = Rinfty *1.d2                                                     ! Convert to cm.
C----------------------------------------------
      write(2,*) Rmi,dR,dA/pi*180.d0,dZ,
     >'     !  Rmi/cm, dR/cm, dA/deg, dZ/cm'
      write(2,*) '# Field map generated using geneAVFMap.f '
      write(2,fmt='(a)') '# AT/rd,  AT/deg, Rmi/cm, Rma/cm, RM/cm,'
     >//' NR, dR/cm, NX, RdA/cm, dA/rd : '
      write(2,fmt='(a,1p,5(e16.8,1x),2(i3,1x,e16.8,1x),e16.8)')
     >'# ',AT, AT/pi*180.d0,Rmi, Rma, RM, NR, dR, NX, RdA, dA
      write(2,*) '# For TOSCA: ',NX,NR,' 1 22.1 1.  !IZ=1 -> 2D ; '
     >//'MOD=22 -> polar map ; .MOD2=.1 -> one map file'
      write(2,*) '# R*cosA (A:0->360), Z==0, R*sinA, BY, BZ, BX '
      write(2,*) '# cm                 cm    cm      kG  kG  kG '
      write(2,*) '# '
      do jr = 1, NR
        R = Rmi + dble(jr-1)*dR
        do ix = 1, NX
          A = A1 + dble(ix-1)*dA
          BR = 1.d0 / sqrt(1.d0 - (R/Rinfty)**2)
C          BR = 1.d0 !  case B=Cst
          BZ = B0 * BR * (1.d0+f*sin(4.d0*A +pi/2.d0))
          X = R * sin(A); Y = R * cos(A)
          write(2,fmt='(1p,6(e16.8),2(1x,i0),4(1x,e16.8))')
     >     Y,Z,X,BY,BZ,BX,ix,jr,A,R,BR,BZ/T2kG*c**2/(BR*am)
        enddo
      enddo
      stop
     >' Job complete ! Field map stored in geneAVFMapIsochro.out.'
      end
```

Top and bottom sections of the field map file geneAVFMapIsochro.out. The file starts with an 8*-line header, the first one of which is effectively used by zgoubi, the following* 7 *are just comments*

```
# Field map generated using geneAVFMapIsochro.f
# AT/rd,  AT/deg, Rmi/cm, Rma/cm, RM/cm, NR, dR/cm, NX, RdA/cm, dA/rd :
#   6.28318531E+00   3.6E+02   1.0E+00   7.6E+01   5.0E+01 151   5.0E-01 315   1.050721E+00   2.101443E-02
# For TOSCA:          315         151  1 22.1 1.  !IZ=1 -> 2D ; MOD=22 -> polar map ; .MOD2=.1 -> one map file
#
# R*cosA (A:0->360), Z==0, R*sinA, BY, BZ, BX
# cm                 cm    cm      kG  kG  kG
  1.00000000E+00  0.00000000E+00  0.00000000E+00  0.00000000E+00  9.49798755E+00  0.00000000E+00 1 1
  9.99799804E-01  0.00000000E+00  2.00088090E-02  0.00000000E+00  9.48358368E+00  0.00000000E+00 2 1
  9.99199295E-01  0.00000000E+00  4.00096065E-02  0.00000000E+00  9.44046431E+00  0.00000000E+00 3 1
  9.98198715E-01  0.00000000E+00  5.99943846E-02  0.00000000E+00  9.36890554E+00  0.00000000E+00 4 1
  ................................
  7.58631023E+01  0.00000000E+00 -4.55957323E+00  0.00000000E+00  9.43869378E+00  0.00000000E+00 312 151
  7.59391464E+01  0.00000000E+00 -3.04073010E+00  0.00000000E+00  9.51078559E+00  0.00000000E+00 313 151
  7.59847851E+01  0.00000000E+00 -1.52066948E+00  0.00000000E+00  9.55422615E+00  0.00000000E+00 314 151
  7.60000000E+01  0.00000000E+00 -1.86146313E-14  0.00000000E+00  9.56873731E+00  0.00000000E+00 315 151
```

A gnuplot script to obtain a similar field plot to Fig. 5.4 *can be found in Tab.* 20.31.

Table 20.36 Orbit length (C), revolution period ($T_{\rm rev}$) and revolution frequency ($f_{\rm rev} = T_{\rm rev}^{-1}$) at injection, as a function of AVF modulation. Closed orbit length, and thus revolution period, tends to decrease with increasing modulation

f	C (cm)	$T_{\rm rev}$ (μs)	$f_{\rm rev}$ (MHz)
Constant B	81.20948	0.13121691	7.6209688
0	81.20948	0.13121691	7.6209688
0.2	81.1014	0.13104242	7.6311163
0.9	79.12344	0.12784631	7.8218917

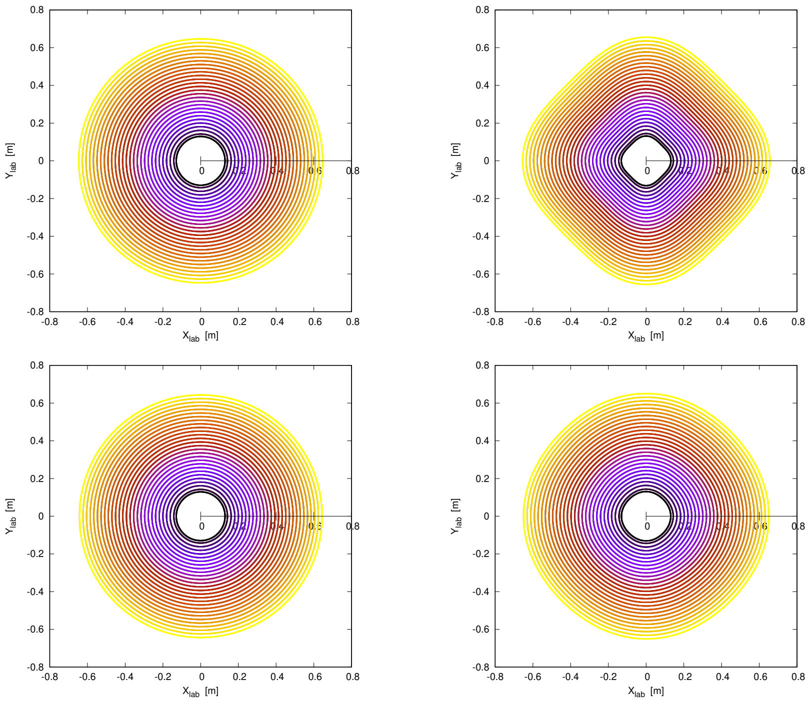


Fig. 20.37 Twenty eight closed orbits in the field of a cyclotron. Top left: constant field $B_Z = 5\,$kG; top right: isochronous $B(R)$ field profile (Eq. 5.14) together with 4-periodic modulation (Eq. 5.3 with N=4) with f=0.9; bottom right: same, with f=0.2; bottom left: same, with f=0. The f=0.9 and f=0.2 cases (right column) satisfy AVF focusing principles, the other two (case of constant B and case f=0, left column) yield unstable optics due to the absence of axial focusing

Table 20.37 Simulation input data file: raytrace a series of ions with different rigidities, spanning 200 keV to 5 MeV

```
Find closed orbits in an [isochronous] 360 degree AVF dipole.
'MARKER'   FieldMapIsochro_S                                         ! Just for edition purposes.
'OBJET'
64.62444403717985                     ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
1 1
12.9248888 0.  0. 0. 0.  1. 'm'       ! Closed orbit coordinates at BR=64.6244440 kG.cm for constant B.
1
'PARTICUL'
PROTON
'TOSCA'
0 2                                ! IL=2 to log step-by-step coordinates, spin, etc., in zgoubi.plt.
1. 1. 1. 1.  ! Normalization coefficients applied to field, and X, Y Z coordinate values read from map.
HEADER_8                                           ! The field map file starts with an 8-line header.
315 151 1 22.1 1.              ! IZ=1 for 2D map; MOD=22 for polar frame; .MOD2=.1: only one map file.
geneAVFMap_360deg_f.2_isochro.out     ! Or [...]_360deg_f.9_isochro.out, or [...]_360deg_Bcst.out, etc.
0 0 0 0                 ! Possible boundaries within the field map, to start/stop stepwise integration.
2
.2 ! cm                                                                     ! Integration step size.
2                                                                      ! Magnet positionning option.
0. 0. 0. 0.                                                                   ! Magnet positionning.
'FAISCEAU'                                                             ! Particle coordinates, here.
'FIT'
1
2  30  0 [5.,100.]                    ! Vary Y0 at OBJET, to allow fulfilling the following constraint:
2  1e-15 200                                   ! Penalty 1e-15; a maximum of 200 calls to the function.
3.1 1 2 5 0. .1 0                                                   ! request same radius after 360 deg;
3   1 3 5 0. 1. 0                                          ! request orbit angle after 360 deg to be zero.

'FAISCEAU' CHECK   ! Allows quick check of particle coordinates, in zgoubi.res: final should = initial.

'FAISTORE'                                             ! Log turn-by-trun particle data in zgoubi.fai.
zgoubi.fai
1

'REBELOTE'                                                                  ! Repeat what precedes,
27  0.1  0 1                                                                               ! 27 times.
1
OBJET 35  1.1:5           ! Change the value of parameter 30 (namely, Y) in OBJET (prior to repeating).
'SYSTEM'
2
gnuplot < ./gnuplot_Zplt_traj.gnu &
gnuplot < ./gnuplot_Zfai_Trev.gnu &
'MARKER'   FieldMapIsochro_E                                         ! Just for edition purposes.
'END'
```

---

gnuplot script to obtain Fig. 20.37:

```
# gnuplot_Zplt_traj.gnu
set xtics; set ytics; set xlabel 'X_{lab}  [m]'; set ylabel 'Y_{lab}  [m]'; unset colorbox;
set size ratio -1; set polar; cm2m=1e-2 ; nrblt1=27 ; FITlast=1
plot for [FITnb=1:nrblt1] 'zgoubi.plt' u \
($49==FITnb && $51==FITlast ? $22 :1/0):($10*cm2m):($41) w p pt 4 ps .2 lc palette notit ; pause 1
```

---

gnuplot script to obtain Fig. 20.38. *All four cases*: *B*=*constant, f*=0*,* 0.2*,* 0.9*, are plotted together, the respective zgoubi.fai files have been saved under different names for that purpose*:

```
# ./gnuplot_Zfai_Trev_all.gnu
set xtics; set ytics nomirror; set y2tics; set ylabel "{/Symbol D}T_{rev}/T_{rev}"; set y2label "T_{rev} [{/Symbol m}s]"
set xlabel "R [m]"; cm2m=0.01; set key c r maxcol 1; set log y; cm2m=.01; set y2range [.1275:.1325]
# Revolution time on lowest rigidity orbits: 1.6402..., 1.636..., etc. are taken from the respective zgoubi.fai files.
plot \
"zgoubi.Bcst" u ($10*cm2m):(abs($15-1.640561029E-01)/1.640561029E-01 ) w lp pt 11 lc rgb "black"  tit "B=Cst" ,\
"zgoubi.f0"   u ($10*cm2m):(abs($15-1.640211407E-01)/1.640211407E-01 ) w lp pt 5 lc rgb "red"  tit "f=0" ,\
"zgoubi.f2"   u ($10*cm2m):(abs($15-1.638030334E-01)/1.638030334E-01 ) w lp pt 7 lc rgb "brown"  tit "f=0.2" ,\
"zgoubi.f9"   u ($10*cm2m):(abs($15-1.598078879E-01)/1.598078879E-01 ) w lp pt 9 lc rgb "blue"  tit "f=0.9" ,\
"zgoubi.Bcst" u ($10*cm2m):($15) axes x1y2 w lp pt 10 ps 1.2 lc rgb "black" notit ,\
"zgoubi.f0"   u ($10*cm2m):($15) axes x1y2 w lp pt  4 ps 1.1 lc rgb "red"  notit  ,\
"zgoubi.f2"   u ($10*cm2m):($15) axes x1y2 w lp pt  6 ps 1.1 lc rgb "brown" notit ,\
"zgoubi.f9"   u ($10*cm2m):($15) axes x1y2 w lp pt  8 ps 1.2 lc rgb "blue" notit ; pause 1
```

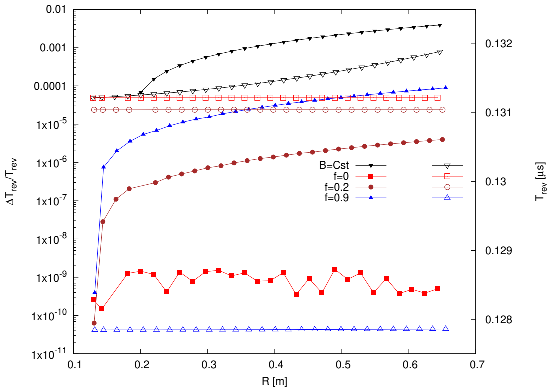


Fig. 20.38 Left vertical scale, solid markers: departure from isochronism in the case of constant field $B_Z = 5\,\mathrm{kG}$ at all (R, θ) (top curve; this is a "classical cyclotron" case) and (from bottom up) of isochronous $B(R)$ (Eq. 5.13) with f=0, f=0.2 and f=0.9 (Eq. 5.3). Right vertical scale, empty markers: revolution time; the "classical cyclotron" case (top curve) features steady increase of revolution time due to mass increase

5.3

Acceleration to 200 MeV in an AVF Cyclotron

(a) Sufficient modulation has to be considered, for the axial focusing to be efficient up to highest γ (compensating the increase in $k(R)$), namely (Eq. 5.8),

$$f > \beta\gamma\sqrt{2}$$

Assume acceleration of protons, up to over 100 MeV, *i.e.* $\beta\gamma \gtrsim 0.474$, axial focusing thus requires $f > \beta\gamma\sqrt{2} = 0.67$. A value of f=0.9 will be taken here.

This results in the 90 deg sector definition given in Tab. 20.38, which uses a field map with a sufficiently large radial extent, geneAVFMap_90deg_f.9_isochro.out, created using Tab. 20.35 program. Note that some cyclotron designs feature negative valley field [3] to further increase the flutter (Eq. 5.5) and thus the axial focusing (Eq. 5.6), so potentially allowing higher $k(R)$ and higher (Eq. 5.1).

Table 20.38 This file provides the simulation of a 90 degree AVF sector, with modulation f=0.9. It defines an #S_AVFMag_90d_f9 to #E_AVFMag_90d_f9 segment subject to INCLUDE in the input data file of Tab. 20.39. The END statement is mandatory at the end of an INCLUDE file

```
! fieldMap90deg_f9.inc
'MARKER'   #S_AVFMag_90d_f9
'TOSCA'
0 0              ! IL=20 to log coordinates, spin, etc., in zgoubi.plt every other 10 integration step.
1. 1. 1. 1.  ! Normalization coefficients applied to field, and X, Y Z coordinate values read from map.
HEADER_8                                              ! The field map file starts with an 8-line header.
80 151 1 22.1 1.                ! IZ=1 for 2D map; MOD=22 for polar frame; .MOD2=.1: only one map file.
geneAVFMap_90deg_f.9_isochro.out                    ! Or [...]_f.1_isochro.out, or [...]_Bcst.out, etc.
0 0 0 0                 ! Possible boundaries within the field map, to start/stop stepwise integration.
2
.2 ! cm                                                                     ! Integration step size.
2                                                                    ! Magnet positionning option.
0. 0. 0. 0.                                                                 ! Magnet positionning.
'MARKER'   #E_AVFMag_90d_f9
'END'
```

The voltage gap is simulated using CAVITE[IOPT=7]. Referring to Tab. 20.36 or Fig. 20.39, the RF frequency has to be around 7.82 MHz, a little tweaking shows that $f_{\rm rf} = 7.7952$ MHz yields efficient use of the RF. A 400 kV peak voltage is applied to the electrode gap. This results in the input data file given in Tab. 20.39. Acceleration cycles (and deceleration, beyond an RF phase of π) are shown in Figs. 20.40, 20.41.

(b) Energy dependence of wave numbers.

The energy dependence of wave numbers is displayed in Fig. 20.42 in the two modulation cases f=0.1 and f=0.9. This simulation has been performed using the input data file of Tab. 20.40. Two field maps have been generated for that purpose, using the Fortran program in Tab. 20.35 with proper f values (the latter is the same as used in exercise 5.2). The argument PRINT under MATRIX causes logging of MATRIX computation outcomes in zgoubi.MATRIX.out, including the wave numbers as plotted in Fig. 20.42.

The theoretical upper limit in energy, for axial stability, is determined by $\beta\gamma < f/\sqrt{2}$, *i.e.*,

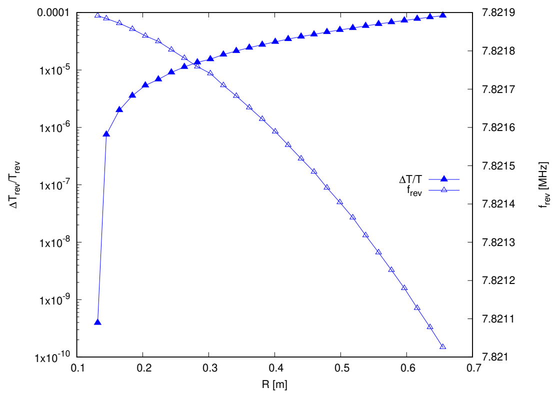


Fig. 20.39 Left vertical scale, solid markers: departure from isochronism as a function of closed orbit radius, case of f=0.9. Right vertical scale, empty markers: revolution frequency

Table 20.39 Simulation input data file: acceleration gaps (two CAVITE) are added between two 180 deg sectors

```
Uniform field sector. INCLUDE file FieldMapSector.inc.
'MARKER'   FieldMapAVFAccel_S                                          ! Just for edition purposes.
'OBJET'
64.62444403717985                      ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
1 1
13.1650 0. 2. 0. 0. 1. 'm'    ! Closed orbit coord. at BR=64.6244440 kG.cm for f=0.9, isochronous B(R).
1
'MARKER'   AVFAccel400kV_S
'PARTICUL'
PROTON
'FAISTORE'
zgoubi.fai
1
'INCLUDE'
1
2 * fieldMap90deg_f9.inc[#S_AVFMag_90d_f9:#E_AVFMag_90d_f9]                         ! 180 deg sector.
'FAISCEAU'                                                                ! Particle coordinates, here.
'CAVITE'   GAP1
7  PRINT                                 ! PRINT: log CAVITE computational data in zgoubi.CAVITE.out.
0.00  7.7952e6
400e3    -1.57079632679                              ! Peak voltage;,  relative phase of 1st cavity.
'INCLUDE'
1
2 * fieldMap90deg_f9.inc[#S_AVFMag_90d_f9:#E_AVFMag_90d_f9]                         ! 180 deg sector.
'FAISCEAU'                                                                ! Particle coordinates, here.
'CAVITE'   GAP1
7  PRINT                                 ! PRINT: log CAVITE computational data in zgoubi.CAVITE.out.
0.00  7.7952e6
400e3    +1.57079632679                              ! Peak voltage;,  relative phase of 2nd cavity.
'MARKER'   AVFAccel400kV_E
'REBELOTE'                                 ! Repeat what preceds,  for a total of 399+1=400 passes.
399 0.1 99
'FAISTORE'
zgoubi.fai
1
'FAISCEAU'
'SYSTEM'
1                                                                          ! 1 SYSTEM command follows.
/usr/bin/gnuplot < ./gnuplot_CAVITE.gnu         ! Plot Ek versus phase, as read from zgoubi.CAVITE.out.
'MARKER'   FieldMapAVFAccel_E                                          ! Just for edition purposes.
'END'
```

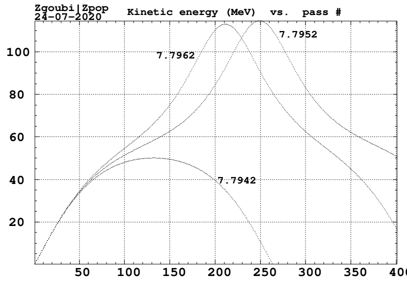


Fig. 20.40 Acceleration followed by deceleration, case of f=0.9, for three different RF frequencies: 7.7942, 7.7952 and 7.7962 MHz

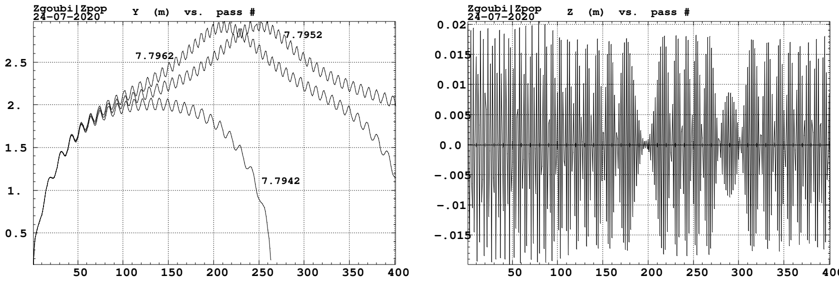


Fig. 20.41 Left: radial excursion during acceleration and deceleration, case of f=0.9, for the three different RF frequencies 7.7942, 7.7952 and 7.7962 MHz. Right: axial excursion, case of $f_{\rm rf}$ = 7.7952 MHz

- a theoretical 2.4 MeV for f=0.1, confirmed in this simulation, and
- a much higher 175 MeV for f=0.9, whereas this simulation yields 280 MeV (as Eq. 5.6 is a weak modulation approximation).

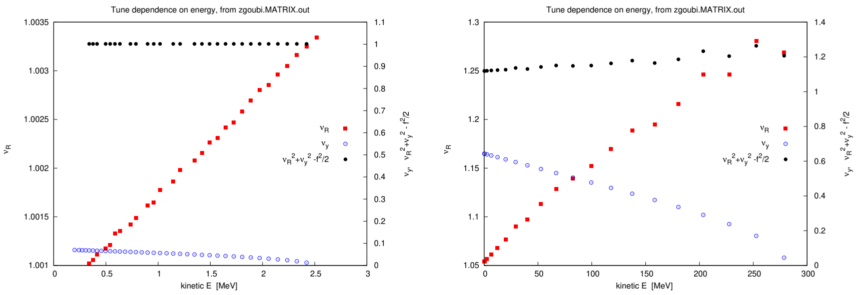


Fig. 20.42 The left and right graphs are for respectively f=0.1 and f=0.9 modulation factor. Left vertical scale: radial wave number. Right vertical scale: axial wave number and $\nu_R^2 + \nu_y^2 - f^2/2$, the latter expected constant and close to 1 in the small scalloping/weak modulation approximation (Eq. 5.8). The upper limit in energy is determined by ν_y decreasing to zero, namely, around 2.4 MeV for f=0.1, around 280 MeV for f=0.9

Table 20.40 Simulation input data file: energy dependence of wave numbers. The INCLUDE uses the TOSCA segment defined in Tab. 20.38

```
Scan momentum.
'MARKER'   FieldMapAVFEdep_S                                              ! Just for edition purposes.
'OBJET'
64.62444403717985                      ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
5
.01 .1  .01 .1 .001 .0001
13. 0. 0. 0. 0. 1.
'PARTICUL'
PROTON
'SPNTRK'
4.1
0. 0. 1.
'INCLUDE'
1
4 * fieldMap90deg_f9.inc[#S_AVFMag_90d_f9:#E_AVFMag_90d_f9]                     ! 90 deg field extent.
'FAISCEAU'                                                               ! Particle coordinates, here.
'FIT'
1
2  30  0  [1.,1e3]
1  1e-10 49
3.1 1 2 7 0. 1. 0
'FAISCEAU'
'MATRIX'
1 11   PRINT
'FAISTORE'
zgoubi.fai
1
'REBELOTE'                                                                    ! Repeat what precedes,
37  0.1  0 1                                                                              ! 37 times.
1
OBJET 35  1.1:3.55        ! Change the value of parameter 35 (namely, D) in OBJET (prior to repeating).
'SYSTEM'
1
gnuplot <./gnuplot_MATRIX_Qxy.gnu
'MARKER'   FieldMapAVFEdep_E                                              ! Just for edition purposes.
'END'
```

---

gnuplot script to obtain Fig. 20.42:

```
set xlabel "kinetic E  [MeV]"; set ylabel "{/Symbol n}_R,    {/Symbol g}"  # font "roman,24"
set y2label "{/Symbol n}_y,   {/Symbol n}_R^2+{/Symbol n}_y^2 - f^2/2"; set key c r maxcol 1;
set key spacin 1.9; set xtics; set ytics nomirror;  set y2tics nomirror; V2MV=1e-6; BORO = 64.62444403717985;
am=938.27208e6; am2=am*am; BrRef = BORO *1e-3; V2MV = 1e-6; intQx = 1.; c= 2.99792458e8; f = 0.1 # or 0.9
plot  \
"zgoubi.MATRIX.out" u ((sqrt(($47*BrRef*c)**2+am2)-am)*V2MV):(intQx+$56) w p pt 5 lc rgb "red" tit "{/Symbol n}_R" ,\
"zgoubi.MATRIX.out" u ((sqrt(($47*BrRef*c)**2+am2)-am)*V2MV):($57) axes x1y2 w p pt 6 lc rgb "blue" tit "{/Symbol n}_y" ,\
"zgoubi.MATRIX.out" u ((sqrt(($47*BrRef*c)**2+am2)-am)*V2MV):((intQx+$56)**2+$57**2-f**2/2.) \
axes x1y2 w p pt 7 lc rgb "black" tit "{/Symbol n}_R^2+{/Symbol n}_y^2 -f^2/2"; pause 1
```

5.4

Thomas-BMT Spin Precession in Thomas Cyclotron

Simulations use files developed in exercise 5.3, with *ad hoc* modifications.

Helion is specified using PARTICUL. This determines the value of the gyromagnetic anomaly, as well as mass and charge as they are needed to solve the spin motion differential equation (Eq. **??**). PARTICUL results in the following, in zgoubi.res execution listing:

```
    Particle  properties :
    HELION
                   Mass          =    2808.39        MeV/c2
                   Charge        =   3.204353E-19    C
                   G  factor     =   -4.18415

             Reference  data :
                  mag. rigidity (kG.cm)   :   64.624444      =p/q, such that dev.=B*L/rigidity
                  mass (MeV/c2)           :   2808.3916
                  momentum (MeV/c)        :   38.747842
                  energy, total (MeV)     :   2808.6589
                  energy, kinetic (MeV)   :  0.26729246
                  beta = v/c              :  1.3795851874E-02
                  beta*gamma              :  1.3797164913E-02
                  G*gamma                 :  -4.184552032
```

(a) Resonant $G\gamma$.

A preliminary scan of motion wave numbers is performed, using the input file of Tab. 20.41. This scan shows that, over a kinetic energy range E_k : 50 → 300 MeV,

Table 20.41 Simulation input data file: a scan of wave numbers, computed using OBJET[KOBJ=5] and MATRIX, in 74 steps over a relative rigidity range $D\,:\,1 \to 36$, *i.e.*, helion rigidity $B\rho$: $64.624444 \to 36 \times 64.624444$ T m, energy $E\,:\,0.267292 \to 2.326479$ MeV. The INCLUDE uses the TOSCA segment defined in Tab. 20.38. FIT finds particle closed orbit and spin $\mathbf{n}_0$ vector, prior to MATRIX computation

```
MATRIX scan.
'OBJET'
64.6244440                              ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
5                                             ! KOBJ=5 to define an 11 paticle sample for use by MATRIX.
.001 .01  .001 .01 .001 .0001
13. 0. 0. 0. 0. 1.  ! Initial Y is taken close to its periodic value, FIT will find its precise value),
                                                 ! whereas REBELOTE changes it at each of the 73 repeat.
'PARTICUL'
HELION
'INCLUDE'
1
4* ./fieldMap90deg_f9.inc[#S_AVFMag_90d_f9:#E_AVFMag_90d_f9]
'FIT'
1
1 30  0  [1.,1e3]                                      ! Vary Y0 (parameter 30) in OBJET (element 1).
2  1e-5 49     ! Periodic orbit constraints apply after a half-turn (i.e., after first 180 deg sector):
3.1 1 2 7 0. 1. 0                                                     ! particle 1 radius unchenged,
3   1 3 7 0. 1. 0                                                          ! particle 1 angle T=0.

'FAISCEAU'
'MATRIX'
1 11   PRINT
'REBELOTE'                                                                 ! Repeat what precedes,
73  0.1  0 1                                                                          ! 73 times.
1
OBJET 35  1.001:36       ! Change the value of parameter 30 (namely, D) in OBJET (prior to repeating).

'SYSTEM'                                ! SYSTEM is executed in sequence, i.e., when REBELOTE is done.
1
/usr/bin/gnuplot < ./gnuplot_MATRIX_Qxy.gnu
'END'
```

---

gnuplot script to obtain Fig. 20.43:

```
set xlabel "|G{/Symbol g}|"; set x2label "kinetic E  [MeV]"; set ylabel "{/Symbol n}_Z,  |G{/Symbol g}|-4"; set y2label "S_Z"
set key c l maxcol 2; set key spacin +1.5; set xtics nomirror; set x2tics; set ytics nomirror; set y2tics
# Particle data:
BORO = 64.62444403717985; q = 2.; amu = 931.4940954e6; am=3.01493224673 * amu; G = 4.1841538; am2=am*am
BrhoRef = BORO *1e-3; eV2MeV = 1e-6; c = 2.99792458e8
# Scales:
Gg1=4.25 ; Gg2=4.65; E1=(Gg1/G-1.)*am/1e6 ; E2=(Gg2/G-1.)*am/1e6; set xrange [Gg1:Gg2] ; set x2range [E1:E2]
plot  \
"zgoubi.fai_spin" u ($25/$29*G):($22) axes x1y2 w lp ps .4 tit "S_Z" ,\
"zgoubi.MATRIX.out_73Qs" u (((sqrt(($47*BrhoRef*c*q)**2 + am2)-am)+am)/am*G):($57) axes x1y1 w p pt 6 lc rgb "blue" tit "{/Symbol n}_Z " ,\
"zgoubi.MATRIX.out_73Qs" u ((sqrt(($47*BrhoRef*c*q)**2 + am2)-am)/1e6):(((sqrt(($47*BrhoRef*c*q)**2 + am2)-am)+am)/am*G-4.) axes x2y1 w p pt 7 lc rgb "red" tit "|G{/Symbol g}|-4}
```

the axial wave number ν_Z decreases from 0.6 to 0.25 about while $G\gamma\,:\,-4.25 \to -4.65$ (Fig. 20.43). It results that at a particular location over that energy range, the relationship

$$G\gamma + \nu_Z = \text{integer} = -4$$

is satisfied, namely here: $G\gamma = -4.4375$.

(b) Helion spin precession.

A spin tracking is launched with the helion ion injected near the $B\rho$ = 64.624444 T m radial closed orbit, namely, $R_{\text{inj}} \approx 13$ cm and angle $T\text{inj} = 0$, with non-zero axial amplitude in order to excite the spin resonance, namely, $Z_{\rm inj} = 2\,$cm (leaving angle $P_{\rm inj} = 0$). The simulation file is given in Tab. 20.42. Acceleration is over $G\gamma : -4.18 \rightarrow -4.75$, the axial wave number decreases from 0.64312 to 0.23011.

Table 20.42 Simulation input data file: spin tracking through the $G\gamma + \nu_Z = -4$ resonance

```
Track spin through resonance.
'OBJET'
64.62444403717985                        ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
1 1
13.1650 0. 2. 0. 0. 1. 'm'    ! Closed orbit coord. at BR=64.6244440 kG.cm for f=0.9, isochronous B(R).
1
'PARTICUL'
HELION
'SPNTRK'
3
'FAISTORE'
zgoubi.fai
1
'INCLUDE'
1
2* ./fieldMap90deg_f9.inc[#S_AVFMag_90d_f9:#E_AVFMag_90d_f9]
'CAVITE'   GAP1
3
0.00  0.
20e3  +1.57079632679                              ! Peak voltage;,  relative phase of 1st cavity.
'INCLUDE'
1
2* ./fieldMap90deg_f9.inc[#S_AVFMag_90d_f9:#E_AVFMag_90d_f9]
'CAVITE'   GAP2
3
0.00  0.
20e3  +1.57079632679                              ! Peak voltage;,  relative phase of 2nd cavity.
'REBELOTE'
3999 1.1 99
'SYSTEM'
1                                                                    ! 1 SYSTEM command follows.
/usr/bin/gnuplot < ./gnuplot_MATRIX_Qxy.gnu
'END'
```

Figure 20.43 displays the vertical spin component, flipping from +1 to -1; a close inspection of raytracing outcomes confirms the location of the resonance at $G\gamma_R = -4 - 0.4375$.

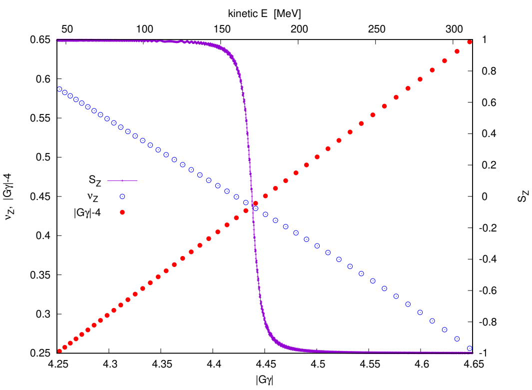


Fig. 20.43 Spin resonance crossing. The graph shows the evolution of the axial wave number ν_Z and of the quantity $|G\gamma| - 4$ (left vertical axis), and of the helion ion spin, initially vertical, $S_Z = 1$ (right vertical axis), as a function of $G\gamma$ (lower horizontal axis) and of energy (upper horizontal axis). ν_Z and $|G\gamma| - 4$ curves cross at $G\gamma = -4.4375$

(c) Spin resonance crossings. Resonance strength.

This exercise is performed by repeating the simulation of Tab. 20.42 for a series of different Z_0 values (an external program can do it); outcomes are displayed in Fig. 20.44.

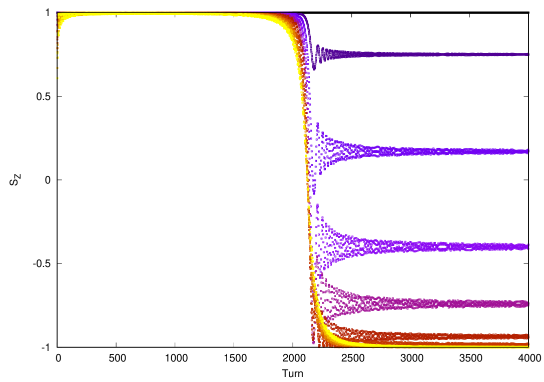


Fig. 20.44 Evolution of S_Z during resonance crossing, for a series of values of the initial axial particle coordinate Z_0. Spin flip occurs at larger Z_0 values

As expected (Eq. 5.18) $S_{Z,f}/S_{Z,i}$ tends toward 1 (respectively, toward 0), as the strength of the resonance tends toward zero (respectively, goes $\gg 0$), Fig. 20.45. The former case corresponds to absence of resonance, *i.e.*, B_Z axial always, as $Z_0 \equiv 0$: the ion motion is in the median plane of the cyclotron dipole. Increasing Z_0 increases the strength of the non-vertical field experienced by the ion as it cycles around the accelerator, and spins to undergo greater tilt at traversal of the resonance, toward spin flip with sufficient vertical excursion.

A match of $S_{Z,f}/S_{Z,i}(Z_0)$ to Eq. 5.18 shows that these raytracing outcomes satisfy $|\epsilon_R| \propto Z_0$.

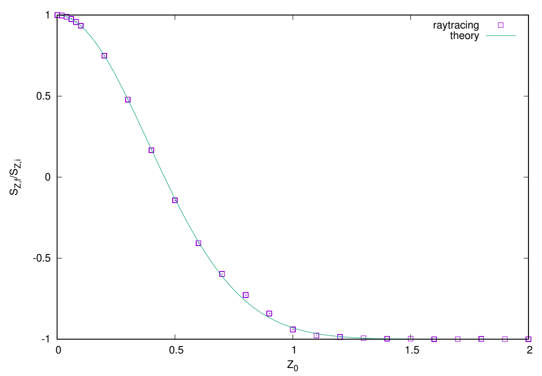


Fig. 20.45 Evolution of $S_{Z,f}/S_{Z,i}$ toward spin flip as the axial motion excursion increases

(d) Changing the crossing speed.

The method to answer this question is the same as in (c), repeating the simulation of Tab. 20.42 for a series of different acceleration rates (*i.e.* gap voltage, V) in CAVITE, to get $S_{y,f}/S_{y,i}(V)$.

Finally, the relationship to the crossing speed (Eq. 5.19) can be established using the $d\nu_Z/dt$ data produced in (b) (Fig. 20.43).

5.5

Isochronism and Edge Focusing in a Separated Sector Cyclotron

An separated sector isochoronous cyclotron modeled using DIPOLE.

(a) DIPOLE allows to account for the field fall-off extent at dipole EFBs, which determines the flutter. The input data file for this simulation is given in Tab. 20.43.

Across the 30 deg sector dipole, a 4-periodic closed orbit undergoes a 90 deg deviation, whatever the rigidity. Due to the periodicity and to the field symmetry (the dipole is symmetric with respect to a vertical plane at 45 deg to its EFBs), the closed orbits enter and exit the magnetic sector with angles $TE = TS = 0$.

FIT is used to find the closed orbit at a particular rigidity, the process is repeated (using REBELOTE) for a series of different rigidities, in the following way:

- the first constraint under FIT imposes that particle 1 be on a periodic orbit. That constraint is enforced with a weight of 0.1, *i.e.* greater compared to 1 for the second constraint;

- for that, FIT allows varying B_0, and ends up with the same B_0 always, as expexted given $k = 0$. This first constraint is maintained unchanged during the REBELOTE process (repeat with different rigidity);

- the second constraint concerns the radial extent of closed orbits: it requires that the initial Y coordinate (Y coordinate at OBJET) of particle 1, be equal to its final coordinate (after DIPOLE).

The REBELOTE process repeats, yet by first changing the relative rigidity D of particle 2 (datum at position 45 in OBJET).

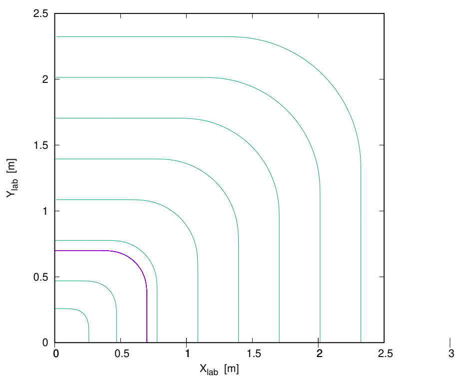


Fig. 20.46 Closed obits across a quadrant, at a few different rigidities

Table 20.43 Simulation input data file 90degEdgeFocusSector.inc: analytical modeling of a 30 degree magnetic sector of a 4-period separated sector cyclotron. This simulation file includes a search of cyclotron orbits for eight different energies in [0.87, 72] MeV. The LABEL_1s #S_90degCycloSector and #E_90degCycloSector define the dipole segment, for further use in subsequent exercises.

```
90degEdgeFocusSector.inc
! Closed orbits and field acroos a 90 degree sector of a 4-period cyclotron
'MARKER'  ProbEdgeFocus_S                                                  ! Just for edition purposes.
'OBJET'
1.2493976131130E3        ! Reference Brho, kG.cm ("BORO" in the users' guide) -> case of 72MeV proton.
2
2 1
70. 0. 0. 0. 0. .3 'o'      ! 6.704673 MeV
25.915 0. 0. 0. 0. 0.10789921779517307 'i'     ! Relative rigidity D=0.10789... -> a 0.870 MeV proton.
1 1
'DIPOLE'   #S_90degCycloSector                          ! Analytical field modeling of a dipole magnet.
2    ! IL=2, purpose: log stepwise particle data in zgoubi.plt. Avoid if unused as I/Os take CPU time.
90. 100.                                                        ! Sector angle AT; reference radius R0.
45.  12.789066 0.  0. 0.         ! Reference azimuthal angle ACN; BM field at R0; indices, N, N', N''.
7.  0.                                                                       ! EFB 1 is  hard-edge,
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.             ! hard-edge only possible with sector magnet.
15. 0.  1.E6  -1.E6  1.E6  1.E6
7.   0.                                                                                     ! EFB 2.
4  .1455   2.2670  -.6395  1.1558  0. 0.  0.
-15. 0.  1.E6  -1.E6  1.E6  1.E6
0. 0.                                                                              ! EFB 3 (unused).
0  0.      0.      0.      0.      0. 0.  0.
0. 0.  1.E6  -1.E6  1.E6  1.E6 0.
2   10.
1.                        ! Integration step size. The smaller, the more accurately the orbits close.
2  0. 0. 0. 0.                                                    ! Magnet positionning RE, TE, RS, TS.
'MARKER'    #E_90degCycloSector
'FAISCEAU'  CHECK                                ! Expect intial coordinates = local coordinates, here.
'FIT'
2
3  5 0  1.                            ! Vary field in DIPOLE (constraint is, below: Y_final=Y_OBJET).
2 40 0 [.1,300.]      ! Vary initial coordinate of particle 2 (constraint is, below: Y_final=Y_OBJET).
2
3.1 1 2 #End 0. 0.1 0  ! Constrain particle 1, R=70cm, to being on periodic prbit; great weight (0.1).
3.1 2 2 #End 0. 1.0 0           ! Constrain particle 2 to be on a periodic orbit; weaker weight (1.0).
'REBELOTE'
7 0.1 0 1                   ! IOPT=1 here allows the change of value of parameter 45 in OBJET, below.
1
OBJET 45 .2:1.       ! 7 additional rigidities (follows from REBELOTE[NPASS=7]), from 2.986 to 72 MeV.
'SYSTEM'
2
gnuplot <./gnuplot_Zplt_orbits.gnu                                             ! Plot orbits in a quadrant.
gnuplot <./gnuplot_Zplt_field.gnu                                ! Plot field along orbits in a quadrant.
'MARKER'  ProbEdgeFocus_E                                                  ! Just for edition purposes.
'END'
```

---

gnuplot script to obtain Fig. 20.46:

```
set xtics; set ytics; set xlabel 'X_{lab}  [m]'; set ylabel 'Y_{lab}  [m]'; cm2m=0.01; set polar
# in zgoubi.plt, col. 19: prticle number; col. 51=1: final past after FIT; col. 22: angle; col. 10: radius
set xrange [0:2.5]; set yrange [0:2.5]; set size ratio -1
plot for [p=1:8] 'zgoubi.plt' u ($19==p && $51==1 ? $22 :1/0):($10 *cm2m) w l notit; pause 1
```

---

gnuplot script to obtain Fig. 20.47:

```
set xtics; set ytics; set xlabel 'X_{lab}  [m]'; set ylabel 'Y_{lab}  [m]'
# in zgoubi.plt, col. 19: prticle number; col. 51=1: final past after FIT; col. 22: angle; col. 25: BZ
plot  for [p=1:8] 'zgoubi.plt' u ($19==p && $51==1 ? $22 :1/0):($25) w l notit; pause 1
```

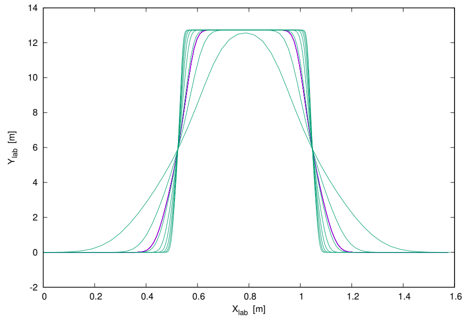


Fig. 20.47 Field along closed obits at different rigidities, over a quadrant

(b) Isochronous B(R).

A similar problem is treated in exercise 20.2, thus just indications are given here, as to determining a proper radial field law for isochronism.

The indices in Eq. 5.20 can be expressed under the form $b_1 = \frac{R_0}{B_0}\frac{\partial B}{\partial R}$, $b_2 = \frac{R_0^2}{2B_0}\frac{\partial^2 B}{\partial R^2}$, $b_3 = \frac{R_0^3}{6B_0}\frac{\partial^3 B}{\partial R^3}$, etc. Expand the $(R-R_0)^i$ terms in Eq. 5.20 and re-organize in increasing powers of R, so writing the radial dependence of the field under the form

$$\mathcal{R}(R) = (1 - b_1 + b_2 - b_3 + b_4 + \ldots) + \frac{R}{R_0}(b_1 - 2b_2 + 3b_3 - 4b_4 + \ldots) + \left(\frac{R}{R_0}\right)^2 (b_2 - 3b_3 + 6b_4 + \ldots) + \left(\frac{R}{R_0}\right)^3 (b_3 - 4b_4 + \ldots) + \ldots \tag{20.11}$$

On the other hand, the Taylor series development of the R-dependent factor of the magnetic field for isochronism, Eq. 5.13, writes

$$\mathcal{R}(R) \approx \frac{1}{\sqrt{1-(\frac{R}{R_\infty})^2}} = 1 + \frac{(R/R_\infty)^2}{2} + \frac{3(R/R_\infty)^4}{8} + \frac{5(R/R_\infty)^6}{16} + \ldots \tag{20.12}$$

Identify term by term with Eq. 20.11, this yields the indices b_i in terms of powers of $1/R_0$ (R_0 is a known quantity), the very values to be used in defining the field and indices in DIPOLES. Accuracy on isochronism can be improved using FIT[2]: require isochronism (the constraint in FIT[2]) and allow varying the b_i indices in DIPOLES (the variables in FIT[2]) starting from initial values obtained as described above.

(c) Changing field fall-off extent.

Indications:

Changing the fringe field extent λ impacts both the closed orbit landscape and the isochronism. The latter can then be re-optimized by means of FIT, varying the b_i coefficients and constraining, concurrently and over the energy extent of concern, both the orbit periodicity and the isochronism of these orbits. Such a FIT is performed in exercise 5.6, the same method can be applied here.

(d, e) Flutter, vertical wave number.

Indications:

Graphs of R-dependence of wave numbers, and relationship to the flutter, are produced in exercise 5.1, the same techniques can be applied here.

5.6

A Model of PSI Ring Cyclotron Using CYCLOTRON

CYCLOTRON provides a realistic analytical modeling of the field in a radial or spiral sector magnet of a separated sector cyclotron [1, Sect. 6.3 & Part B]. CYCLOTRON keyword belongs in the DIPOLE[S] and FFAG[-SPI] families, with some specificities. The large number of field indices available is one, it simulates pole shaping and allows fine tuning of the isochronism.

(a) CYCLOTRON data list.

A sketch of a PSI cyclotron spiral sector as simulated here, and corresponding to CYCLOTRON input data list of Tab. 5.2, is given in Fig. 20.48. A commented version, in answer to question (a), is given in (Tab. 20.44). A note on the origin of

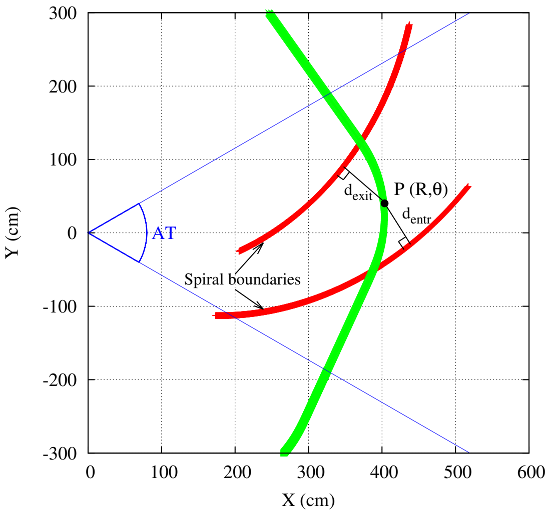


Fig. 20.48 A representation of the EFBs in CYCLOTRON. AT is the total angle of the sector: $2\pi/AT$ defines the number of sectors of the entire ring. The value of the flutter $\mathcal{F}(R,\theta)$ at ion location $P(R,\theta)$ is determined from the distance d to the EFBs. If there are several dipoles within AT, all EFBs are accounted for, in computing the field at $P(R,\theta)$ [5]

the data used to simulate a cyclotron sector, in Tab. 5.2:

(i) The parameters needed in the equation of the spiral effective field boundaries [1, Eq. 6.3.15], and to determine the effective magnetic field length, have been obtained using the magnetic field map of PSI cyclotron [4]. This yielded, as part of the simulation input file data (Tab. 5.2)

Table 20.44 Simulation input data file: a period of PSI eight-sector CYCLOTRON model. The data file is set up for a scan of the closed orbits, from radius R=204.1171097 cm to R=383.7131468 cm, in 15 steps. Comments have been added, line by line, as a guidance

```
PSI CYCLOTRON    ! Title. Need one comment line at top of file. More comment lines requires a commnent
!                                                                              sign, '!', like this one.
'MARKER'   ProbPSICYCLOTRON_S                                                ! Just for edition purposes.
'OBJET'                                                   ! Definition of initial particle coordinates.
1249.382414                                                                        ! Rigidity [kG cm].
2
1 1
2.67042304E+02 -1.50516664E+01  0.  0.  0.  1.4    'o'
1
'PARTICUL'   ! Type of particle. The only interest here is its allowing computation of time of flight,
PROTON                               ! otherwsie, zgoubi does not need it: it works with the rigidity.

'CYCLOTRON'                        ! Analytical modeling of the field in a separated sector cyclotron.
2                    ! Next line: N, AT, R0 (reference radius), type of sector (radial, spiral, both).
1 45. 276. 1.                    ! Next line: ACENT, dR0, FAC, HNORM, K, Rref, field indices b1 to b4.
0. 0. 0.99212277 51.4590015 0.5 800. -0.476376328 2.27602517e-03 -4.8195589e-06 3.94715806e-09
18.3000E+00  1.  28. -2.0                                              ! lambda=gap, gap's k  g10 g11.
8 1.1024358 3.1291507 -3.14287154 3.0858059 -1.43545 0.24047436 0. 0. 0.  ! NBCOEF, COEFS_C0-7, NORME.
11.0  3.5 35.E-3  0.E-4 3.E-8 0. 0. 0.                  ! Entrance EFB: OMEGA, XI0, XI1, XI2, XI3, a,b,c.
18.3000E+00  1.  28. -2.0                                              ! lambda=gap, gap's k  g10 g11.
8 0.70490173 4.1601305 -4.3309575 3.540416 -1.3472703 0.18261076 0. 0. 0. ! NBCOEF, COEFS_C0-5, SHIFT.
-8.5  2.  12.E-3 75.E-6 0.    0. 0. 0.                      ! Exit EFB: OMEGA, XI0, XI1, XI2, XI3, a,b,c.
0. -1                                                                            ! Lateral EFB, unused.
0  0.  0.  0.  0.  0.  0.  0.                                                 ! NBCOEF, C0...C5, shift.
0.  0.  0.   0.  0.  0.                                                      ! omega+, xi, 4 dummies.
2  10.  ! Numerical method for field & derivatives,  flying mesh size is xpas/10=0.4/10. (KIRD,RESOL).
0.4                                                                           ! Integration step size.
2 0.  0. 0.  0.                                                                 ! magnet positioning.

'FIT2'                                                                               ! FIT procedure.
2                                                                                    ! 2 variables:
2 31 0 [-300.,100]                                                ! vary initial angle T0 in OBJET,
2 35 0 [.1,3.]                                                ! vary relative momentum D in OBJET.
2                                                                                   ! 2 constraints:
3.1 1 2 #End 0. 1. 0                                                                  ! get Y-Y0=0,
3.1 1 3 #End 0. 1. 0                                                                  ! get T-T0=0.
'FAISCEAU'

'FAISTORE'                                     ! Store coordinates at each pass, in file orbits.fai.
orbits.fai
1

'REBELOTE'                                 ! Repeat the complete sequence above, from OBJET, 14 times.
14 0.2  0 1                                                                  ! IOPT=1 will cause change
1                                                                 ! of NPRM=1 paameters, as follows:
OBJET 30 281.258209:353.20117   ! in OBJET parameter 30 (Y0) will take 14 values from 281... to 353...

'SYSTEM'                                                               ! A "call system". Will execute
1                                                                           ! 1 command, as follows:
gnuplot <./gnuplot_orbits.gnu
'MARKER'   ProbPSICYCLOTRON_E                                                ! Just for edition purposes.
'END'                                           ! End of the sequence. Whatever follows is ignored.
```

---

gnuplot script to obtain Figs. 20.51 *and* 20.52:

```
# gnuplot_orbits.gnu
set key c t; set xtics; set xlabel "R [cm]"; set ylabel "(T_{rev}-T_{R=314})/T_{R=314}"
colY0=3; T314=1.4737924713529E-02
plot 'orbits.fai' u colY0:(($15-T314)/T314) w lp lt 3 dt 7 lw 2 pt 4 lc rgb "black"; pause 1
#
set xtics nomirror; set xlabel "Y0  [cm]"; set ytics nomirror; set ylabel "Y  [cm]"
set x2tics; set x2label "T0  [mrad]"; set y2tics; set y2label "T  [mrad]"; colY0=3;  colY=10;  colT0=4;  colT=11
plot 'orbits.fai' u colY0:colY w lp lt 3 dt 7 lw 2 pt 4 lc 1,'orbits.fai' u colT0:colT axes x2y2 w lp lt 3 dt 7 lw 2 pt 4 lc 2
```

(ii) The radial field law $\mathcal{R}(R)$ have been obtained by fitting the field fall-off along a series of closed orbits at different radii in the magnetic field map, which yielded the polynomial coefficients b_0 to b_4. A fitting of the 137 MeV closed orbit in particular provided the fringe field coefficients C_0 to C_5.

(b) Pole and field profiles, closed orbits.

The CYCLOTRON input data file, Tab. 20.44, can be run for various particle rigidities (change D in the particle coordinates under OBJET[KOBJ=2]), possibly with several particles (OBJET[KOBJ=2,IMAX>1]) IL=2 under CYCLOTRON takes care of storing stepwise particle data in zgoubi.plt, to produce graphs.

Outcomes are illustrated in Fig.20.49 which shows a series of trajectories across a 45 deg sector and the magnetic field along these trajectories.

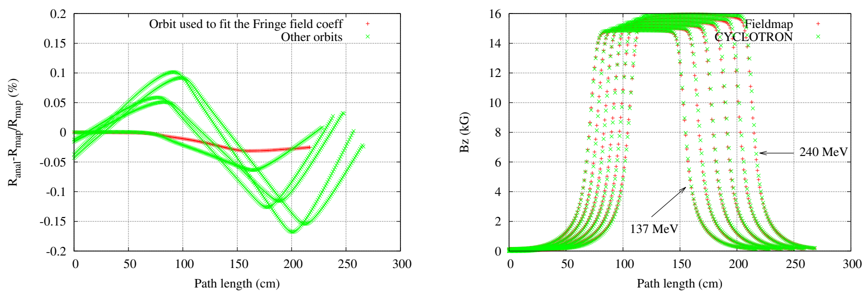


Fig. 20.49 Left: trajectories obtained using the analytical model CYCLOTRON. Right: field profiles along different trajectories, at 137, 156, 176, 196, 218 and 240 MeV, as obtained by raytracing using indifferently (no noticeable difference at this scale) the analytical model CYCLOTRON, or the field map from which it originates [4]

- at entrance EFB: ω = 11 deg and $\xi\,[deg] = 3.5 + 35.10^{-3}\,r + 3.10^{-8}\,r^3$;
- at exit EFB: $\omega = -8.5$ deg and $\xi\,[deg]\; = 2 + 12.10^{-3}\,r + 75.10^{-6}\,r^2$.

Closed orbits at a very large number of different rigidities can be raytraced (use OBJET[KOBJ=2,IMAX≫1]), from which a graph of isomagnetic field lines can be produced, by reading from zgoubi.plt. This allows producing Fig.20.50 in which the EFBs and a series of closed orbits are superposed on a field scale background. Note: the closed orbit for a particular rigidity can be found using FIT, the process can be repeated using REBELOTE (as in Tab. 20.44).

(c) Revolution period.

The Simulation input data file of Tab. 20.44 performs the scan needed here, comments therein explain the method, which is based on FIT to find the proper rigidity and periodic orbit angle for a particle with periodic radius defined by OBJET, and on REBELOTE to repeat the FIT procedure for a new value of the orbit radius, NPASS times.

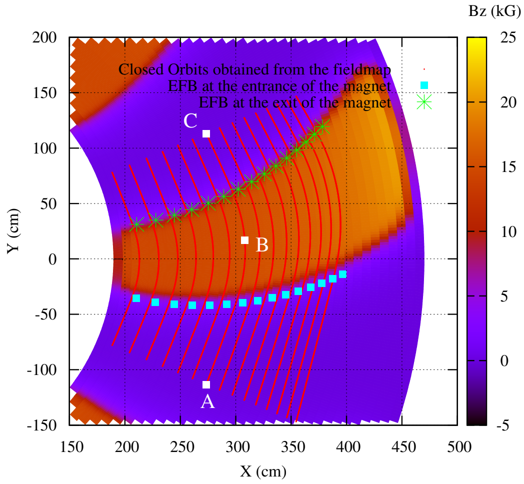


Fig. 20.50 EFBs and field scale, obtained from raytracing using CYCLOTRON or ,indifferently, PSI sector field map [4]. B is the location of the maximum field value along the 137 MeV orbit, C is the location of the minimum value in the field valley

Figure 20.51 checks the proper completion of the FIT procedure, showing that final orbit radius $R = Y$ (down the magnetic sector) is identical to initial $R = Y_0$ (at OBJET) and final orbit angle T is identical to initial T_0. Note that a global check is provided by the penalty value, an outcome also of the FIT procedure.

Figure 20.51 displays the relative time difference to that of a reference orbit, taken to be orbit number 7 at R=314 cm in the middle region of the range, bottom of the time of flight parabola.

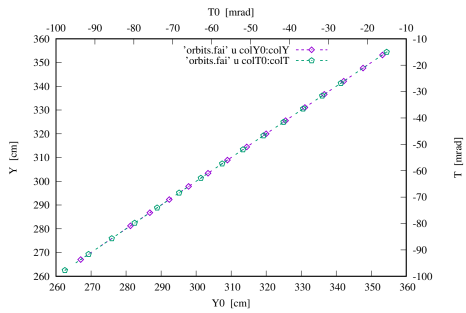


Fig. 20.51 Checking the proper completion of the FIT procedure: final orbit radius $R = Y$ (down the magnetic sector) is identical to initial $R = Y_0$ (at OBJET). The constraint is similar orbit angles: final orbit angle T identical to initial T_0

(d) Improved isochronism.

The number of field indices accounted for in $\mathcal{R}(R)$ (Eq. 5.20) in setting up CYCLOTRON modeling (four only in the case of questions (a) and (b), b_1 to b_4) is increased iteratively, one additional index at a time, and the FIT procedure is re-run each time, until it shows convergence to a series of index values which result in the required degree of isochronism. Eight additional indices come out to allow an

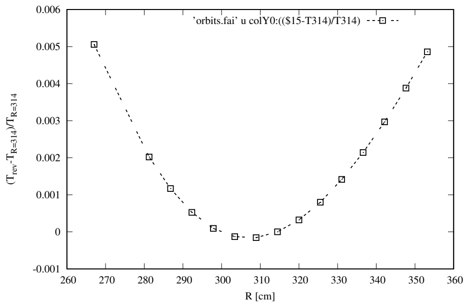


Fig. 20.52 Time of flight difference as a function of closed orbit radius, relative to time of flight $T(R = 314.46264\,cm) = 0.01473792\,\mu s$

improvement of the isochronism by a factor 50. The main field B_0 is also part of the variables, as it allows the orbits to adjust to periodic condition (closed orbits). The simulation data file is given in Tab. 20.45, including the FIT procedure at the last stage of the iteration on the number of field indices.

Note in the FIT procedure constraints: the time of flight on orbit 4, middle region of the range, bottom of the time of flight parabola, is taken as a reference. It does not act as a constraint in the FIT as its weight is 10^9, compared to 10^{-4} and less for the other 6 orbits. That orbit 4 ends up reaching the revolution time value $T_{\rm ref} = 0.014744215772216707$.

The resulting relative time of flight difference $dT_{\rm rev}/T_{\rm ref} = (T_{\rm rev} - T_{\rm ref})/T_{\rm ref}$ as a function of closed orbit radius is displayed in Fig. 20.54, much improved by comparison to the 4 index case (Fig. 20.52).

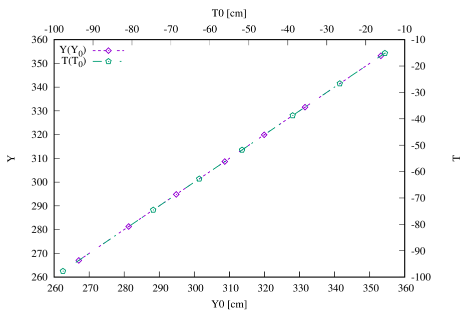


Fig. 20.53 Checking the proper completion of the 12-index FIT procedure: the final orbit radius $R = Y$ (down the magnetic sector) is identical to the initial $R = Y_0$ (at OBJET). Same constraint for orbit angles: the final orbit angle T comes out identical to the initial T_0

Table 20.45 Simulation input data file: a period of PSI separated sector cyclotron, using CYCLOTRON for an analytical modeling of the field. The file is setup to FIT 12 field indices, b_1 to b_{12} for improved isochronism. The constant field B_0 is part of FIT variables, to allow for the constraint of orbit periodicity. OBJET[KOBJ=2,IMAX=7] creates 7 particles which span the momentum range of interest, via $D : 1.4 \rightarrow 2.1$

```
PSI using CYCLOTRON. Fit isochronism with 12 indices.
'MARKER'   ProbPSICYCLOTRON_c_S                                              ! Just for edition purposes.
'OBJET'                                                    ! Definition of initial particle coordinates.
1249.382414                                                                         ! Rigidity [kG cm].
2
7 1             ! A set of 7 particles covering the periodic orbit range of interest R=267 to 351 cm).
2.67042304E+02  -1.50516664E+01   0.  0.   0.00  1.40000001E+00 'o'
2.81258209E+02  -2.66331145E+01   0.  0.   0.00  1.50000001E+00 'o'
2.94829572E+02  -3.87557743E+01   0.  0.   0.00  1.60000001E+00 'o'
3.08680463E+02  -5.17104469E+01   0.  0.   0.00  1.70778172E+00 'R'
3.19924957E+02  -6.27126203E+01   0.  0.   0.00  1.80000001E+00 'o'
3.31557533E+02  -7.45532152E+01   0.  0.   0.00  1.90000001E+00 'o'
3.53201171E+02  -9.77647037E+01   0.  0.   0.00  2.10037893E+00 'o'
1 1 1 1 1 1 1                         ! 7 times 1, for 7 particles (-1 instead, to inhibit tacking).
'PARTICUL'   ! Type of particle. The only interest here is its allowing computation of time of flight,
PROTON                              ! otherwsie, zgoubi does not need it: it works with the rigidity.
'CYCLOTRON'                        ! Analytical modeling of the field in a separated sector cyclotron.
2
1   45.0  276.  1.0              ! N, AT, R0 (reference radius), type of sector (radial, spiral, both).
0. 0. 0.992122800 51.4311902 0. 0. -4.48715507E-01 2.09658166E-03 -4.52609250E-06 3.95913656E-09
-5.68972605E-14 8.48686076E-17 -2.51326976E-19 4.87639705E-22 -1.54248545E-25 1.75499497E-27
-3.23761721E-29 2.75094168E-32    ! ATTENTION: this line and the previous 2, and the following comment
                                   ! line must actually all be on a single line - no carriage return.
18.3000E+00  1.  28. -2.0                                               ! lambda=gap, gap's k  g10 g11.
8 1.1024358 3.1291507 -3.14287154 3.0858059 -1.43545 0.24047436 0. 0. 0.  ! NBCOEF, COEFS_C0-7, NORME.
11.0  3.5 35.E-3  0.E-4 3.E-8 1. 1. 1.      ! OMEGA,XI0entr,XI1entr,XI2entr,XI3entr,aentr,bentr,centr.
18.3000E+00  1.  28. -2.0                                               ! lambda=gap, gap's k  g10 g11.
8 0.70490173 4.1601305 -4.3309575 3.540416 -1.3472703 0.18261076 0. 0. 0. ! NBCOEF, COEFS_C0-5, SHIFT.
-8.5  2.  12.E-3 75.E-6 0.E-6 1. 1. 1.      ! OMEGA,XI0exit,XI1exit,XI2exit,XI3exit,aexit,bexit,cexit.
0. -1                                                                           ! Lateral EFB, unused.
0  0.  0.  0.  0.  0.  0.  0.                                                 ! NBCOEF, C0...C5, shift.
0.  0.  0.   0.  0.  0.                                    ! Lateral face: omega+, xi, R1, U1, U2, R2.
2  10.  ! Numerical method for field & derivatives,  flying mesh size is xpas/10=0.4/10. (KIRD,RESOL).
1.                                                                             ! Integration step size.
2 0.  0. 0.  0.                                                                   ! magnet positioning.
 'FIT'
 27
 2 30 0 [200,400]             ! The following first 14 variables are initial periodic radius and angle
 2 31 0 [-100,0]                                                                 ! of the seven orbits.
 2 40 0 [200,400]
 2 41 0 [-100,0]
 2 50 0 [200,400]
 2 51 0 [-100,0]
 2 60 0 [200,400]
 2 61 0 [-100,0]
 2 70 0 [200,400]
 2 71 0 [-100,0]
 2 80 0 [200,400]
 2 81 0 [-100,0]
 2 90 0 [200,400]
 2 91 0 [-100,0]
 4 9 0 .2                                                                                  ! Vary B0.
 4 12 0 1.                                     ! Next twelve variables: the 12 field indices B1 to B12.
 4 13 0 1.
 4 14 0 1.
 4 15 0 1.
 4 16 0 5.
 4 17 0 5.
 4 18 0 9.
 4 19 0 9.
 4 20 0 9.
 4 21 0 9.
 4 22 0 9.
 4 23 0 9.
 21 1e-15
 3.1 1 2 #End 0. 1. 0                 ! The following 14 constraints request periodic radius and angle
 3.1 1 3 #End 0. 1. 0                                                           ! for the seven orbits.
 3.1 2 2 #End 0. 1. 0
 3.1 2 3 #End 0. 1. 0
 3.1 3 2 #End 0. 1. 0
 3.1 3 3 #End 0. 1. 0
 3.1 4 2 #End 0. 1. 0
 3.1 4 3 #End 0. 1. 0
 3.1 5 2 #End 0. 1. 0
 3.1 5 3 #End 0. 1. 0
 3.1 6 2 #End 0. 1. 0
 3.1 6 3 #End 0. 1. 0
 3.1 7 2 #End 0. 1. 0
 3.1 7 3 #End 0. 1. 0
 3.4 1 7 #End 0. .00001 1 4            ! 6 constraints request equal rev. period for all 7 particles.
 3.4 2 7 #End 0. .00001 1 4
 3.4 3 7 #End 0. .00005 1 4
 3 4 7 #End 1.4743416128820E-02 1.e9 1 4     ! Reference time is that of particle 4, not a constraint.
 3.4 5 7 #End 0. .0001 1 4
 3.4 6 7 #End 0. .0001 1 4
 3.4 7 7 #End 0. .00001 1 4
'FAISTORE'          ! Store the 7 particle data once FIT is done, for plot by gnuplot_Trev.gnu, below.
FITted.fai
1
'SYSTEM'
1
gnuplot < gnuplot_Trev.gnu             ! Plot T_rev vs radius, reading particle data from FITted.fai.
'MARKER'   ProbPSICYCLOTRON_c_E                                              ! Just for edition purposes.
'END'                              ! End of the sequence. Whatever might follow is ignored by zgoubi.
```

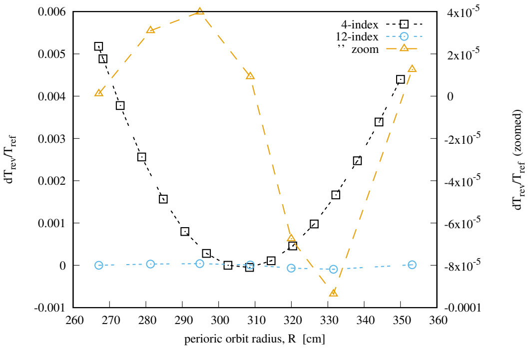


Fig. 20.54 A graph of the improved isochronism with 12 field indices (circles and left vertical axis, and a zoom-in: triangles and right axis; data are read from the file FITted.fai), compared to results obtained in question (b) (squares) where 4 indices were used. The isochronism is improved by a factor of 50 about

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